

Dual magnetic and electric dipole symmetry: Pseudoangular momentum in parity space and the electric Landé g factor

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Electric dipole moments (EDMs) are sensitive probes of fundamental symmetries and central to searches for physics beyond the standard model. We present a symmetry-based, Zeeman-analog operator framework that places magnetic and electric dipole physics on parallel footing under electromagnetic duality and introduce a polar-sector pseudo-angular-momentum degree of freedom in parity space together with an associated electric Landé g factor, g_E , that organizes induced orbital dipoles. Following Ohanian's effective-current formulation of the Zeeman effect, we construct its electric dual: The wavefunction's microscopic polarization admits an equivalent effective magnetic probability-current representation, providing a field-equivalence description of parity-mixed charge displacement. In this notation, the total EDM expectation takes the unified form $\langle \hat{d}_{\text{tot}} \rangle = d_B (g_E \frac{\langle \hat{J}_p \rangle}{\hbar} + g_E^e \frac{\langle \hat{S} \rangle}{\hbar})$, with $g_E^e = \frac{2d_{\text{int}}}{d_B}$, where $\hat{J}_p$ captures Stark-induced pseudoangular momentum and $\hat{S}$ encodes any intrinsic (spin-aligned) EDM d_{int} from symmetry-violating interactions. We define a natural electric dipole unit (the “Bohr EDM”) as $d_B \equiv ea_0 = \frac{2\mu_B}{c\alpha}$ (a_0 the Bohr radius and μ_B the Bohr magneton). As a canonical analytic benchmark, we show in the hydrogenic problem that a static electric field couples within a fixed n manifold through the scaled Runge-Lenz structure, yielding a compact Landé-like description and reproducing the Stark doublet (e.g., $|\langle d_{\text{orb}} \rangle| = 3d_B$ for the $2s$ – $2p_{m=0}$ mixing).

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I. INTRODUCTION

Electric dipole moments (EDMs) are exceptionally sensitive probes of fundamental symmetries and play a central role in the search for physics beyond the standard model. In atomic and molecular systems, EDMs can arise through two distinct mechanisms: as *intrinsic EDMs*, aligned with the particle's real spin and signaling new *CP*-violating physics, or as *induced EDMs*, generated by parity mixing of orbital states and as we show in this paper, may naturally be described by a pseudo-angular-momentum degree of freedom, as in the linear Stark effect. Although these mechanisms are conceptually different, one tied to spin in position space, the other to pseudoangular momentum in parity space, both manifest experimentally as level shifts under external electric fields, a signature exploited in high-precision spectroscopy and EDM searches.

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The linear Stark effect is a well-known example of an EDM arising from orbital dynamics and is the electric analog of the Zeeman effect. This phenomenon describes the splitting or shifting of atomic energy levels in a static electric field [1]. For a nonzero first-order shift to occur, the system must contain states of opposite parity that are sufficiently degenerate. In excited hydrogen, for instance, the $n = 2$ manifold includes the degenerate $2s$ and $2p$ orbitals. An applied electric field mixes these states, distorting the electron cloud and inducing a dipole moment aligned with the field, which breaks inversion symmetry and couples to the orbital motion, producing a first-order energy shift [2,3].

In contrast, EDMs associated with intrinsic spin are expected to be permanent and arise from violations of discrete symmetries, particularly parity (P) and time-reversal (T) invariance. Within quantum field theory, such EDMs emerge from CP -violating operators and are predicted to couple directly to the spin vector via terms such as

$$\mathcal{L}_{\text{EDM}} = -\frac{1}{2}d_{\text{int}}\bar{\psi}\sigma^{\mu\nu}\gamma^5\psi F_{\mu\nu}. \quad (1)$$

Here, d_{int} is the intrinsic EDM of the particle (in units of C m), ψ is the Dirac spinor field, $\sigma^{\mu\nu} = \frac{i}{2}\gamma^\mu\gamma^\nu$ is the antisymmetric spin tensor, $F_{\mu\nu}$ is the electromagnetic field strength tensor, and γ^5 ensures that the interaction violates both parity (P) and time-reversal (T) symmetry. The detection of a spin-aligned EDM in a fundamental particle like the electron or neutron would constitute direct evidence of physics beyond the standard model and provide insights into the baryon asymmetry of the universe.

Despite decades of effort, no intrinsic EDM has been observed to date. Current experiments place stringent limits on possible CP -violating effects [4–8], driving continued interest in EDM searches across atomic, molecular, and solid-state platforms. Theoretical frameworks beyond the standard model such as supersymmetry, left-right symmetric models, and theories of baryogenesis often predict EDMs just below current experimental sensitivity.

Complementary to these quantum approaches, insights from condensed matter and electromagnetic energy generation provide an alternative view of EDM-like phenomena. For example, modern polarization theory models permanent electric polarization in materials using the Berry phase formalism, where each unit cell contributes a quantized dipole moment [9–11]. This macroscopic charge separation is analogous to intrinsic EDMs in fundamental systems. Active voltage sources also create permanent polarizations [12–14] and reflect symmetry-breaking mechanisms similar in form to those studied in particle physics [15,16].

In this work, we build on the magnetic-electric analogy to develop a semiclassical model of EDMs, inspired by the Bohr magneton and by Ohanian’s analysis [17]. Ohanian established a connection between magnetic moment and electron spin by showing that spin can be viewed as arising from a circulating flow of energy in the wave field, analogous to the angular momentum carried by a classical circularly polarized wave. In doing so, he introduced an effective probability current density, analogous to a bound current in a macroscopic magnetized medium, defined as the curl of an effective microscopic magnetization field constructed directly from the electron wavefunction.

By electromagnetic duality, we construct an electric analog of Ohanian’s picture: We introduce a “Bohr EDM” and a dual-current description in which an effective probability magnetic current arises from the curl of a microscopic polarization field constructed from the wavefunction. In this framework, induced EDMs, such as those of the Stark-mixed hydrogen doublet, are organized by a polar-sector pseudoangular momentum in parity space (with an explicit Runge-Lenz connection in the hydrogenic case). This pseudoangular momentum is not a physical rotation in space, but a measure of parity mixing that determines the induced polarization. The same Zeeman-analog operator language accommodates both induced orbital EDMs (from parity mixing/charge displacement) and intrinsic spin-aligned EDMs within a unified framework; hydrogen is used as the canonical analytic benchmark where the algebra can be demonstrated in closed form, so the familiar Stark eigenstates are reproduced by construction while the main contribution is the underlying duality-driven operator structure and its physical interpretation.

II. MAGNETIC DIPOLE MOMENTS

A. The semiclassical Bohr magneton

The Bohr magneton arises naturally from a semiclassical model of an electron orbiting a nucleus, as in the Bohr model of the atom. In this picture, an electron of charge $-e$ and mass m_e moves in a circular orbit of radius r with speed v , generating a current, I :

$$I = \frac{e}{T} = \frac{ev}{2\pi r}. \quad (2)$$

To draw a classical parallel, consider a loop of current I encircling an area $\vec{A}$. The magnetic dipole moment associated with this loop is given by

$$\vec{\mu} = I\vec{A}. \quad (3)$$

For an electron in a circular orbit of radius r and speed v , the current is $I = e/T = ev/(2\pi r)$, and the area is $A = \pi r^2$, yielding

$$\mu = \frac{evr}{2} = \frac{eL}{2m_e}, \quad (4)$$

where $L = m_e vr$ is the orbital angular momentum.

