

Quantum thermalization mechanism and the emergence of symmetry-breaking phases

Sebastián Gómez ^{1,2,*}, Ángel L. Corps ^{3,4,†} and Armando Relaño ^{1,4,‡}

¹*Departamento de Estructura de la Materia, Física Térmica y Electrónica, Universidad Complutense de Madrid, Avenida Complutense s/n, E-28040 Madrid, Spain*

²*Instituto de Estructura de la Materia, IEM-CSIC, Serrano 123, E-28006 Madrid, Spain*

³*Institute of Particle and Nuclear Physics, Faculty of Mathematics and Physics, Charles University, V Holešovičkách 2, 180 00 Prague, Czech Republic*

⁴*Grupo Interdisciplinar de Sistemas Complejos (GISC), Universidad Complutense de Madrid, Avenida Complutense s/n, E-28040 Madrid, Spain*



(Received 29 August 2025; revised 1 December 2025; accepted 1 May 2026; published 8 May 2026)

We propose a generalization of the eigenstate thermalization hypothesis accounting for the emergence of symmetry-breaking phases. It consists of two conditions that any system with a degenerate spectrum must fulfill in order to thermalize. The failure of each condition generates a different nonthermalizing scenario. One is due to the absence of chaos and may indicate that extra constants of motion are required to describe equilibrium states. The other one implies the existence of initial conditions evolving toward symmetry-breaking equilibrium states. If it spreads across an entire spectral region, then this region gives rise to a symmetry-breaking phase. We explore the applicability of this formalism by means of numerical experiments on a three-site Bose-Hubbard model with two noncommuting discrete symmetries.

DOI: [10.1103/w77n-lyjd](https://doi.org/10.1103/w77n-lyjd)

I. INTRODUCTION

Understanding how an isolated quantum system thermalizes has been a fundamental goal since von Neumann's times [1]. Today, we have a solid theory for its underlying mechanism. The eigenstate thermalization hypothesis (ETH) states that every Hamiltonian eigenstate is thermal in chaotic quantum systems [2–7]. In its strongest version, the eigenstate expectation value of any few-body physical observable, $O_{nn} = \langle E_n | \hat{O} | E_n \rangle$, agrees with the microcanonical ensemble in the thermodynamic limit (TL) [8–11],

$$\max_n \Delta_n(E) \equiv \max_n \frac{|O_{nn} - O_{\text{ME}}(E)|}{\eta_O} \xrightarrow{N \rightarrow \infty} 0. \quad (1)$$

Here, $O_{\text{ME}}(E)$ is the microcanonical average and η_O is the spectral width of the observable $\hat{O}$, $\eta_O \equiv O_{\text{max}} - O_{\text{min}}$, with O_{max} and O_{min} being the largest and smallest eigenvalues of $\hat{O}$, respectively.

* Contact author: sebastian.gomez@csic.es

† Contact author: corps.angel.l@gmail.com

‡ Contact author: armando.relano@fis.ucm.es

Published by the American Physical Society under the terms of the [Creative Commons Attribution 4.0 International](https://creativecommons.org/licenses/by/4.0/) license. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI.

The reason why this guarantees thermalization lies in the time-dependent expectation values for physical observables, $\langle \hat{O}(t) \rangle = \sum_{n,m} c_m^* c_n e^{-i(E_n - E_m)t/\hbar} O_{nm}$, from a pure initial condition, $|\Psi(0)\rangle = \sum_n c_n |E_n\rangle$. Assuming no degeneracies, off-diagonal terms in this expression dephase in the long-time limit, and hence the system approaches an effective equilibrium state, $\hat{\rho}_{\text{eq}} = \sum_n |c_n|^2 |E_n\rangle \langle E_n|$, after a transient regime. If Eq. (1) holds and the initial condition is sufficiently narrow in energy, expectation values of physical observables in this effective equilibrium state coincide with microcanonical averages. There are strong arguments supporting that deviations from this equilibrium state are rare and short-lived [12–18].

Despite the universality of this mechanism [19–32], whose exceptions are mainly due to integrability [33–38], many-body localization [39–44], the presence of scarred states [45–49], and certain other effects [50,51], the scenario behind the nondegeneration hypothesis is highly involved. The level spacing in realistic quantum systems decreases exponentially with the number of particles, and thus dephasing times may exceed the age of the Universe even for modest system sizes. In many quantum systems, this conundrum is ruled out because the matrices of physical observables in the Hamiltonian eigenbasis are banded [52], with off-diagonal elements decaying fast with the number of particles, as in Srednicki’s ansatz for the ETH [3,53], based on the random matrix theory (RMT) [54], where $O_{nm} \propto e^{-S(E)/2}$, for $m \neq n$, and $S(E)$ is the thermodynamic entropy.

Although there is an abundance of numerical evidence supporting the above thermalization mechanism [29,31,32,55–58], it cannot be satisfied in some important physical situations, like symmetry-breaking phases. These are characterized by nonzero equilibrium values for order parameters, whose diagonal expectation values are necessarily zero [59]. This implies that some off-diagonal elements must contribute to the final equilibrium state, like the *tower of states* becoming degenerate in the TL when a continuous symmetry is broken [60–65]. To complicate things further, the number of these states is typically very small [66], and their role becomes relevant only under very specific circumstances—for the right values of the system parameters and below the energy corresponding to the critical temperature of the transition. Thus, a number of fundamental questions emerge: What are the physical properties of the relevant degeneracies? Is there a simple way to predict in which spectral regions symmetry-breaking equilibrium states may appear? And considering that the distinction of diagonal and off-diagonal matrix elements is ill-defined in the presence of degeneracies, what can be said about the thermalization mechanism in this case?

In this article, we provide answers to these questions. Our main result is a generalization of the ETH, which accounts for the role of degeneracies and has the following properties: (1) It is basis independent; (2) it reduces to the standard ETH in the absence of degeneracies; and (3) it distinguishes between two qualitatively different situations in which thermalization fails: the absence of chaos and symmetry breaking.

II. GENERALIZED ETH

We focus on isolated quantum systems with one or more symmetries. We label their eigenstates by two quantum numbers, $|E_{n,\alpha}\rangle$, where α refers to the eigenvalues of a (sub)set of commuting symmetries. We assume that our system is large enough so that $E_{n,\alpha} = E_{n,\beta} \equiv E_n$, $\forall \alpha, \beta$, for all practical purposes. We rule out remaining degeneracies by assuming that off-diagonal matrix elements between them fulfill Srednicki’s ansatz.

Within this framework, let us consider (1) the subspaces $\mathcal{H}_n$ spanned by Hamiltonian eigenstates with the same energy E_n , $\mathcal{H}_n \equiv \text{span}\{|E_{n,\alpha}\rangle : \hat{H}|E_{n,\alpha}\rangle = E_n|E_{n,\alpha}\rangle\}$, with $\alpha = 1, \dots, d_n$, where $d_n = \dim \mathcal{H}_n$; and (2) the projection of a physical observable, $\hat{O}$, onto them, $O_{\alpha\beta}^{(n)} = \langle E_{n,\alpha} | \hat{O} | E_{n,\beta} \rangle$. Our generalized ETH consists of the following two conditions:

$$\max_n \Delta_n^T(E) \equiv \max_n \frac{|\bar{\lambda}_n^{(O)} - O_{\text{ME}}(E)|}{\eta_O} \xrightarrow{N \rightarrow \infty} 0, \quad (2)$$

$$\max_{n,\alpha} \Delta_{n,\alpha}^\lambda \equiv \max_{n,\alpha} \frac{|\lambda_{n,\alpha}^{(O)} - \bar{\lambda}_n^{(O)}|}{\eta_O} \xrightarrow{N \rightarrow \infty} 0. \quad (3)$$

Here, $\lambda_{n,\alpha}^{(O)}$ are the eigenvalues of the matrix $O_{\alpha,\beta}^{(n)}$ and $\bar{\lambda}_n^{(O)}$ is the average of these eigenvalues. These equations reduce to Eq. (1) in the absence of degeneracies because $d_n = 1$, $\bar{\lambda}_n^{(O)} = O_{nn}$, and Eq. (3) is identically zero.

Our formulation states that, within a given energy region, a system thermalizes if and only if every few-body physical observable fulfills both Eqs. (2) and (3). The proof is given in Appendix A, and some practical examples are given in the Supplemental Material [67]. Deeper physical insight is provided by the following two scenarios.