In the Bohr model, the quantization condition imposes $L = \hbar$, for the lowest energy level, which yields a magnetic moment

$$\mu_B = \frac{e\hbar}{2m_e}, \quad (5)$$

defining the Bohr magneton. This derivation illustrates how the classical concept of a circulating charge loop, combined with quantum angular momentum quantization, leads to a fundamental unit of magnetic moment in atomic physics.

B. Zeeman effect due to intrinsic and orbital magnetic moments

In the presence of an external static and uniform magnetic field $\vec{B}$, the perturbation Hamiltonian arises from the interaction of the electron magnetic moments with $\vec{B}$:

$$\hat{H}_Z = -\vec{\mu} \cdot \vec{B}, \quad (6)$$

where the magnetic moment, $\vec{\mu} = \vec{\mu}_L + \vec{\mu}_S$, the sum of the orbital and intrinsic spin magnetic moments,

$$\begin{aligned} \hat{\mu}_L &= -\frac{\mu_B}{\hbar} \hat{L}, \\ \hat{\mu}_S &= -g_S \frac{\mu_B}{\hbar} \hat{S}. \end{aligned} \quad (7)$$

Here, g_S is the electron spin g factor, which in Dirac theory equals 2, and in QED $\simeq 2.002\,319\,2$. While orbital moments follow from classical current loops, the spin moments are considered to have no classical counterpart.

This perturbation leads to lines splitting into several components in the presence of the magnetic field, with Zeeman corrections characterized by

$$\hat{H}_M = g_M \frac{\hat{J}_z}{\hbar} \mu_B B_z, \quad \Delta E_M = g_M m_j \mu_B B_z, \quad (8)$$

where $\hat{H}_M$ is the Zeeman Hamiltonian, $\hat{J}_z$ is the z component of the total angular momentum, $\hat{J}$, while ΔE_M is the energy shift of each Zeeman sublevel, g_M is the magnetic Landé g factor, and $m_j = -J, -J+1, \dots, J$ is the magnetic quantum number associated with the total angular momentum projection.

C. Ohanian's model of spin: Magnetic moment from internal magnetization

In his seminal paper “What is Spin?” [17], Ohanian provides a physically motivated description of the electron's magnetic moment by modeling it in terms of an effective probability quantum current distribution, despite the fact that spin has no classical analog in terms of rotating charge. His treatment shows that the electromagnetic properties of spin, specifically the magnetic moment, can be derived from a localized magnetization density constructed from the Dirac wavefunction.

Ohanian introduces a microscopic magnetization vector field $\vec{M}$ defined as

$$\vec{M} = \mu_B \psi^\dagger \vec{\Sigma} \psi, \quad (9)$$

where ψ is the Dirac spinor and $\vec{\Sigma}$ represents the spin matrices. This magnetization leads to an effective bound probability current density,

$$\vec{J}_b = \nabla \times \vec{M}, \quad (10)$$

analogous to classical magnetization currents. The magnetic moment $\vec{\mu}$ associated with this probability current is

$$\vec{\mu} = \frac{1}{2} \int \vec{r} \times \vec{J}_b d^3r. \quad (11)$$

Using the relation to magnetization

$$\vec{\mu} = \int \vec{M} d^3r, \quad (12)$$

Ohanian recovers the quantum result

$$\hat{\mu}_S = -g_S \frac{\mu_B}{\hbar} \hat{S}, \quad g_S = 2. \quad (13)$$

Ohanian's construction reveals that although spin cannot be interpreted as a literal circulating charge, its associated magnetic moment behaves as if it originates from an effective probability current density. This provides a bridge between classical electromagnetism (magnetization and bound currents) and quantum field theory (spinor structure and symmetry). The result reinforces the idea that the magnetic moment of a spin- $\frac{1}{2}$ particle is not merely a formal operator feature, but corresponds to a real, observable electromagnetic quantity tied to internal degrees of freedom in the quantum field.

III. ELECTRIC DIPOLE MOMENTS

So far, no particle has been observed to have a nonzero intrinsic EDM, and are predicted to exist but at levels far below current experimental sensitivity (e.g., the electron EDM is expected to be about 10^{-38} e cm, whereas the neutron is about 10^{-31} e cm). State-of-the-art experiments constrain the electron EDM to be smaller than 4.1×10^{-30} e cm [18] and the neutron EDM below 1.8×10^{-26} e cm [19], with similar null results for the muon and proton. These increasingly tight bounds are crucial because a confirmed intrinsic EDM would signal new CP -violating physics beyond the standard model, possibly helping to explain the matter-antimatter asymmetry of the Universe. In contrast, the electric dipole moments that appear in the linear Stark effect of hydrogen or polar molecules are *not intrinsic but induced*. For example, in hydrogen they arise from parity mixing of nearly degenerate orbitals, such as the $2s$ and $2p_0$ states. In this case, the dipole operator acts on a pseudo-angular-momentum degree of freedom, defined in the two-level parity subspace, and the resulting EDM depends on the external field and the mixing angle.

For atomic scale, we define the atomic dipole unit, the Bohr EDM as

$$d_B \equiv e a_0. \quad (14)$$

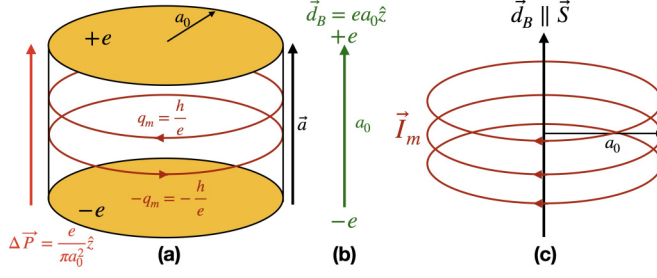


FIG. 1. Semiclassical representations of the Bohr EDM scale, $\vec{d}_B$. (a) Idealized quasi-1D “unit cell” of quantized permanent polarization, $\Delta\vec{P} = \frac{e}{\pi a_0} \hat{z}$, for which $\nabla \times \vec{P} \neq 0$ implies an *effective* magnetic current on the radial surface (a dual representation of the polarization texture, not a literal magnetic-charge flow). For mnemonic purposes, this surface current can be pictured as two fictitious opposite magnetic charges $\mp q_m = \mp \frac{h}{e}$ circulating in opposite senses, associated with pseudospin (pseudoangular momentum) in parity space rather than mechanical orbital motion. (b) Classical Bohr EDM: Two point charges $\pm e$ separated by the Bohr radius a_0 . (c) Schematic of an intrinsic Bohr-scale EDM connected to the position-space spin vector $\vec{S}$, represented equivalently by an effective azimuthal magnetic current $\vec{I}_m$ at radius $\sim a_0$.

In the Coulomb problem, this scale is consistent with the hydrogenic SO(4) (Runge-Lenz) structure within a fixed n manifold, and it also has an immediate classical interpretation: It is the dipole moment of two point charges $\pm e$ separated by a distance a_0 [Fig. 1(b)]. Furthermore, the same dipole unit appears in a quasi-one-dimensional (1D) “cell” formulation governed by modern polarization theory, when the size of the cell spacing is of order a_0 .

The following toy model is included only as a *heuristic mnemonic* (not as a physical derivation) to connect this dipole scale to a dual-current picture. Similar to the semiclassical Bohr magneton estimate, we imagine an effective surface magnetic current associated with the curl of the polarization. For mnemonic purposes, this may be visualized as two fictitious opposite “magnetic charges” $\pm \frac{h}{e}$ circulating at the Bohr radius $a_0 = \frac{\hbar}{m_e c \alpha}$, where $\alpha = \frac{e^2}{4\pi\epsilon_0 \hbar c}$.