S1: Failure of Eq. (2), which is due to nonchaotic behavior. In fully chaotic systems, Eq. (1) is expected to hold independently in each symmetry sector [7,68–71], and thus, Eq. (2) is immediately fulfilled. Therefore, the failure of this equation implies that RMT does not work, so the system is not fully chaotic [72]. This may be linked to the presence of additional constants of motion, implying that a generalized Gibbs ensemble (GGE) [73–75] is required to describe equilibrium states.

S2: Failure of Eq. (3), which can occur under two different conditions:

(1) The diagonal elements differ across symmetry sectors. An example is provided by the elements of the symmetry group. If these are local operators, they must be included in a GGE or in a non-Abelian thermal state [77–84] to correctly describe symmetric equilibrium states. This scenario follows from the standard ETH.

(2) The diagonal elements are identical across symmetry sectors, as expected for discrete symmetries [51]. Then, failure of Eq. (3) is due to coherence between different sectors, so that applying the standard ETH independently to each one cannot diagnose this situation. A paradigmatic example is an order parameter $\hat{O} = \hat{M}$, for which $M_{ME} = 0$ and $\langle E_{n,\alpha} | \hat{M} | E_{n,\alpha} \rangle = 0$ for all n, α , thus trivially satisfying Eq. (1) in each symmetry sector. Since the diagonal elements vanish, $\bar{\lambda}_n^{(M)} = 0$ for all n [59]. However, some values of $\lambda_{n,\alpha}^{(M)}$ may remain finite in the thermodynamic limit, allowing equilibrium states with $\langle \hat{M} \rangle \neq 0$, signaling symmetry breaking.

We thus arrive at the following condition for symmetry-breaking phases:

A spectral region is a symmetry-breaking phase if and only if it has no subspaces fulfilling $\Delta_{n,\alpha}^\lambda \xrightarrow{N \rightarrow \infty} 0$, for the corresponding order parameter $\hat{O} = \hat{M}$.

Under such circumstances, and if $\hat{M}$ is a few-body local operator, each of the possible values of $\lambda_{n,\alpha}^{(M)}$ represents a different branch of the order parameter. That is, these values indicate the possible symmetry-breaking equilibrium states that may be observed in the corresponding energy regions; other possible ways of symmetry breaking are not allowed [85]. On the contrary, if some subspaces fulfill $\Delta_{n,\alpha}^\lambda \xrightarrow{N \rightarrow \infty} 0$, there exist initial conditions relaxing to symmetric equilibrium states, and therefore, the spectral region does not give rise to a symmetry-breaking phase.

III. PHYSICAL INSIGHT

To illustrate our theory, we consider the three-site Bose-Hubbard model with periodic boundary conditions,

$$\hat{H} = \sum_{i=1}^3 \left[-J(\hat{a}_{i+1}^\dagger \hat{a}_i + \hat{a}_{i+1} \hat{a}_i^\dagger) + \frac{U}{N} \hat{a}_i^\dagger \hat{a}_i^\dagger \hat{a}_i \hat{a}_i \right], \quad (4)$$

where N is the number of particles, and $\hat{a}_i^\dagger$ and $\hat{a}_i$ are the creation and annihilation bosonic operators at site i . Equation (4) commutes with the total number of particles, $\hat{N} = \sum_i \hat{a}_i^\dagger \hat{a}_i$.

The symmetries of this model are described by the dihedral group D_3 , whose elements can be obtained from two noncommuting operators: a $2\pi/3$ rotation, $\hat{R}|n_1, n_2, n_3\rangle = |n_3, n_1, n_2\rangle$, where n_i indicates the number of particles in the site i ; and a reflection, $\hat{S}|n_1, n_2, n_3\rangle = |n_3, n_2, n_1\rangle$. $\hat{R}$ has three eigenvalues, $r \in \{1, e^{2\pi i/3}, e^{4\pi i/3}\}$, and $\hat{S}$ has just two, $s \in \{1, -1\}$. This model also has a classical limit that can be easily obtained by taking $\hat{a}_j = \sqrt{N/2}(\hat{q}_j + i\hat{p}_j)$, and $N \rightarrow \infty$. See Appendix B for details. The result is a classical Hamiltonian with three coordinates, q_i , and momenta, p_i , but just

two degrees of freedom due to the conservation of particle number implying $\sum_i (q_i^2 + p_i^2) = 2$. In this way, $1/N$ plays the role of an effective Planck constant, $\hbar_{\text{eff}} \propto 1/N$, and therefore, the TL can be studied by means of the classical equations. The physics of this model is quite complex, giving rise to persistent currents and vortex formation [86,87], fragmented condensation [88], and a transition from regularity to chaos [89–94]; it is also a minimal model for a superfluid circuit [95].

To explore the physical insight of our generalized ETH, we focus on two order parameters. First, the persistent current $\hat{C} = i \sum_j (\hat{a}_{j+1}^\dagger \hat{a}_j - \hat{a}_j^\dagger \hat{a}_{j+1})$. As $\langle \hat{C} \rangle = 0$ in eigenstates of $\hat{S}$, it is a good order parameter for an ordered phase in which the reflection symmetry is broken.

And second, the generalized imbalance $\hat{I} = \hat{n}_1 + e^{2\pi i/3} \hat{n}_2 + e^{4\pi i/3} \hat{n}_3$, which is such that $\langle \hat{I} \rangle/N = 1, e^{2\pi i/3}$, or $e^{4\pi i/3}$ if all the particles are located at the first, second, or third site. As $\langle \hat{I} \rangle = 0$ in the eigenstates of $\hat{R}$, it is a good order parameter for an ordered phase in which the $2\pi/3$ rotation symmetry is broken. Both these order parameters will be used to study just Eq. (3), since order parameters trivially satisfy Eq. (2).

To study Eq. (2), we rely on the hopping between the first two sites, $\hat{O} = \hat{h}_{12} = \hat{a}_1^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_1$. Its microcanonical value is obtained from the classical version of Eq. (4) (see Appendix C for details).

Throughout the rest of this article, we always refer to the intensive versions of these three operators, $\hat{C}/N$, $\hat{I}/N$, and $\hat{h}_{12}/N$, whose spectral widths do not depend on system size. Their classical versions, representing the TL of the quantum model, are denoted by $C(\mathbf{q}, \mathbf{p})$, $I(\mathbf{q}, \mathbf{p})$, and $h_{12}(\mathbf{q}, \mathbf{p})$.

The first step to explore the consequences of Eqs. (2) and (3) is to build the subspaces $\mathcal{H}_n$. We use the eigenstates of the operator $\hat{R}$ for this purpose, and we consider that $\alpha = 1$ labels the eigenvalue $r = 1$, $\alpha = 2$ the eigenvalue $r = e^{2\pi i/3}$, and $\alpha = 3$ the eigenvalue $r = e^{4\pi i/3}$; therefore, $d_n = 3$, $\forall n$. In the subspaces with $r = e^{2\pi i/3}$ and $r = e^{4\pi i/3}$, $\hat{S}$ and $\hat{R}$ do not commute. As a consequence, $E_{n,2} = E_{n,3}$, $\forall n$, so we can identify the breaking of both rotation and reflection with this choice. In the subspace with $r = 1$, $\hat{R}$ and $\hat{S}$ commute.

For our numerical experiments, we set $U = -5$ and $J = 1$. We focus on $\lambda_n^{(I)}$, $\lambda_n^{(C)}$, and $\bar{\lambda}_n^{(h_{12})}$. Similar results are obtained in other cases. For example, $\lambda_n^{(h_{12})}$ behaves in the same way as $\lambda_n^{(I)}$.

We start analyzing the region with $E/N \lesssim -3.9$. In Fig. 1(a), the mean eigenvalue of $\hat{h}_{12}$ satisfies Eq. (2) in some subspaces while not in others, giving a mixed behavior. In contrast, Fig. 1(b) shows that none of the subspaces satisfy Eq. (3) for $\hat{I}$. We refer to this region as the *rotation-breaking region*, anticipating the analysis below. Finally, Fig. 1(c) shows the opposite behavior, where all subspaces satisfy Eq. (3).

A similar phenomenon occurs in the $E/N \gtrsim -0.9$ region. Figure 1(c) shows that none of the subspaces satisfy Eq. (3) for $\hat{C}$. We refer to this region as the *reflection-breaking region*. Figures 1(a) and 1(b) are less conclusive in this regime.