In this picture, the net electric charge, net magnetic charge, and total angular momentum of the cell are zero, but an effective magnetic surface current exists, which we associate with pseudospin (parity mixing) rather than mechanical orbital motion. With $q_m = \frac{h}{e}$, the effective magnetic current at the boundary is written as

$$\vec{I}_m = -\frac{2q_m v}{2\pi a_0} \hat{\phi} = -\frac{e}{2\pi a_0^2 m^* \epsilon_0 \alpha c} (a_0 m^* v) \hat{\phi}, \quad (15)$$

where m^* is an effective inertia parameter for the internal (parity-space) degree of freedom. For a Bohr-scale dimensional estimate, we take the usual quantization scale $a_0 m^* v \sim \hbar$ with $m^* \sim \mathcal{O}(m_e)$ (the precise value of m^* is not used elsewhere in the article). This gives

$$\vec{I}_m \approx -\frac{e\hbar}{\pi a_0^2 m_e \epsilon_0 \alpha c} \hat{\phi} = -\frac{2\mu_B}{\pi a_0^2 \epsilon_0 \alpha c} \hat{\phi}, \quad (16)$$

and we may express the Bohr EDM in terms of the effective azimuthal magnetic current (left-hand rule) as

$$d_B = -\epsilon_0 I_m \pi a_0^2 = \frac{2\mu_B}{c\alpha} = ea_0, \quad (17)$$

the electromagnetic dual of the Bohr magneton in Eq. (5). Thus, Figs. 1(a) and 1(b) represent the same Bohr-scale dipole unit: $\vec{a} = a_0 \hat{z}$ implies $d_B = ea_0$.

In modern polarization theory, the bulk polarization is defined “modulo a quantum,” along with a Berry phase defined modulo 2π , and is given by

$$\Delta \vec{P} = \frac{e \vec{a}}{V}, \quad (18)$$

where $\vec{a}$ is any lattice vector and V the unit-cell volume. This follows from an electron in a periodic potential, which gives the $U(1)$ gauge freedom (Bloch’s theorem); $\psi_{n\vec{k}} \rightarrow e^{-i\vec{k}\cdot\vec{a}} \psi_{n\vec{k}}$, which shifts the Berry phase by 2π and $\vec{P}$ by $e\vec{a}/V$. Here, we consider a quasi-1D “cell” as shown in Fig. 1 of length a (along $\hat{z}$) and cross section A_{eff} , so the effective volume is $V_{\text{eff}} = A_{\text{eff}}a$. In our construction 2π , winding of the internal phase is equivalent to choosing $\vec{a} = a\hat{z}$. Then, from Eq. (18),

$$\Delta P_z = \frac{ea}{V_{\text{eff}}} = \frac{e}{A_{\text{eff}}} = \frac{e}{\pi a_0^2}. \quad (19)$$

This is equivalent to a single cell of a supercell in modern polarization theory [10], with a quantum of permanent polarization ΔP_z set by $A_{\text{eff}} = \pi a_0^2$, as shown in Fig. 1. With $\vec{a} = a_0\hat{z}$, the dipole moment follows as $\vec{d}_B = \int \Delta P_z d^3r = ea_0\hat{z} = d_B\hat{z}$, consistent with Eq. (17).

For completeness when discussing the hydrogenic problem, we note that, for a finite-mass nucleus, the mass parameter in the internal (relative-coordinate) Coulomb problem should be understood as the reduced mass

$$\mu = \frac{m_e m_N}{m_e + m_N}, \quad (20)$$

rather than the electron mass m_e , so that the Bohr radius is replaced by

$$a_\mu = \frac{4\pi\epsilon_0\hbar^2}{\mu Ze^2}, \quad (21)$$

and hence the Bohr dipole scale used throughout becomes $d_B = ea_\mu$ (rather than ea_0 being defined using m_e). This substitution rescales lengths and energies but does not modify the underlying symmetry structure: The $SO(4) \simeq SU(2) \times SU(2)$ algebra and associated commutation relations remain unchanged (see Appendix C), with $m \rightarrow \mu$ in the Hamiltonian. In practice, for hydrogen the correction is very small,

$$\frac{\mu}{m_e} = \frac{1}{1 + m_e/m_p} \approx 0.99946, \quad (22)$$

so a_μ (and thus d_B) differs from the m_e value by only $\sim 5.4 \times 10^{-4}$ (about 0.054%), and either convention may be adopted depending on the desired level of precision. For clarity, we emphasize that throughout the paper we use the internal (relative-coordinate) formulation of the two-body problem, and therefore all “densities/fields” refer to quantities defined in the nucleus/centre-of-mass frame.

IV. DUAL OHANIAN MODEL: DIPOLE MOMENT FROM INTERNAL POLARIZATION

A. Bohr electric dipole moment from spin

As discussed previously, in Ohanian’s seminal work, the intrinsic magnetic moment of the electron was interpreted semiclassically as arising from an effective circulating energy flow of the electron wavefunction, generating an angular momentum and associated magnetic dipole [17]. The effective magnetization density was defined from the bilinear spinor quantity $\psi^\dagger \vec{\Sigma} \psi$, leading to the construction of an effective probability current density $\vec{J}_b = \nabla \times \vec{M}$, from which the Bohr magneton, μ_B , naturally arises.

In analogy, we pose a putative dual picture in which a permanent EDM arises from an internal effective probability magnetic current distribution, which creates a permanent polarization by a

left-hand rule. This provides a complementary viewpoint in which the EDM can be understood as the macroscopic manifestation of internal field circulations analogous to those producing magnetic dipole moments. Thus, by invoking an analog dual construction, one may define a spin operator, $\hat{\vec{S}}$, associated with the following internal microscopic polarization density:

$$\vec{P}(\vec{r}) = d_B \psi^\dagger(\vec{r}) \vec{\Sigma} \psi(\vec{r}), \quad d_B \equiv ea_0, \quad (23)$$

$$= \frac{2d_B}{\hbar} \psi^\dagger(\vec{r}) \hat{\vec{S}} \psi(\vec{r}), \quad \hat{\vec{S}} \equiv \frac{\hbar}{2} \vec{\sigma}, \quad (24)$$

where $\vec{\sigma}$ denotes the vector of Pauli operators acting on the two-level parity space, $\vec{\Sigma}$ is the Pauli spin matrix vector, and ψ is the Dirac or Pauli spinor. Adopting the dual (Weber) convention for magnetic current, the curl of $\vec{P}$ defines an effective probability *magnetic* current density,

$$\vec{J}_m = -\frac{1}{\epsilon_0} \nabla \times \vec{P}, \quad (25)$$

which plays the role dual to Ohanian's effective current from $\vec{M}$. Since, $\|\langle \vec{\Sigma} \rangle\| = 1$, the expectation value of the electric dipole moment is then

$$\langle \vec{d} \rangle = \int \vec{P}(\vec{r}) d^3r = d_B, \quad (26)$$

consistent with

$$\hat{\vec{d}} = \frac{g_E^S d_B}{\hbar} \hat{\vec{S}} g_E^S = 2, \quad (27)$$

where g_E^S is the electric Landé g factor. Thus, a fully polarized single-particle state carries the ‘‘Bohr’’ electric dipole moment $d_B = ea_0$. This gives rise to an effective electric dipole moment associated with the spin degree of freedom. The spin carries an intrinsic dual moment generated by the internal microscopic polarization of the wavefunction. This construction provides a dual Ohanian model that allows a semiclassical interpretation of an effective intrinsic electric dipole moment.