Next, we focus on the region $-3.8 \lesssim E/N \lesssim -3.1$, where Figs. 1(a) and 1(b) clearly indicate again the mixed behavior similar to the one observed previously: Our observables satisfy Eqs. (2) and (3) in some subspaces while violating them in others. A similar behavior is found in the region $-2.3 \lesssim E/N \lesssim -0.9$; in this case, Eq. (3) is also violated by $\hat{C}$ in some subspaces. We therefore group these two regions and refer to them as the *mixed regions*.

According to our generalized ETH, none of these four energy regions is thermal, since there exist physical observables that do not meet Eq. (2), Eq. (3), or both. This means that microcanonical and canonical ensembles cannot provide a good description of the equilibrium states within them.

We turn to the interval $-3.1 \lesssim E/N \lesssim -2.3$, where Fig. 1(a) shows that $\bar{\lambda}_n^{(h_{12})}$ fluctuates around the microcanonical average. A stronger test is provided in Fig. 2. In panel (a), the width of the distribution of $\bar{\lambda}_n^{(h_{12})} - h_{12}(E)$ decreases as a power law, $\sigma \propto N^{-0.419}$. This decreasing behavior is close to Srednicki's ansatz, in which diagonal expectation values fluctuate around the microcanonical with a width proportional to $e^{-S(E)/2} = [\rho(E)]^{-1/2} \propto N^{-1/2}$. In Fig. 2(b), we study the total number of subspaces in which $|\bar{\lambda}_n^{(h_{12})} - h_{12}(E)|$ is larger than several bounds. Equation (2) requires that this number goes to zero in the TL. Results in Fig. 2(b) suggest the existence of a general pattern: First, this number increases with N , then it reaches a maximum, and finally it decreases for larger system

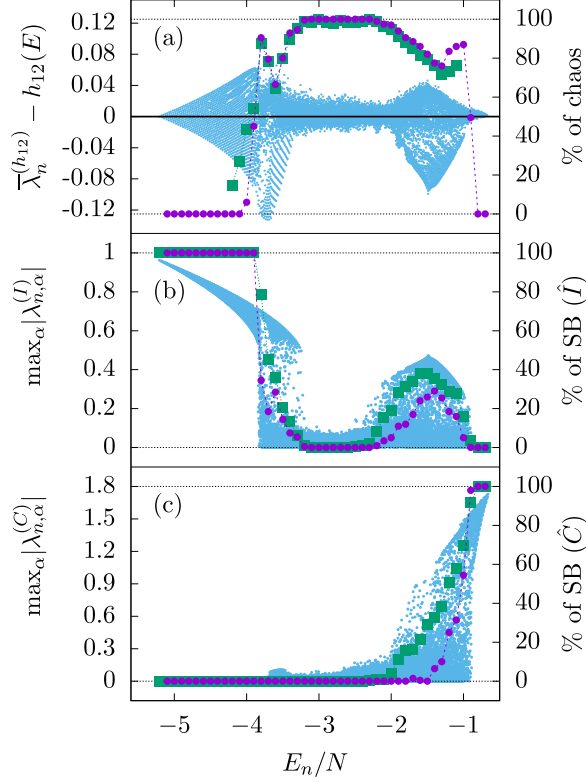


FIG. 1. All the results are obtained with $N = 320$, except where indicated. (a) Difference between the mean eigenvalue of $\hat{h}_{12}$ and its classical microcanonical value (blue points, left axis), and percentage of chaotic quantum and classical trajectories (purple circles and green squares, right axis). (b) Maximum value of $|\lambda_{n,\alpha}^{(I)}|$ (blue points, left axis), percentage of quantum and classical trajectories in which $\hat{I}$ is broken (purple points, right axis), and percentage of symmetry-breaking eigenvalues (green squares, right axis). (c) Same as panel (b), for the current $\hat{C}$. The spectral statistics is performed with a system size of $N = 800$.

sizes; the smaller the bound, the larger the number of particles required to reach the maximum. Although larger system sizes would be required to extend this result to lower bounds, the overall picture suggests that Eq. (2) is satisfied. In addition, $\lambda_{n,\alpha}^{(h_{12})}$ is consistent with Eq. (3) (not shown).

Similarly, Figs. 1(b) and 1(c) are consistent with $\lambda_{n,\alpha}$ fluctuating around zero for both observables. Figure 2(a) shows that the width of the corresponding distributions decreases as a power law in both cases. Figures 2(c) and 2(d) suggest a similar pattern for $\lambda_{n,\alpha}^{(I)}$ and $\lambda_{n,\alpha}^{(C)}$ as for the mean eigenvalue of $\hat{h}_{12}$, although the results for $\hat{C}$ are less conclusive due to finite-size effects (diagonalizing larger systems would be necessary to get a clearer image). All these results suggest that both Eqs. (2) and (3) are fulfilled within this region. For this reason, we refer to it as *thermalizing region*. According to our generalized ETH, we expect any initial condition to relax to a microcanonical symmetric equilibrium state.

We now turn to the physical interpretation of these results. In particular, we formulate the following question: If we prepare an initial condition with different number of particles in each of the three wells and a given energy, what is the nature of the ensuing equilibrium state?

A. Rotation-breaking region

We have already seen that $\hat{I}$ does not meet Eq. (3). Figure 3(a) provides a deeper physical insight. The values of $\lambda_{n,\alpha}^I$ at $E/N \sim -4.2$ (the same results are obtained at other energies within this region)

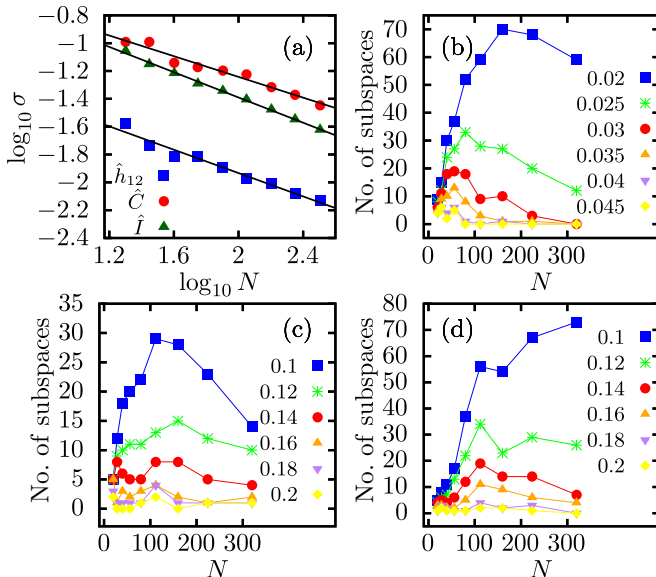


FIG. 2. (a) Variance of the difference between the numerical (quantum) and microcanonical (classical) values for subspaces in the energy range $[-3.1, -2.3]$ for different observables. We find the following power-law behaviors: $\sigma \propto N^{-0.419}$ for $\hat{\lambda}_n^{(h_{12})}$ (squares), $\sigma \propto N^{-0.372}$ for the maximum eigenvalue of $\hat{C}$ (circles), and $\sigma \propto N^{-0.454}$ for the maximum eigenvalue of $\hat{I}$ (triangles). Number of subspaces such that the difference between exact and microcanonical averages is below a given arbitrary bound (see legend), for $\hat{\lambda}_n^{(h_{12})}$ [panel (b)], $\hat{I}$ [panel (c)], and $\hat{C}$ [panel (d)].

are distributed around 0 , $2\pi/3$, and $4\pi/3$ angles. This means that the only possible equilibrium states within this region have two wells with equal and smaller population than the third one. Long-time averages of $I(\mathbf{q}, \mathbf{p})$ for 1000 classical trajectories perfectly match this prediction. Furthermore, Fig. 3(c) shows that the distributions for the classical and the quantum results also agree very well.

Results in Figs. 1(a) and 1(b) show that there is no clear link between the failure of Eq. (3) and the absence of chaos. The degree of both classical and quantum chaos changes within this region, from apparent integrability in the lowest-energy region to around 50% of chaos at $E/N \sim -3.9$. We do not show the result for spectral statistics below $E/N \sim -4.2$ because the level density is too low to obtain significant values with steps of $\Delta E/N = 0.1$; a global fit to $E/N < -4.2$ results in 0% of quantum chaos. Finally, results in Fig. 1(c) show that the long-time average of $C(\mathbf{q}, \mathbf{p})$ is always zero, as predicted by our generalized ETH.