In this framework, the magnetic moment $\vec{\mu}$ due to a circulating electric current is mirrored by an electric dipole moment $\vec{d}$ due to a circulating magnetic current:

$$\langle \vec{d} \rangle = \frac{\epsilon_0}{2} \int \vec{r} \times \vec{J}_m d^3r, \quad (28)$$

with a semiclassical picture shown in Fig. 1(c). This construction preserves the correct transformation properties under parity (P) and time-reversal (T), making it consistent with known symmetry-breaking signatures of EDMs. However, in reality, no EDMs coupled to intrinsic spin have yet to be observed, and if they do exist are significantly smaller than the Bohr EDM.

B. Intrinsic and orbital polarization

If the electron carries an intrinsic electric dipole moment, d_{int} , the local microscopic polarization density of the electron wavefunction would be of the form

$$\vec{P}_e(\vec{r}) = d_{\text{int}} \psi^\dagger(\vec{r}) \vec{\Sigma} \psi(\vec{r}) \approx \frac{2d_{\text{int}}}{\hbar} \vec{s}(\vec{r}), \quad (29)$$

consistent with

$$\hat{\vec{d}}_e = \frac{g_E^e d_B}{\hbar} \hat{\vec{S}} g_E^e = \frac{2d_{\text{int}}}{d_B}, \quad (30)$$

where $\vec{s}(\vec{r}) = \psi^\dagger(\vec{r}) \frac{\hbar}{2} \vec{\sigma} \psi(\vec{r})$ is the nonrelativistic spin density and g_E^e is the effective electric Landé g factor of the electron. To understand how the electron behaves in an electric field, we consider the

interaction term in Eq. (1). To interpret this interaction physically, we examine its nonrelativistic limit and begin by noting the electromagnetic field strength tensor:

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}. \quad (31)$$

The term $\bar{\psi}\sigma^{\mu\nu}\gamma^5\psi F_{\mu\nu}$ contracts to

$$2\bar{\psi}\sigma^{0i}\gamma^5\psi F_{0i} + \bar{\psi}\sigma^{ij}\gamma^5\psi F_{ij}, \quad (32)$$

where the dominant term at low velocities is the electric coupling:

$$\bar{\psi}\sigma^{0i}\gamma^5\psi \approx \psi^\dagger \Sigma^i \psi, \quad F_{0i} = -E^i. \quad (33)$$

Thus,

$$\bar{\psi}\sigma^{\mu\nu}\gamma^5\psi F_{\mu\nu} \rightarrow -2\psi^\dagger \vec{\Sigma} \psi \cdot \vec{E}. \quad (34)$$

Substituting back into the Lagrangian (1), we obtain

$$\mathcal{L}_{\text{EDM}} \rightarrow d_{\text{int}} \psi^\dagger \vec{\Sigma} \psi \cdot \vec{E} = -\vec{P}(\vec{r}) \cdot \vec{E}, \quad (35)$$

which corresponds to the energy density term.

Generalizing to an atomic state $\psi_\pm$, the total polarization may be expanded to the sum of the orbital (charge-displacement) part and the EDM (spin) part,

$$\vec{P}_{\text{tot}}(\vec{r}) = \underbrace{\rho_{\text{charge}}(\vec{r}) \vec{r}}_{\text{orbital (Stark)}} + \underbrace{d_{\text{int}} \psi_\pm^\dagger(\vec{r}) \vec{\Sigma} \psi_\pm(\vec{r})}_{\text{spin EDM}}. \quad (36)$$

Integrating over space gives the total permanent dipole moment

$$\langle \vec{d}_{\text{tot}} \rangle = \int \vec{P}_{\text{tot}} d^3r = \langle \vec{d}_{\text{orb}} \rangle + d_{\text{int}} \langle \vec{\Sigma} \rangle_{\psi_\pm}, \quad (37)$$

$$\langle \vec{d}_{\text{orb}} \rangle \simeq \int \rho_{\text{charge}}(\vec{r}) \vec{r} d^3r. \quad (38)$$

Since no electron EDM has been observed, it either vanishes or the intrinsic value d_{int} lies below current experimental sensitivity. Throughout this work, we set $d_{\text{int}} \simeq 0$, so that the polarization obtained from Eq. (36) reduces to $\vec{P}(\vec{r}) \simeq \rho_{\text{charge}}(\vec{r}) \vec{r}$. To calculate orbital EDMs (e.g., the linear Stark effect), we use the effective magnetic probability current density [Eq. (25)] together with the polarization relations (26) and (28). Importantly, the induced orbital EDM is produced physically by parity mixing (charge displacement); the associated magnetic-current description is a topological field equivalence (a dual representation of the polarization texture), not an additional causal mechanism. We further show that this charge displacement can be organized in terms of an *effective* angular-momentum-like degree of freedom, a pseudoangular momentum in parity space, that emerges when the system possesses an internal two- (or few-) level structure governed by a hidden symmetry.

V. THE STARK EFFECT FOR THE HYDROGEN ATOM

To make the analogy between magnetic and electric dipole moments concrete, we consider the classic example of the linear Stark effect in the hydrogen atom. The perturbing Hamiltonian is

$$H_S = -\vec{d} \cdot \vec{E} = -q\vec{r} \cdot \vec{E}, \quad (39)$$

where $\vec{d} = q\vec{r}$ is the electric dipole operator and $\vec{E}$ is the applied electric field. For a field along the z direction, $\vec{E} = E_z \hat{z}$, this becomes $H_S = -qzE_z$.

In first-order perturbation theory, the energy shift is

$$\Delta E_S^{(1)} = \langle n, l, m | H_S | n, l, m \rangle, \quad (40)$$

which vanishes unless the state has mixed parity. In this representation, the electron's orbital quantum numbers are the principal quantum number $n = 1, 2, 3, \dots$, the orbital angular momentum $\ell = 0, 1, \dots, n-1$ (with spectroscopic labels $s, p, d, f, \dots$ for $\ell = 0, 1, 2, 3, \dots$), and the magnetic quantum number $m = -\ell, -\ell+1, \dots, \ell$, which set $L^2 = \ell(\ell+1)\hbar^2$ and $L_z = m\hbar$.

In a degenerate subspace of opposite parity states, such as the $n = 2$ manifold in hydrogen, the electric field couples $2s$ ($\ell = 0$) and $2p$ ($\ell = 1$) orbitals, leading to level splitting. Diagonalizing the perturbation in this subspace gives a permanent induced dipole moment and energy shifts linear in E_z .

This behavior mirrors the Zeeman effect, where magnetic dipole moments couple to an applied magnetic field. In both cases, an external field breaks an internal symmetry (parity or time reversal), revealing hidden structure and inducing a dipole aligned to the field.

A. Stark effect for the $n = 2$ orbital

The hydrogen eigenstates $\psi_{n\ell m}$ separate into a radial part $R_{n\ell}(r)$ and an angular part $Y_{\ell m}(\theta, \phi)$ in spherical coordinates (r, θ, ϕ) by separation of variables in the Schrödinger equation with the Coulomb potential $V(r) = -\frac{e^2}{4\pi\epsilon_0 r}$. Explicitly,

$$\psi_{n\ell m}(r, \theta, \phi) = R_{n\ell}(r) Y_{\ell m}(\theta, \phi). \quad (41)$$

Here, we focus on the $n = 2$ degenerate manifold that includes the following states: the $2s$ state, ψ_{200} , and the $2p$, ψ_{21m} states, with $m = -1, 0, +1$. From Eq. (40), only the matrix element between ψ_{200} and ψ_{210} contributes at first order, given by

$$\langle 2s | H_S | 2p_0 \rangle = -eE_z \langle 2s | z | 2p_0 \rangle. \quad (42)$$

The normalized radial wavefunctions given in Eq. (41) for $n = 2$ are known to be

$$R_{20}(r) = \frac{1}{2\sqrt{2}a_0^{3/2}} \left(2 - \frac{r}{a_0} \right) e^{-r/(2a_0)}, \quad (43)$$

$$R_{21}(r) = \frac{1}{2\sqrt{6}a_0^{3/2}} \left(\frac{r}{a_0} \right) e^{-r/(2a_0)}, \quad (44)$$

and the angular wavefunctions to be

$$Y_{00}(\theta, \phi) = \frac{1}{\sqrt{4\pi}}, \quad (45)$$

$$Y_{10}(\theta, \phi) = \sqrt{\frac{3}{4\pi}} \cos \theta, \quad (46)$$

so

$$\psi_{2s}(r, \theta) = R_{20}(r) Y_{00} = \frac{R_{20}(r)}{\sqrt{4\pi}}, \quad (47)$$

$$\psi_{2p_0}(r, \theta) = R_{21}(r) Y_{10} = \sqrt{\frac{3}{4\pi}} R_{21}(r) \cos \theta. \quad (48)$$

The integral splits into radial and angular parts:

$$\langle 2s | z | 2p_0 \rangle = \int \psi_{200}^*(\vec{r}) z \psi_{210}(\vec{r}) d^3r = \left(\int_0^\infty R_{20}(r) r^3 R_{21}(r) dr \right) \left(\int Y_{00}^* \cos \theta Y_{10} \sin \theta d\theta d\phi \right). \quad (49)$$

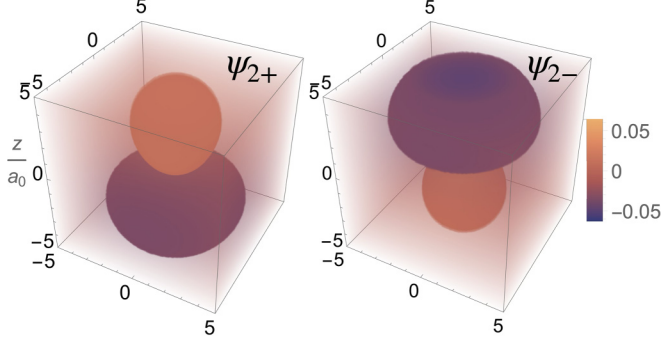


FIG. 2. Density plot of the Stark state wavefunctions, $\psi_{2\pm}$. The asymmetry of the electron cloud around the nucleus generates an electric dipole moment.

Using the known integrals

$$\int_0^\infty R_{20}(r) r^3 R_{21}(r) dr = -3a_0, \quad (50)$$

$$\int_0^{2\pi} d\phi \int_0^\pi Y_{00}^* \cos \theta Y_{10} \sin \theta d\theta = 1, \quad (51)$$

we find

$$\langle 2s | z | 2p_0 \rangle = -3a_0. \quad (52)$$

Given the direction of the electric field, $\vec{E}$, defines the z direction, the perturbation matrix in the $\{2s, 2p_0\}$ basis is

$$H = \begin{pmatrix} 0 & 3eE_z a_0 \\ 3eE_z a_0 & 0 \end{pmatrix}. \quad (53)$$

Diagonalizing this into the hybridized Stark basis,

$$\psi_{2\pm}(r, \theta) = \frac{1}{\sqrt{2}}[\psi_{2s}(r, \theta) \pm \psi_{2p_0}(r, \theta)], \quad (54)$$

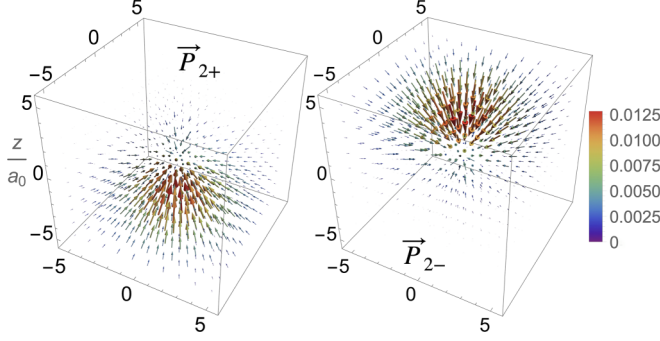
yields the well-known $n = 2$ first-order energy Stark shifts,

$$\Delta E_S^{(1)} = \pm 3eE_z a_0 = \pm 3E_z d_B = \pm E_z d_2, \quad (55)$$

where d_2 is the magnitude of the dipole moment, with the wavefunctions given by Eq. (54), and plotted in Fig. 2. A comparison of parameters for the $n = 2$ and $n = 3$ Stark states is shown in Table I.

TABLE I. Calculated values for $n = 2, 3$, where the Stark states have nonzero k values.

n	$g_E(n) = \frac{3}{2}n$	k values	$\int \frac{z}{r} \psi ^2, d^3r$	$\langle J_{p,z} \rangle$	$\langle d_z \rangle$
2	3	1, 0, -1	$-\frac{1}{2}, 0, \frac{1}{2}$	$\hbar, 0, -\hbar$	$3d_B, 0, -3d_B$
3	$\frac{9}{2}$	2, 1, 0, -1, -2	$-\frac{2}{3}, -\frac{1}{3}, 0, \frac{1}{3}, \frac{2}{3}$	$2\hbar, \hbar, 0, -\hbar, -2\hbar$	$\frac{9}{2}d_B, \frac{9}{4}d_B, 0, -\frac{9}{4}d_B, -\frac{9}{2}d_B$


 FIG. 3. Effective microscopic polarization vector fields for $n = 2$ Stark states.

B. Dual Ohanian model for the $n = 2$ orbital

Applying the dual Ohanian model introduced in Sec. IV, we compute the *internal* (relative-coordinate) electron charge density from the Stark wavefunctions. The hydrogen atom is a two-body system and separates into center-of-mass and relative coordinates, $\Psi(\vec{r}_e, \vec{r}_p) = \Phi(\vec{R})\psi(\vec{r})$, with $\vec{r} = \vec{r}_e - \vec{r}_p$. The corresponding unconditional laboratory-frame one-particle density depends on the center-of-mass state $\Phi(\vec{R})$, where the center-of-mass coordinate is $\vec{R} = (m_e\vec{r}_e + m_p\vec{r}_p)/(m_e + m_p)$. For a freely propagating atom with a delocalized center-of-mass wavefunction, the unconditional laboratory-frame density becomes translationally invariant (uniform) and therefore does not encode the internal structure; it is not the appropriate quantity for visualizing internal polarization fields. Throughout, we work in the internal (centre-of-mass/relative-coordinate) description, so that the associated microscopic “effective” fields are internal fields defined in the atomic rest frame. Considering this, we calculate the effective electron charge density directly from the internal wavefunction,

$$\rho_{2\pm}(r, \theta) = -e |\psi_{2\pm}(r, \theta)|^2 \quad (56)$$

and thus the effective microscopic polarization from the charge density,

$$\vec{P}_{2\pm}(r, \theta) = r \rho_{2\pm}(r, \theta) \hat{r} \quad (57)$$

$$= -\frac{e r}{8\pi} [R_{20}^2 + 3R_{21}^2 \cos^2 \theta \pm 2\sqrt{3} R_{20} R_{21} \cos \theta] \hat{r}, \quad (58)$$

which is plotted in Fig. 3. Substituting Eq. (58) into Eq. (25), the effective magnetic current density may be calculated to be

$$\vec{J}_{m2\pm} = -\frac{1}{\epsilon_0} \nabla \times \vec{P}_{2\pm} = -\frac{e \sin \theta R_{21}}{4\pi \epsilon_0} [3R_{21}(r) \cos \theta \pm \sqrt{3} R_{20}(r)] \hat{\phi}. \quad (59)$$