B. Reflection-breaking region

Similarly to the previous case, Fig. 1(c) together with condition S2(2) indicates that this is a small reflection-breaking region, characterized by the presence of persistent currents. We show in the same panel of Fig. 1 that the long-time average of $C(\mathbf{q}, \mathbf{p})$ is always different from zero, as our generalized ETH predicts. On the contrary, as Eq. (3) is fulfilled by $\hat{I}$, long-time averages of $I(\mathbf{q}, \mathbf{p})$ must always be equal to zero, as observed in Fig. 1(b).

C. Thermalizing region

Figure 1(a) suggests that Eq. (2) is satisfied. This is checked as the spectral statistics and classical trajectories in Fig. 1(a) both agree that there is full chaos. Since RMT does not constrain correlations between different symmetry sectors, full chaos does not imply Eq. (3). Nevertheless, Figs. 1(b) and 1(c) also indicate that Eq. (3) is satisfied in this region. Thus, our generalized ETH predicts that

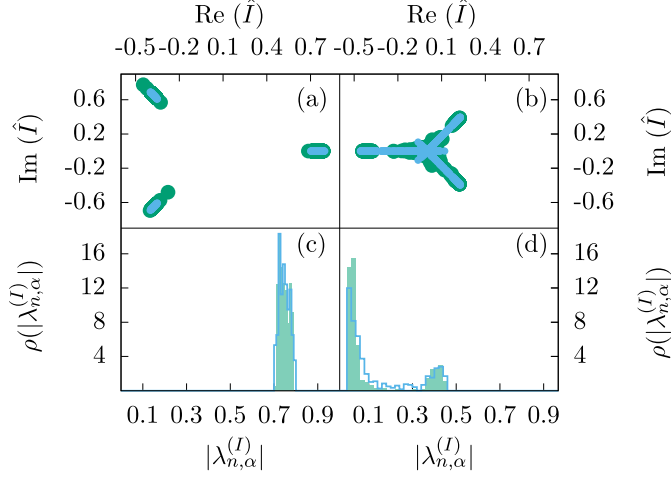


FIG. 3. Complex eigenvalues of $\hat{I}$ (blue) and long-time average of different classical trajectories (green) for $E/N = -4.2$ [panel (a)] and $E/N = -1.4$ [panel (b)]. Classical results correspond to 1000 trajectories in an interval of width 0.1 around E/N . (c), (d) Distribution of absolute value of the eigenvalues of $\hat{I}$ (blue) and classical average (green). Quantum results correspond to $N = 320$.

microcanonical ensemble (with no symmetry breaking) works within this region. This is confirmed by the classical results shown in Fig. 1. We find no classical trajectories with long-time averages of $I(\mathbf{q}, \mathbf{p})$ or $C(\mathbf{q}, \mathbf{p})$ different from zero. And, although not shown here for the sake of clarity, long-time averages of $h_{12}(\mathbf{q}, \mathbf{p})$ also concentrate around the microcanonical prediction.

D. Mixed regions

We have already seen that this is a mixed region, in which both symmetric and symmetry-breaking equilibrium states may be observed. In Figs. 3(b) and 3(d), which show results for $E/N \sim -1.4$, we focus on $\hat{I}$. The values of $\lambda_{n,\alpha}^{(I)}$ show two different patterns: They either distribute around $\pi/3, \pi$, or $5\pi/3$ angles, with $|\lambda_{n,\alpha}^{(I)}| \sim 0.4$, or they have a very small value for $|\lambda_{n,\alpha}^{(I)}|$. Thus, according to our generalized ETH, we may observe either symmetry-breaking equilibrium states with one well less populated than the other, equally populated two or a symmetric equilibrium state. Long-time averages of $I(\mathbf{q}, \mathbf{p})$ perfectly match this prediction; even the probability distributions of $|I(\mathbf{q}, \mathbf{p})|$ and $|\lambda_{n,\alpha}^{(I)}|$ perfectly agree.

The behavior of $\bar{\lambda}^{(h_{12})}$, shown in Fig. 1(a), is a bit more complicated. Some subspaces show an ordered pattern, typical of integrable systems, whereas others show erratic and narrow fluctuations, typical of chaos [96]. This is compatible with the degree of chaos found in both the classical and quantum versions of the system. However, the microcanonical average does not coincide with the erratic part of $\bar{\lambda}^{(h_{12})}$. This is probably because the microcanonical average is obtained by integrating over all the available phase space, including both the chaotic and the nonchaotic regions.

Finally, Fig. 1(c) shows that there are also classical trajectories such that the long-time average of $C(\mathbf{q}, \mathbf{p})$ is different from zero, as predicted by our generalized ETH.

IV. DISCUSSION

The main contribution of this article is an extension of the ETH accounting for the emergence of symmetry-breaking phases. As an example of its applications, we have performed numerical experiments on a quantum system with a clear classical analog. As a result, we have shown that such equilibrium states may appear in two different situations. First, they can completely characterize a

spectral region, giving rise to a symmetry-breaking phase. Second, they may be surrounded by symmetric equilibrium states, thus playing a role similar to many-body scars within a chaotic sea. We have also shown that our theory successfully predicts the precise way in which a $\mathbb{Z}_3$ symmetry is broken in such equilibrium states. This cannot be captured by the standard ETH when applied independently within each symmetry sector as coherence between sectors plays a key role.

This generalization of the ETH explains why and when a symmetric system becomes unstable upon the introduction of a small symmetry-breaking term [97]: In such circumstances, Eq. (3) must fail. It can also be used to predict the existence of particular initial conditions evolving toward stable symmetry-breaking states in the absence of an ordered phase [98]. And it may be extended to the study of topological order. For example, in the ground-state subspace of the paradigmatic model considered in Ref. [99], Eq. (3) must be fulfilled by all local observables, like the magnetization and the staggered magnetization along every direction, but it must fail for observables linked to the spin-1/2 degree of freedom localized at the edges of the chain.

ACKNOWLEDGMENTS

A.L.C. acknowledges fruitful discussions with P. Cejnar, P. Stránský, and J. Novotný. A.L.C. and A.R. acknowledge financial support by the Spanish Grant No. PID2022-136285NB-C31 funded by Ministerio de Ciencia e Innovación/Agencia Estatal de Investigación MCIN/AEI/10.13039/501100011033 and FEDER “A Way of Making Europe.” A.L.C. also acknowledges financial support from the Czech Science Foundation under Project No. 25-16056S as well as the JUNIOR UK Fund of the Faculty of Mathematics and Physics, Charles University.

DATA AVAILABILITY

The data that support the findings of this article are openly available [100].

APPENDIX A: PROOF OF THE MAIN RESULT

Here, we prove that Eqs. (2) and (3) hold if and only if all the initial conditions narrow enough in energy thermalize.

1. Forward implication

Let us consider that the system starts in an initial condition, $|\Psi(0)\rangle = \sum_{n,\alpha} c_{n,\alpha} |E_{n,\alpha}\rangle$, where $c_{n,\alpha} \neq 0$ only in a subextensive interval $[E - \delta E, E + \delta E]$, and let us calculate the resulting long-time average of a physical observable $\hat{O}$. To do so, we consider the matrix for $\hat{O}$ in a subspace of energy E_m , $\hat{O}_{\alpha\beta}^{(m)} = \langle E_{m,\alpha} | \hat{O} | E_{m,\beta} \rangle$, with $\alpha, \beta \in \{1, \dots, d_m\}$. Then, we diagonalize it as $\hat{O}^{(m)} = \hat{P}^{(m)} \hat{D}^{(m)} (\hat{P}^{(m)})^{-1}$, where $\hat{P}$ is a block-diagonal matrix that acts as the identity in every subspace with energy different from E_m . The result of repeating this procedure on every subspace is a new eigenbasis for the Hamiltonian, $\{|\tilde{E}_{m,\alpha}\rangle\}$, in which $\hat{O}_{\alpha,\beta}^{(m)}$ is diagonal in every subspace. Then, writing the initial condition in this eigenbasis, $|\Psi(0)\rangle = \sum_{n,\alpha} \tilde{c}_{n,\alpha} |\tilde{E}_{n,\alpha}\rangle$, we get

$$\langle \Psi(t) | \hat{O} | \Psi(t) \rangle = \sum_{n,m} \sum_{\alpha,\beta} \tilde{c}_{m,\beta}^* \tilde{c}_{n,\alpha} e^{-i(E_n - E_m)t/\hbar} \langle \tilde{E}_{m,\beta} | \hat{O} | \tilde{E}_{n,\alpha} \rangle. \quad (\text{A1})$$