Note that $\vec{J}_{m2\pm} = \mp J_{m2} \hat{\phi}$ is purely azimuthal; it vanishes on the symmetry axis (where $\sin \theta = 0$), and the $\pm$ sign reverses the direction between the two Stark states, as shown in Fig. 4. Furthermore, Eq. (28) implies a left-hand rule relating the magnetic current to the dipole moment, d_2 . Substituting Eq. (59) into Eq. (28), one readily confirms $d_2 = 3d_B$.

VI. THE ELECTRIC g FACTOR

The hidden Runge-Lenz (Kepler) symmetry of the Coulomb problem (see Appendix A) explains why hydrogen shows a *linear* Stark effect. To calculate expectation values, we can start from the

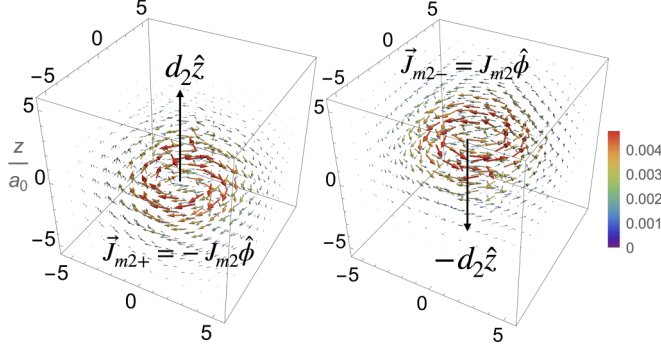


FIG. 4. Magnetic current vector field, $\vec{J}_{m2\pm}$ calculated from Eq. (59), acting as a generator of the $\pm$ EDMs of magnitude d_2 , via the left-hand rule for the $n = 2$ Stark states.

Runge-Lenz density,

$$\vec{\mathcal{A}}(\vec{r}) = \frac{\hbar^2}{m} \text{Im}[(\vec{\nabla}\psi^*) \times (\vec{r} \times \vec{\nabla}\psi)] - \kappa \frac{\vec{r}}{r} |\psi(\vec{r})|^2, \quad \int \vec{\mathcal{A}} d^3r = \langle \hat{\vec{A}} \rangle. \quad (60)$$

For the hydrogen Stark eigenstates, we can choose ψ real (no vector potential), so the first currentlike term in Eq. (60) vanishes and the expectation value of the z component of the Runge-Lenz vector becomes

$$\langle A_z \rangle = \int \mathcal{A}_z d^3r = -\kappa \int \frac{z}{r} |\psi(\vec{r})|^2 d^3r, \quad (61)$$

and the scaled vector in units of angular momentum (see Appendix A) becomes

$$\langle A_{\text{sc},z} \rangle = -\hbar n \int \frac{z}{r} |\psi(\vec{r})|^2 d^3r = \langle \hat{J}_{p,z} \rangle, \quad (62)$$

where we define $\hat{J}_{p,z}$ as the z component pseudoangular momentum, $\hat{J}_p$, in parity space, which is essentially defined as the scaled Runge-Lenz vector. It follows that $\hat{J}_{p,z}$ is quantized in units of $\hbar$ so that,

$$\langle J_{p,z} \rangle = \hbar k, k = n_1 - n_2 \in \{-n + 1, \dots, n - 1\}, \quad (63)$$

so

$$\int \frac{z}{r} |\psi(\vec{r})|^2 d^3r = -\frac{k}{n}. \quad (64)$$

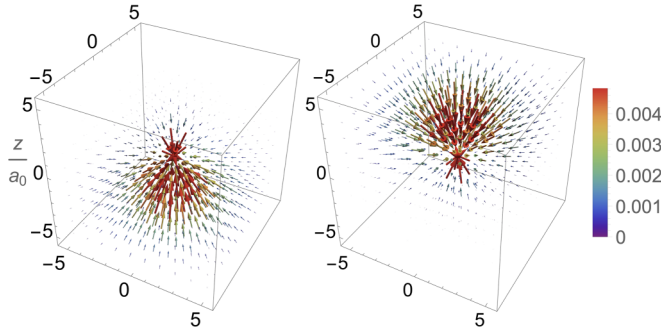


FIG. 5. The Runge-Lenz vector density, $\vec{\mathcal{A}}$, for the $n = 2$ Stark states, given by Eq. (60).

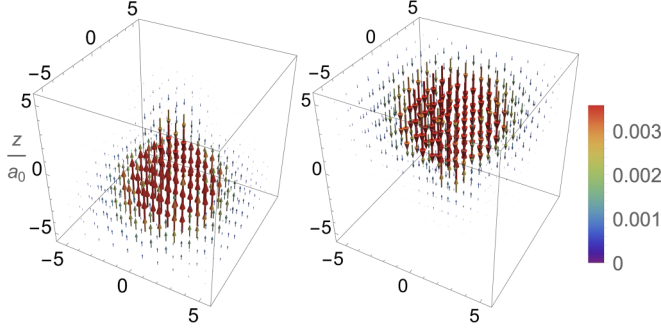


FIG. 6. The z component of the Runge-Lenz vector density, $\vec{A}_z$, for the $n = 2$ Stark states, defined in Eq. (61).

For the $n = 2$ Stark states, the Runge-Lenz vector density, along with the z -component is plotted in Fig. 5 and 6 respectively. Using the analogous relation to the magnetic Landé factor defined by Eqs. (7) and (8),

$$\hat{d} = \frac{g_E d_B}{\hbar} \hat{J}_p, \quad (65)$$

where $g_E(n) = \frac{3}{2}n$ is the electric Landé factor and $d_B = ea_0$ from Eq. (17) defines the Bohr dipole scale, the expectation value of the dipole moment along the field direction, and the Stark energy level shift, ΔE_S , becomes

$$\begin{aligned} \langle d_z \rangle &= \frac{g_E d_B}{\hbar} \langle J_{p,z} \rangle = g_E d_B k, \\ \Delta E_S &= g_E k d_B E_z. \end{aligned} \quad (66)$$

These expressions show that the induced dipole moment and observed energy level shift are proportional to the projection quantum number k , scaled by the electric Landé factor and the fundamental dipole unit. Also, the density sum rule in Eq. (64) is a compact way to verify that the *local* Runge-Lenz density reproduces the *global* SU(2) quantization, giving a direct bridge to the electric Landé factor via Eq. (65). Just as magnetic g factors differ between systems depending on their internal structure and dynamics, the electric g factor encodes the efficiency with which the electron orbital in parity space is converted to a permanent polarization through the effective magnetic current loop.

Electric spin

To include an intrinsic dipole moment with the induced orbital effects, one can combine Eqs. (30) and (65) to obtain

$$\hat{d}_{\text{tot}} = d_B [g_E \hat{J}_p \hbar + g_E^e \langle \hat{S} \rangle / \hbar], \quad g_E^e = \frac{2d_{\text{int}}}{d_B}. \quad (67)$$

In this case, the intrinsic EDM is aligned with the particles spin and is related to a probability magnetic current produced in position space by the spin operator. Spin is a pseudovector (axial vector), it behaves like an ordinary vector under rotations, but under spatial inversion (parity), it does not change sign, unlike a true polar vector. Under time reversal, it does flip sign. Consequently, an intrinsic EDM is parity odd and time-reversal odd, so detecting a nonzero intrinsic EDM would signal CP violation.

TABLE II. Duality between the magnetic Zeeman and electric Stark effects.

	Magnetic sector (Zeeman)	Electric sector (linear Stark)
External field	$\vec{B}$ (pseudovector)	$\vec{E}$ (polar vector)
Coupled operator	Total angular momentum $\hat{J}$	Pseudoangular momentum $\hat{J}_p$
Conserved generators	z component, $\hat{J}_z$	z component, Runge-Lenz component, $\hat{A}_{sc,z}$
Bohr model	$\mu_B = \frac{e\hbar}{2m_e}$ Bohr magneton	$d_B = \frac{2\mu_B}{e\alpha} = ea_0$ Bohr EDM
Good quantum number	m_J (magnetic)	k (Stark or parabolic)
Dipole moment	$\hat{\mu} = g_M \mu_B \hat{J}/\hbar$	$\hat{d} = g_E d_B \hat{J}_p/\hbar$
Energy level shift	$\Delta E_M = g_M m_J \mu_B B_z$	$\Delta E_S = g_E k d_B E_z$
Hamiltonian	$H_M = -\vec{\mu} \cdot \vec{B}$	$H_S = -\vec{d} \cdot \vec{E}$
Field coupling (total)	$H_M = -(\vec{\mu}_L + \vec{\mu}_S) \cdot \vec{B}$	$H_S = -(\vec{d}_{orb} + \vec{d}_e) \cdot \vec{E}$
Linear response	Zeeman splitting	Stark splitting
Residual symmetry	$SO(4) \rightarrow SO(2)$	$SO(4) \rightarrow SO(2) \times SO(2)$
Parity symmetry	Conserved (no mixing)	Broken, mixed opposite parity states
Time reversal sym	Broken	Conserved
Basis states	$ \uparrow\rangle, \downarrow\rangle$	$ +\rangle, -\rangle$ (even/odd parity)
Source	Circulating <i>charge current</i> density	Circulating <i>magnetic current</i> density
Intrinsic operator	$\hat{S} = \frac{\hbar}{2}\vec{\sigma}$ spin operator	$\hat{S} = \frac{\hbar}{2}\vec{\sigma}$ spin operator
Physical interpretation	Precession in $\vec{B}$; magnetic polarization	Polarization in $\vec{E}$; parity mixing
Geometric picture	Rotation in real space	Rotation in parity space

In contrast, there exist many *fermionic* quasiparticles whose two-component “spinor” resides in an *internal* space and couples *electrically*. Some canonical electric-sector realizations include graphene’s Dirac carriers, where the spinor acts on the A/B *sublattice* degree of freedom and couples to inversion-breaking scalar potentials introduced by substrates or gating [20,21]; the closely related *valley* pseudospin associated with the two inequivalent Brillouin-zone corners, which can be driven by electric fields through inversion asymmetry and Berry-curvature (valley Hall) effects [22]; and bilayer graphene’s *layer/orbital* pseudospin, controllable by a perpendicular displacement field (interlayer bias) that opens and tunes an energy gap [23]. Cold-atom Fermi gases provide another electric route: Two-level *hyperfine* manifolds act as a tunable pseudospin- $\frac{1}{2}$ coherently manipulated via optical (electric-dipole) Raman coupling and AC-Stark shifts [24].

In this sense, the Stark/EDM pseudoangular momentum $\hat{J}_p$ is of the same kind: an *orbital*, internal SU(2) (Runge-Lenz) that couples opposite-parity components within a fixed n manifold, while the underlying particles remain bona fide fermions.

VII. DISCUSSION AND IMPLICATIONS

The framework presented here provides a concrete route to visualizing electric dipole moments in close analogy to magnetic dipole moments (Table II). Introducing an effective magnetic current associated with permanent polarization yields a dual description of a permanent EDM and motivates an electric Bohr moment and electric g factor. This electromagnetic-duality viewpoint highlights how internal circulations, whether represented as electric or magnetic currents, produce observable moments coupled to external fields, and how these couplings encode (and, in symmetry-violating cases, break) fundamental symmetries. The quantized boundary-current picture also connects naturally to modern polarization theory in materials, bridging Berry-phase descriptions and classical electromagnetic analogies.

A key point is that the pseudoangular momentum used here is general at the level of an internal two- (or few-) level subspace: Whenever the dynamics is confined to an opposite-parity manifold coupled by an electric-dipole interaction, an SU(2) pseudospin (Pauli-operator) algebra acting in that parity subspace naturally organizes Stark mixing and the associated induced dipoles. What

is special to the hydrogenic Coulomb problem is the *explicit realization* of this SU(2) generator in terms of continuous-space symmetry operators: Within a fixed hydrogenic n manifold the pseudoangular momentum can be identified in closed form with a generator proportional to the scaled Runge-Lenz structure [via the hidden $\text{SO}(4) \simeq \text{SU}(2) \times \text{SU}(2)$ symmetry]. Thus, the SU(2) parity-subspace organization is the general layer, while the Runge-Lenz proportionality provides the canonical hydrogenic realization used here as an analytic benchmark.

This viewpoint mirrors standard classical impressed-source and field-equivalence formalisms [25,26]: The driving EMF that separates charge may be represented as an impressed electric field, equivalently as a polarization field with nonzero curl. Because the EMF is localized, the boundary carries (1) dipolar charge that terminates the polarization and (2) a fictitious magnetic surface current parallel to the polarization, reproducing external fields and terminal responses without modeling the internal workings of the source; this underpins our analyses of active classical dipoles using *magnetic* equivalence currents [12,13], and effective EMF generation in axion electrodynamics [14,27]. The Stark-sector theory developed here is the quantum dual of this equivalence-current picture: Starting from the wavefunction's microscopic polarization $\vec{P}(\vec{r})$, we define an *effective magnetic probability current* $\vec{J}_m = -\epsilon_0^{-1} \nabla \times \vec{P}$, providing a dual-current representation of the polarization texture. Importantly, the induced EDM arises physically from charge displacement via parity (Stark) mixing; $\vec{J}_m$ is not an additional causal mechanism, but a convenient field-equivalent description of the same internal structure. In this sense, the dual-Ohanian formulation makes the classical-quantum correspondence explicit: An impressed electric excitation is represented by a magnetic equivalence current in the classical dipole [12,13], while Stark mixing is represented by the magnetic probability-current *representation* of the quantum dipole.