Thus, if we consider that $\langle \tilde{E}_{m,\beta} | \hat{O} | \tilde{E}_{n,\alpha} \rangle$ decays exponentially with the system size for $n \neq m$, we can dismiss these off-diagonal terms in the calculation of the long-time average. Therefore,

$$\begin{aligned} \bar{\hat{O}} &= \sum_n \sum_{\alpha,\beta} \tilde{c}_{n,\beta}^* \tilde{c}_{n,\alpha} \langle \tilde{E}_{n,\beta} | \hat{O} | \tilde{E}_{n,\alpha} \rangle \\ &= \sum_{n,\alpha} |\tilde{c}_{n,\alpha}|^2 \lambda_{n,\alpha}, \end{aligned} \quad (\text{A2})$$

where $\lambda_{m,\alpha}$, $\alpha \in \{1, \dots, d_m\}$ are the eigenvalues of $\hat{O}_{\alpha,\beta}^{(m)}$.

Finally, if both Eqs. (2) and (3) are satisfied, then

$$\max_{n,\alpha} \frac{|\lambda_{n,\alpha}^{(O)} - O(E)|}{\eta_O} \rightarrow 0, \quad N \rightarrow \infty. \quad (\text{A3})$$

And therefore, the long-time average given in Eq. (A2) coincides with the microcanonical average.

2. Backward implication

This is equivalent to showing that if (1) $\max_n |\bar{\lambda}_n^{(O)} - O_{\text{ME}}(E)| \not\rightarrow 0$ or (2) $\max_{n,\alpha} |\lambda_{n,\alpha}^{(O)} - \bar{\lambda}_n^{(O)}| \not\rightarrow 0$ when $N \rightarrow \infty$, then there exists at least one nonthermalizing initial condition.

Let us begin by assuming that condition (1) is true. This means that there exists at least one subspace with energy E_m in which

$$|\bar{\lambda}_m^{(O)} - O_{\text{ME}}(E)| \not\rightarrow 0, \quad N \rightarrow \infty. \quad (\text{A4})$$

Thus, it is enough to find an initial condition $|\Psi(0)\rangle$ such that $\bar{\hat{O}} = \bar{\lambda}_m^{(O)} = T_m^{(O)}/d_m$ to show that condition (1) implies the existence of at least one nonthermalizing initial condition. We can build a simple one by means of $|\Psi(0)\rangle = \sum_{\alpha=1}^{d_m} |\tilde{E}_{m,\alpha}\rangle / \sqrt{d_m}$. Then, we have $\| |\Psi(0)\rangle \| = 1$ and $|\Psi(t)\rangle = \sum_{\alpha=1}^{d_m} e^{-iE_m t/\hbar} |\tilde{E}_{m,\alpha}\rangle / \sqrt{d_m}$. Therefore,

$$\bar{\hat{O}} = \langle \Psi(0) | \hat{O} | \Psi(0) \rangle = \sum_{\alpha,\beta=1}^{d_m} \frac{\langle \tilde{E}_{m,\beta} | \hat{O} | \tilde{E}_{m,\alpha} \rangle}{d_m} = \sum_{\alpha=1}^{d_m} \frac{\lambda_{m,\alpha}}{d_m} = \frac{T_m^{(O)}}{d_m} = \bar{\lambda}_m^{(O)}, \quad (\text{A5})$$

which proves condition (1).

Let us now assume that condition (2) is satisfied. Similarly to case (1), we can assume that there exists an eigenspace of energy E_m and an eigenvalue $\lambda_{n,m}^{(\hat{O})}$ such that

$$|\lambda_{n,m}^{(O)} - \bar{\lambda}_m^{(O)}| \not\rightarrow 0, \quad N \rightarrow \infty. \quad (\text{A6})$$

Due to our previous results, we may assume that condition (1) is not satisfied and that $\max_m |\bar{\lambda}_m^{(O)} - O_{\text{ME}}| \rightarrow 0$. Let $\lambda_{\alpha}^{(m)}$ be the eigenvalue of $\hat{O}^{(m)}$ such that $\langle \tilde{E}_{m,\alpha} | \hat{O} | \tilde{E}_{m,\alpha} \rangle = \lambda_{m,\alpha}$ and let us consider an initial condition $|\Psi(0)\rangle = |\tilde{E}_{m,\alpha}\rangle$. Thus, the long-time average of the expectation value of $\hat{O}$ is simply

$$\bar{\hat{O}} = \langle \tilde{E}_{m,\alpha} | \hat{O} | \tilde{E}_{m,\alpha} \rangle = \lambda_{m,\alpha}^{(O)}, \quad (\text{A7})$$

and this initial state does not thermalize.

APPENDIX B: CLASSICAL LIMIT

The three-site Bose-Hubbard model, Eq. (4), has a well-defined classical limit when $N \rightarrow \infty$. In this limit, the bosonic positions and momenta commute, $[\hat{q}_i, \hat{p}_j] \propto 1/N \rightarrow 0$, so they can be treated as classical variables defined on the phase space, $(\mathbf{q}, \mathbf{p}) \in \mathcal{M}$. The simple substitution $\hat{a}_j \rightarrow \sqrt{N/2}(q_j + ip_j)$ allows us to obtain a classical Hamiltonian from the quantum Hamiltonian, $\hat{H} \rightarrow H(\mathbf{q}, \mathbf{p})$,

$$\frac{H}{N} = \sum_{i=1}^3 \left[-J(q_{i+1}q_i + p_{i+1}p_i) + \frac{U}{4}(q_i^2 + p_i^2)^2 \right], \quad (\text{B1})$$

where $\sum_i (q_i^2 + p_i^2) = 2$. One can similarly obtain classical expressions for the current $\hat{C}$, the imbalance $\hat{I}$, and the hopping $\hat{h}_{12}$, namely,

$$\frac{I}{N} = \frac{1}{2} [(q_1^2 + p_1^2) + e^{2\pi i/3} (q_2^2 + p_2^2) + e^{4\pi i/3} (q_3^2 + p_3^2)], \quad (\text{B2})$$

$$\frac{C}{N} = \sum_j (p_j q_{j+1} - p_{j+1} q_j), \quad (\text{B3})$$

and

$$\frac{h_{12}}{N} = q_1 q_2 + p_1 p_2. \quad (\text{B4})$$

Symmetries

Equation (4) is invariant under $2\pi/3$ and $4\pi/3$ rotations, and under the reflection around any of its symmetry axes. This set of transformations constitutes the dihedral D_3 group, whose elements are given by the operators $\hat{R}$ and $\hat{S}$.

APPENDIX C: NUMERICAL PROTOCOLS

1. Evaluation of quantum chaos

We rely on the distribution of nearest-neighbor level spacings, $P(s)$. In the case of semiclassical systems exhibiting the coexistence of chaotic and regular phase space regions, Berry and Robnik [101] derived an expression for $P(s)$ assuming that eigenlevel sequences from chaotic regions follow the Wigner-Dyson distribution [54,72], while sequences from regular regions follow the Poisson distribution [102],

$$P(s; \rho) = \rho^2 e^{-\rho s} \operatorname{erfc}\left(\frac{1}{2}\sqrt{\pi}\bar{\rho}s\right) + (2\rho\bar{\rho} + \frac{1}{2}\pi\bar{\rho}^3 s) e^{-\rho s - \frac{1}{4}\pi\bar{\rho}^2 s^2}. \quad (\text{C1})$$

where ρ is the fraction of Poissonian (regular) sequences and $\bar{\rho} = 1 - \rho$. Classically, ρ represents the proportion of regular orbits. Taken as a one-parameter function, Eq. (C1) may be fitted to the numerical $P(s)$ to assess the degree of chaos through ρ .

As the universal predictions of the RMT only apply if $\langle s \rangle = 1$, a prior mandatory step is the so-called unfolding procedure, which removes system-dependent features in the energy spectrum [103,104]. Our method consists of a numerical fit to a polynomial of degree 4 to the cumulative level density, which gives rise to a smooth function $\bar{N}(E)$. Then, the unfolded energy levels are obtained as $\epsilon_n = \bar{N}(E_n)$, from which we calculate $s_n = \epsilon_{n+1} - \epsilon_n$, which automatically fulfills $\langle s \rangle = 1$. In our numerical experiments, we have divided the spectrum of the Bose-Hubbard model into equispaced sequences with width $\delta E = 0.1$, performed the unfolding procedure to each of them separately, and calculated the fraction ρ reported in Fig. 1(a). We have worked with $N = 800$, $r = 1$, and $s = \pm 1$.

2. Evaluation of classical chaos

For a given energy E , we randomly generate 200 initial conditions $(\mathbf{q}, \mathbf{p})$ with a uniform distribution fulfilling $\sum_i q_i^2 + p_i^2 = 2$; to do so, we consider all initial conditions such that $H(\mathbf{q}, \mathbf{p})/N \in [E - \delta E, E + \delta E]$, with $\delta E = 0.05$. Then, we numerically solve the Hamilton equations for each initial condition with two different computational precisions (with 20 and 25 significant figures), up to time $t = 1000$. Finally, we compute the time average of the distance of these two trajectories, for each initial condition. We consider the trajectory as chaotic if this average is greater than 0.3. (Similar results are obtained with other bounds.) This is in agreement with one of the most common definitions of classical chaos based on the exponential sensitivity of nearby initial conditions [105]. The percentage of chaotic trajectories for a given energy is reported in Fig. 1(a).

The classical microcanonical average at a given energy E in Fig. 1(a) is

$$\langle h_{12} \rangle_{\text{ME}} = \frac{\int d\mathbf{q}d\mathbf{p} h_{12}(\mathbf{q}, \mathbf{p}) \delta[H(\mathbf{q}, \mathbf{p}) - E]}{\int d\mathbf{q}d\mathbf{p} \delta[H(\mathbf{q}, \mathbf{p}) - E]}, \quad (\text{C2})$$

where $(\mathbf{q}, \mathbf{p}) \in \mathcal{M}$. The integrals in this equation have been computed numerically through Monte Carlo simulations, considering 10^{12} points in phase space, and an energy width $\delta E = 0.01$.

3. Evaluation of symmetry breaking

To compute the percentage of symmetry-breaking classical trajectories in Figs. 1(b) and 1(c), we consider the long-time average of each order parameter. We consider that a trajectory is symmetry breaking if the absolute value of this magnitude is greater than its maximum possible value divided by 10. Thus, this bound is 0.1 for $\hat{I}$ and $\sqrt{3}/10$ for $\hat{C}$.

4. Eigenvalues $\lambda_{n,\alpha}^{(O)}$

In Figs. 1 and 3, we report the values of the eigenvalues $\lambda_{n,\alpha}^{(\hat{O})}$ and the mean eigenvalue $\bar{\lambda}_n^{(O)}$ for $\hat{h}_{12}$, $\hat{I}$, and $\hat{C}$. To compute these quantities, first we diagonalize the Hamiltonian in each of the rotation subspaces. These subspaces are built considering the two eigenstates that are exactly degenerate (states belong to the $r = e^{2\pi i/3}$ and $r = e^{4\pi i/3}$ sectors, which do not commute with the reflection symmetry) and the eigenstate with $r = 1$ such that its energy is closest to that of the previous states. Finally, we compute the matrices $\hat{h}_{12}$, $\hat{I}$, and $\hat{C}$ in these subspaces, and calculate their eigenvalues and traces numerically. An explicit example of this procedure is given in Ref. [67].

-
- [1] J. von Neumann, Beweis des Ergodensatzes und des H-Theorems in der neuen Mechanik, *Z. Phys.* **57**, 30 (1929).
 - [2] J. M. Deutsch, Quantum statistical mechanics in a closed system, *Phys. Rev. A* **43**, 2046 (1991).
 - [3] M. Srednicki, Chaos and quantum thermalization, *Phys. Rev. E* **50**, 888 (1994).
 - [4] M. Rigol, V. Dunjko, and M. Olshanii, Thermalization and its mechanism for generic isolated quantum systems, *Nature (London)* **452**, 854 (2008).
 - [5] A. Polkovnikov, K. Sengupta, A. Silva, and M. Vengalattore, *Colloquium: Nonequilibrium dynamics of closed interacting quantum systems*, *Rev. Mod. Phys.* **83**, 863 (2011).
 - [6] C. Gogolin and J. Eisert, Equilibration, thermalisation, and the emergence of statistical mechanics in closed quantum systems, *Rep. Prog. Phys.* **79**, 056001 (2016).
 - [7] L. D'Alessio, Y. Kafri, A. Polkovnikov, and M. Rigol, From quantum chaos and eigenstate thermalization to statistical mechanics and thermodynamics, *Adv. Phys.* **65**, 239 (2016).
 - [8] P. Reimann, Generalization of von Neumann's approach to thermalization, *Phys. Rev. Lett.* **115**, 010403 (2015).
 - [9] T. Yoshizawa, E. Iyoda, and T. Sagawa, Numerical large deviation analysis of the eigenstate thermalization hypothesis, *Phys. Rev. Lett.* **120**, 200604 (2018).
 - [10] S. Sugimoto, R. Hamazaki, and M. Ueda, Test of the eigenstate thermalization hypothesis based on local random matrix theory, *Phys. Rev. Lett.* **126**, 120602 (2021).
 - [11] S. Sugimoto, R. Hamazaki, and M. Ueda, Eigenstate thermalization in long-range interacting systems, *Phys. Rev. Lett.* **129**, 030602 (2022).
 - [12] P. Reimann, Equilibration of isolated macroscopic quantum systems under experimentally realistic conditions, *Phys. Scr.* **86**, 058512 (2012).
 - [13] P. Reimann, Foundation of statistical mechanics under experimentally realistic conditions, *Phys. Rev. Lett.* **101**, 190403 (2008).
 - [14] N. Linden, S. Popescu, A. J. Short, and A. Winter, Quantum mechanical evolution towards thermal equilibrium, *Phys. Rev. E* **79**, 061103 (2009).