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX A: RUNGE-LENZ OPERATOR

The unperturbed hydrogen atom is described by the Coulomb Hamiltonian

$$\hat{H}_0 = \frac{\hat{p}^2}{2\mu} - \frac{\kappa}{\hat{r}}, \quad \mu = \frac{m_e m_p}{m_e + m_p} \text{ [kg]}, \quad (\text{A1})$$

where μ is the reduced mass of the electron-proton system. The Coulomb coupling constant is given by

$$\kappa = \frac{e^2}{4\pi\epsilon_0} \text{ [J m]}, \quad e \text{ [C]}, \quad \epsilon_0 \text{ [F m}^{-1}\text{]}, \quad (\text{A2})$$

and the characteristic Bohr radius follows as

$$a_0 = \frac{4\pi\epsilon_0\hbar^2}{\mu e^2} \text{ [m]}, \quad \hbar \text{ [J s]}. \quad (\text{A3})$$

The canonical position and momentum operators are defined as

$$\hat{\vec{r}} = (\hat{x}, \hat{y}, \hat{z}), \quad \hat{r} = |\hat{\vec{r}}|, \quad \hat{\vec{r}} = \frac{\hat{\vec{r}}}{\hat{r}}, \quad (\text{A4})$$

$$\hat{\vec{p}} = -i\hbar \vec{\nabla} \text{ [kg m s}^{-1}\text{]}, \quad (\text{A5})$$

and the orbital angular momentum operator takes the standard form

$$\hat{\vec{L}} = \hat{\vec{r}} \times \hat{\vec{p}} \text{ [J s]}. \quad (\text{A6})$$

A less familiar but equally fundamental conserved quantity of the Coulomb problem is the Runge-Lenz vector [28–31], defined (in symmetrized operator form) as

$$\hat{\vec{A}} = \frac{1}{2\mu} (\hat{\vec{p}} \times \hat{\vec{L}} - \hat{\vec{L}} \times \hat{\vec{p}}) - \kappa \frac{\hat{\vec{r}}}{r} \text{ [J m]}. \quad (\text{A7})$$

This operator generates a hidden dynamical symmetry of the hydrogen atom, corresponding to the conserved direction of the elliptical orbit in the classical limit. Its magnitude and commutation relations with $\hat{\vec{L}}$ lead to the well-known $\text{SO}(4) \simeq \text{SU}(2) \times \text{SU}(2)$ symmetry of the bound-state spectrum [28].

To obtain an angular-momentum dimension, we define a scaled Runge-Lenz operator,

$$\hat{A}_{\text{sc}} \equiv \frac{\sqrt{\mu\hat{A}}}{\sqrt{-2\hat{H}_0}} \text{ [J s]}, \quad \hat{J}_p \equiv \hat{A}_{\text{sc}}, \quad (\text{A8})$$

which introduces a pseudoangular momentum $\hat{J}_p$ with the same algebraic structure as $\hat{\vec{L}}$.

On a hydrogenic energy eigenspace $\hat{H}_0|_{E_n} = E_n|_{E_n}$, with

$$E_n = -\frac{\mu\kappa^2}{2\hbar^2 n^2}, \quad (\text{A9})$$

the scaling factor becomes a number:

$$\sqrt{-2E_n/\mu} = \frac{\kappa}{\hbar n} \Rightarrow \hat{A}_{\text{sc}} = \frac{\hbar n}{\kappa} \hat{A} = \frac{n}{c\alpha} \hat{A} \quad (\text{A10})$$

on the n manifold. Thus, within each principal quantum manifold n , the scaled Runge-Lenz vector plays the role of a conserved pseudoangular momentum, forming with $\hat{\vec{L}}$ the generators of the $\text{SU}(2) \times \text{SU}(2)$ symmetry that underlies the degeneracy of the hydrogen spectrum.

APPENDIX B: HYDROGEN QUANTUM NUMBERS (SPHERICAL VERSUS PARABOLIC)

In hydrogen, the electron's orbital quantum numbers are the principal quantum number $n = 1, 2, 3, \dots$, the orbital angular momentum $\ell = 0, 1, \dots, n-1$ (with spectroscopic labels $s, p, d, f, \dots$ for $\ell = 0, 1, 2, 3, \dots$), and the magnetic quantum number $m = -\ell, -\ell+1, \dots, \ell$, which set $L^2 = \ell(\ell+1)\hbar^2$ and $L_z = m\hbar$. In an external electric field, it is convenient to use the *parabolic* set (n_1, n_2, m) , obtained by separating the Schrödinger equation in parabolic coordinates $(\xi = r+z, \eta = r-z, \phi)$. Here, $n_1, n_2 = 0, 1, 2, \dots$ are *not* principal quantum numbers: They count radial-like nodes along the two parabolic directions and, together with m , obey the constraint $n_1 + n_2 + |m| + 1 = n$ (where n remains the single principal quantum number). The combination $k \equiv n_1 - n_2 \in \{-(n-1), \dots, n-1\}$ labels the linear Stark splitting and controls the expectation values $\langle z \rangle = \frac{3}{2} n a_0 k$ and $\langle A_{\text{sc},z} \rangle = \hbar k$. In zero field, the parabolic basis is just another unitary choice within the n manifold, equivalent to the spherical $|n\ell m\rangle$ basis.

APPENDIX C: HYDROGENIC $\text{SO}(4)$ STRUCTURE AND THE PSEUDOANGULAR MOMENTUM

For a hydrogenic atom/ion, the following familiar Coulomb-problem commutation relations encode the hidden $\text{SO}(4)$ symmetry of the bound-state ($H < 0$) spectrum:

$$[L_i, L_j] = i\hbar \epsilon_{ijk} L_k, \quad (\text{C1})$$

$$[L_i, A_j] = i\hbar \epsilon_{ijk} A_k, \quad (\text{C2})$$

$$[A_i, A_j] = -2\mu H i\hbar \epsilon_{ijk} L_k. \quad (\text{C3})$$

For bound states, it is convenient to use the scaled Runge-Lenz operator (A8), so Eq. (C3) implies that

$$[A_{\text{sc},i}, A_{\text{sc},j}] = i\hbar \epsilon_{ijk} L_k, \quad (\text{C4})$$

and $\vec{A}_{\text{sc}}$ has the dimensions of angular momentum (like $\vec{L}$). One may then form the standard $\text{SO}(4) \simeq \text{SU}(2) \times \text{SU}(2)$ generators

$$\vec{J}_{\pm} \equiv \frac{1}{2}(\vec{L} \pm \vec{A}_{\text{sc}}), \quad (\text{C5})$$

which satisfy the $\text{SU}(2)$ algebras

$$[J_{\pm,i}, J_{\pm,j}] = i\hbar \epsilon_{ijk} J_{\pm,k}, \quad (\text{C6})$$

$$[J_{+,i}, J_{-,j}] = 0. \quad (\text{C7})$$

Within a fixed n manifold (degenerate subspace), these relations provide the rigorous $\text{SU}(2)$ structure underpinning the Landé-factor analogy used below.

In the Stark/parity (two- or few-level) subspace relevant for the orbital EDM, we identify the pseudoangular momentum $\vec{J}_p$ with the appropriate projected $\text{SU}(2)$ generator (equivalently, the corresponding component of $\vec{J}_{\pm}$ along the field axis) that labels the parabolic/Stark structure. Consequently, $\vec{J}_p$ obeys the standard $\text{SU}(2)$ commutation relations,

$$[J_{p,i}, J_{p,j}] = i\hbar \epsilon_{ijk} J_{p,k}, \quad (\text{C8})$$

justifying the Landé-factor form $\hat{d}_{\text{orb}} = g_E d_B \hat{J}_p / \hbar$ used in this work.

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