- [15] N. Linden, S. Popescu, A. J. Short, and A. Winter, On the speed of fluctuations around thermodynamic equilibrium, [New J. Phys. **12**, 055021 \(2010\)](#).
- [16] P. Reimann, Canonical thermalization, [New J. Phys. **12**, 055027 \(2010\)](#).
- [17] A. J. Short, Equilibration of quantum systems and subsystems, [New J. Phys. **13**, 053009 \(2011\)](#).
- [18] A. J. Short and T. C. Farrelly, Quantum equilibration in finite time, [New J. Phys. **14**, 013063 \(2012\)](#).
- [19] S. Trotzky, Y.-A. Chen, A. Flesch, I. P. McCulloch, U. Schollwöck, J. Eisert, and I. Bloch, Probing the relaxation towards equilibrium in an isolated strongly correlated one-dimensional gas, [Nat. Phys. **8**, 325 \(2012\)](#).
- [20] A. M. Kaufman, M. Eric Tai, A. Lukin, M. Rispoli, R. Schittko, P. M. Preiss, and M. Greiner, Quantum thermalization through entanglement in an isolated many-body system, [Science **353**, 794 \(2016\)](#).
- [21] G. Clos, D. Porras, U. Warring, and T. Schaetz, Time-resolved observation of thermalization in an isolated quantum system, [Phys. Rev. Lett. **117**, 170401 \(2016\)](#).
- [22] Y. Tang, W. Kao, K.-Y. Li, S. Seo, K. Mallayya, M. Rigol, S. Gopalakrishnan, and B. L. Lev, Thermalization near integrability in a dipolar quantum Newton's cradle, [Phys. Rev. X **8**, 021030 \(2018\)](#).
- [23] V. K. B. Kota, A. Relaño, J. Retamosa, and M. Vyas, Thermalization in the two-body random ensemble, [J. Stat. Mech. \(2011\) P10028](#).
- [24] M. Rigol and M. Srednicki, Alternatives to eigenstate thermalization, [Phys. Rev. Lett. **108**, 110601 \(2012\)](#).
- [25] T. N. Ikeda, Y. Watanabe, and M. Ueda, Finite-size scaling analysis of the eigenstate thermalization hypothesis in a one-dimensional interacting Bose gas, [Phys. Rev. E **87**, 012125 \(2013\)](#).
- [26] W. Beugeling, R. Moessner, and M. Haque, Finite-size scaling of eigenstate thermalization, [Phys. Rev. E **89**, 042112 \(2014\)](#).
- [27] R. Steinigeweg, A. Khodja, H. Niemeyer, C. Gogolin, and J. Gemmer, Pushing the limits of the eigenstate thermalization hypothesis towards mesoscopic quantum systems, [Phys. Rev. Lett. **112**, 130403 \(2014\)](#).
- [28] H. Kim, T. N. Ikeda, and D. A. Huse, Testing whether all eigenstates obey the eigenstate thermalization hypothesis, [Phys. Rev. E **90**, 052105 \(2014\)](#).
- [29] D. Jansen, J. Stoplp, L. Vidmar, and F. Heidrich-Meisner, Eigenstate thermalization and quantum chaos in the Holstein polaron model, [Phys. Rev. B **99**, 155130 \(2019\)](#).
- [30] M. Brenes, T. LeBlond, J. Goold, and M. Rigol, Eigenstate thermalization in a locally perturbed integrable system, [Phys. Rev. Lett. **125**, 070605 \(2020\)](#).
- [31] J. Richter, A. Dymarsky, R. Steinigeweg, and J. Gemmer, Eigenstate thermalization hypothesis beyond standard indicators: Emergence of random-matrix behavior at small frequencies, [Phys. Rev. E **102**, 042127 \(2020\)](#).
- [32] J. Wang, M. H. Lamann, J. Richter, R. Steinigeweg, A. Dymarsky, and J. Gemmer, Eigenstate thermalization hypothesis and its deviations from random-matrix theory beyond thermalization time, [Phys. Rev. Lett. **128**, 180601 \(2022\)](#).
- [33] T. Kinoshita, T. Wenger, and D. S. Weiss, A quantum Newton's cradle, [Nature \(London\) **440**, 900 \(2006\)](#).
- [34] M. Gring, M. Kuhnert, T. Langen, T. Kitagawa, B. Rauer, M. Schreitl, I. Mazets, D. Adu Smith, E. Demler, and J. Schmiedmayer, Relaxation and prethermalization in an isolated quantum system, [Science **337**, 1318 \(2012\)](#).
- [35] T. Langen, S. Erne, R. Geiger, B. Rauer, T. Schweigler, W. Rohringer, I. E. Mazets, T. Gasenzer, and J. Schmiedmayer, Experimental observation of a generalized Gibbs ensemble, [Science **348**, 207 \(2015\)](#).
- [36] M. Rigol, Breakdown of thermalization in finite one-dimensional systems, [Phys. Rev. Lett. **103**, 100403 \(2009\)](#).
- [37] F. H. L. Essler and M. Fagotti, Quench dynamics and relaxation in isolated integrable quantum spin chains, [J. Stat. Mech. \(2016\) 064002](#).
- [38] M. Mierzejewski and L. Vidmar, Quantitative impact of integrals of motion on the eigenstate thermalization hypothesis, [Phys. Rev. Lett. **124**, 040603 \(2020\)](#).
- [39] J.-Y. Choi, S. Hild, J. Zeiher, P. Schauß, A. Rubio-Abadal, T. Yefsah, V. Khemani, D. A. Huse, I. Bloch, and C. Gross, Exploring the many-body localization transition in two dimensions, [Science **352**, 1547 \(2016\)](#).

- [40] J. Smith, A. Lee, P. Richerme, B. Neyenhuis, P. W. Heiss, P. Hauke, M. Heyl, D. A. Huse, and C. Monroe, Many-body localization in a quantum simulator with programmable random disorder, *Nat. Phys.* **12**, 907 (2016).
- [41] V. Oganesyan and D. A. Huse, Localization of interacting fermions at high temperature, *Phys. Rev. B* **75**, 155111 (2007).
- [42] R. Nandkishore and D. A. Huse, Many-body localization and thermalization in quantum statistical mechanics, *Annu. Rev. Condens. Matter Phys.* **6**, 15 (2015).
- [43] A. L. Corps, R. A. Molina, and A. Relaño, Thouless energy challenges thermalization on the ergodic side of the many-body localization transition, *Phys. Rev. B* **102**, 014201 (2020).
- [44] A. L. Corps, R. A. Molina, and A. Relaño, Signatures of a critical point in the many-body localization transition, *SciPost Phys.* **10**, 107 (2021).
- [45] C. J. Turner, A. A. Michailidis, D. A. Abanin, M. Serbyn, and Z. Papić, Weak ergodicity breaking from quantum many-body scars, *Nat. Phys.* **14**, 745 (2018).
- [46] S. Moudgalya, S. Rachel, B. A. Bernevig, and N. Regnault, Quantum scarred eigenstates in a Rydberg atom chain: Entanglement, breakdown of thermalization, and stability to perturbations, *Phys. Rev. B* **98**, 235155 (2018).
- [47] M. Schecter and T. Iadecola, Weak ergodicity breaking and quantum many-body scars in spin-1 XY magnets, *Phys. Rev. Lett.* **123**, 147201 (2019).
- [48] M. Serbyn, D. A. Abanin, and Z. Papić, Quantum many-body scars and weak breaking of ergodicity, *Nat. Phys.* **17**, 675 (2021).
- [49] J. O. Austin-Harris, I. Rana, S. E. Begg, C. Binengar, T. Bilitewski, and Y. Liu, Observation of ergodicity breaking and quantum many-body scars in spinor gases, *Phys. Rev. Lett.* **134**, 113401 (2025).
- [50] A. Relaño, Thermalization in an interacting spin system in the transition from integrability to chaos, *J. Stat. Mech.* (2010) P07016.
- [51] N. Shiraishi and T. Mori, Systematic construction of counterexamples to the eigenstate thermalization hypothesis, *Phys. Rev. Lett.* **119**, 030601 (2017); R. Mondiani, K. Mallayya, L. F. Santos, and M. Rigol, Comment on “Systematic construction of counterexamples to the eigenstate thermalization hypothesis”, *ibid.* **121**, 038901 (2018); N. Shiraishi and T. Mori, Shiraishi and Mori reply, *ibid.* **121**, 038902 (2018).
- [52] T. R. de Oliveira, C. Charalambous, D. Jonathan, M. Lewenstein, and A. Riera, Equilibration time scales in closed many-body quantum systems, *New J. Phys.* **20**, 033032 (2018).
- [53] M. Srednicki, The approach to thermal equilibrium in quantized chaotic systems, *J. Phys. A: Math. Gen.* **32**, 1163 (1999).
- [54] M. L. Mehta, *Random Matrices* (Academic Press, Amsterdam, 2004).
- [55] W. Beugeling, R. Moessner, and M. Haque, Off-diagonal matrix elements of local operators in many-body quantum systems, *Phys. Rev. E* **91**, 012144 (2015).
- [56] R. Mondiani and M. Rigol, Eigenstate thermalization in the two-dimensional transverse field Ising model. II. Off-diagonal matrix elements of observables, *Phys. Rev. E* **96**, 012157 (2017).
- [57] C. Nation and D. Porras, Off-diagonal observable elements from random matrix theory: Distributions, fluctuations, and eigenstate thermalization, *New J. Phys.* **20**, 103003 (2018).
- [58] T. LeBlond and M. Rigol, Eigenstate thermalization for observables that break Hamiltonian symmetries and its counterpart in interacting integrable systems, *Phys. Rev. E* **102**, 062113 (2020).
- [59] A. J. Beekman, L. Rademaker, and J. van Wezel, An introduction to spontaneous symmetry breaking, *SciPost Phys. Lect. Notes* **11** (2019).
- [60] P. W. Anderson, An approximate quantum theory of the antiferromagnetic ground state, *Phys. Rev.* **86**, 694 (1952).
- [61] B. Bernu, C. Lhuillier, and L. Pierre, Signature of Néel order in exact spectra of quantum antiferromagnets on finite lattices, *Phys. Rev. Lett.* **69**, 2590 (1992).
- [62] P. Azaria, B. Delamotte, and D. Mouhanna, Spontaneous symmetry breaking in quantum frustrated antiferromagnets, *Phys. Rev. Lett.* **70**, 2483 (1993).
- [63] B. Bernu, P. Lecheminant, C. Lhuillier, and L. Pierre, Exact spectra, spin susceptibilities, and order parameter of the quantum Heisenberg antiferromagnet on a triangular lattice, *Phys. Rev. B* **50**, 10048 (1994).

- [64] T. Koma and H. Tasaki, Symmetry breaking and finite size effects in quantum many-body systems, *J. Stat. Phys.* **76**, 745 (1994).
- [65] H. Tasaki, Long-range order, “Tower” of states, and symmetry breaking in lattice quantum systems, *J. Stat. Phys.* **174**, 735 (2019).
- [66] The scenario becomes even more complicated when considering that the gap between the states belonging to the tower of states decays as $1/N$, with N being the number of particles, whereas the mean level spacing decays exponentially. In the case of discrete symmetries, both decays are exponential.
- [67] See Supplemental Material at <http://link.aps.org/supplemental/10.1103/w77n-lyjd> for an explicit example of the calculation of matrix elements of observables in the reduced eigenspaces.
- [68] L. F. Santos and M. Rigol, Onset of quantum chaos in one-dimensional bosonic and fermionic systems and its relation to thermalization, *Phys. Rev. E* **81**, 036206 (2010).
- [69] L. F. Santos and M. Rigol, Localization and the effects of symmetries in the thermalization properties of one-dimensional quantum systems, *Phys. Rev. E* **82**, 031130 (2010).
- [70] J. M. G. Gómez, K. Kar. V. K. B. Kota, R. A. Molina, A. Relaño, and J. Retamosa, Many-body quantum chaos: Recent developments and applications to nuclei, *Phys. Rep.* **499**, 103 (2011).
- [71] A. Gubin and L. F. Santos, Quantum chaos: An introduction via chains of interacting spins $1/2$, *Am. J. Phys.* **80**, 246 (2012).
- [72] O. Bohigas, M. J. Giannoni, and C. Schmit, Characterization of chaotic quantum spectra and universality of level fluctuation laws, *Phys. Rev. Lett.* **52**, 1 (1984).
- [73] E. T. Jaynes, Information theory and statistical mechanics, *Phys. Rev.* **106**, 620 (1957).
- [74] E. T. Jaynes, Information theory and statistical mechanics. II, *Phys. Rev.* **108**, 171 (1957).
- [75] M. Rigol, A. Muramatsu, and M. Olshanii, Hard-core bosons on optical superlattices: Dynamics and relaxation in the superfluid and insulating regimes, *Phys. Rev. A* **74**, 053616 (2006).
- [76] A. L. Corps and A. Relaño, General theory for discrete symmetry-breaking equilibrium states, *Phys. Rev. E* **110**, 034137 (2024).
- [77] Y. Guryanova, S. Popescu, A. J. Short, R. Silva, and P. Skrzypczyk, Thermodynamics of quantum systems with multiple conserved quantities, *Nat. Commun.* **7**, 12049 (2016).
- [78] N. Y. Halpern, P. Faist, J. Oppenheim, and A. Winter, Microcanonical and resource-theoretic derivations of the thermal state of a quantum system with noncommuting charges, *Nat. Commun.* **7**, 12051 (2016).
- [79] N. Y. Halpern, M. E. Beverland, and A. Kalev, Noncommuting conserved charges in quantum many-body thermalization, *Phys. Rev. E* **101**, 042117 (2020).
- [80] F. Kranzl, A. Lasek, M. K. Joshi, A. Kalev, R. Blatt, C. F. Roos, and N. Y. Halpern, Experimental observation of thermalization with noncommuting charges, *PRX Quantum* **4**, 020318 (2023).
- [81] S. Majidy, W. F. Braasch, A. Lasek, T. Upadaya, A. Kalev, and N. Y. Halpern, Noncommuting conserved charges in quantum thermodynamics and beyond, *Nat. Rev. Phys.* **5**, 689 (2023).
- [82] C. Murthy, A. Babakhani, F. Iniguez, M. Srednicki, and N. Y. Halpern, Non-Abelian eigenstate thermalization hypothesis, *Phys. Rev. Lett.* **130**, 140402 (2023).
- [83] A. Lasek, J. D. Noh, J. LeSchack, and N. Y. Halpern, Numerical evidence for the non-Abelian eigenstate thermalization hypothesis, [arXiv:2412.07838](https://arxiv.org/abs/2412.07838).
- [84] R. Patil and M. Rigol, Eigenstate thermalization in spin- $1/2$ systems with $SU(2)$ symmetry, *Phys. Rev. B* **111**, 205126 (2025).
- [85] In this case, it is still possible to prepare an initial region relaxing onto a symmetric equilibrium state in which $\langle \hat{M} \rangle = 0$, but this necessarily implies a macroscopic superposition of the different branches of the order parameter [76].
- [86] M. Tsubota and K. Kasamatsu, Josephson current flowing in cyclically coupled Bose-Einstein condensates, *J. Phys. Soc. Jpn.* **69**, 1942 (2000).
- [87] D. R. Scherer, C. N. Weiler, T. W. Neely, and B. P. Anderson, Vortex formation by merging of multiple trapped Bose-Einstein condensates, *Phys. Rev. Lett.* **98**, 110402 (2007).
- [88] A. Gallemi, M. Guilleumas, J. Martorell, R. Mayol, A. Polls, and B. Juliá-Díaz, Fragmented condensation in Bose-Hubbard trimes with tunable tunneling, *New J. Phys.* **17**, 073014 (2015).

- [89] M. A. Garcia-March, S. van Frank, M. Bonneau, J. Schmiedmayer, M. Lewenstein, and L. F. Santos, Relaxation, chaos, and thermalization in a three-mode model of a Bose–Einstein condensate, [New J. Phys.](#) **20**, 113039 (2018).
- [90] J. de la Cruz, S. Lerma-Hernández, and J. G. Hirsch, Quantum chaos in a system with high degree of symmetries, [Phys. Rev. E](#) **102**, 032208 (2020).
- [91] D. Wozniak, J. Kroha, and A. Posazhennikova, Chaos onset in large rings of Bose-Einstein condensates, [Phys. Rev. A](#) **106**, 033316 (2022).
- [92] G. Nakerst and M. Haque, Chaos in the three-site Bose-Hubbard model: Classical versus quantum, [Phys. Rev. E](#) **107**, 024210 (2023).
- [93] K. W. Wittmann, E. R. Castro, A. Foerster, and L. F. Santos, Interacting bosons in a triple well: Preface of many-body quantum chaos, [Phys. Rev. E](#) **105**, 034204 (2022).
- [94] E. R. Castro, K. Wittmann W., J. Chávez-Carlos, I. Roditi, A. Foerster, and J. G. Hirsch, Quantum-classical correspondence in a triple-well bosonic model: From integrability to chaos, [Phys. Rev. A](#) **109**, 032225 (2024).
- [95] G. Arwas, A. Vardi, and D. Cohen, Triangular Bose-Hubbard trimer as a minimal model for a superfluid circuit, [Phys. Rev. A](#) **89**, 013601 (2014).
- [96] A. Peres, New conserved quantities and test for regular spectra, [Phys. Rev. Lett.](#) **53**, 1711 (1984).
- [97] K. R. Fratus and M. Srednicki, Eigenstate thermalization in systems with spontaneously broken symmetry, [Phys. Rev. E](#) **92**, 040103(R) (2015).
- [98] A. L. Corps and A. Relaño, Scarred ferromagnetic phase in the long-range transverse-field Ising model, [Phys. Rev. B](#) **113**, L100304 (2026).
- [99] I. Affleck, T. Kennedy, E. H. Lieb, and H. Tasaki, Rigorous results on valence-bond ground states in antiferromagnets, [Phys. Rev. Lett.](#) **59**, 799 (1987).
- [100] S. Gómez, A. L. Corps and A. Relaño, Data supporting Figs. 1-3 of manuscript at arXiv2506.13370, Zenodo (2026), doi:[10.5281/zenodo.19953058](#).
- [101] M. V. Berry and M. Robnik, Semiclassical level spacings when regular and chaotic orbits coexist, [J. Phys. A: Math. Gen.](#) **17**, 2413 (1984).
- [102] M. Berry and M. Tabor, Level clustering in the regular spectrum, [Proc. R. Soc. A](#) **356**, 375 (1997).
- [103] J. M. G. Gómez, R. A. Molina, A. Relaño, and J. Retamosa, Misleading signatures of quantum chaos, [Phys. Rev. E](#) **66**, 036209 (2002).
- [104] A. L. Corps and A. Relaño, Long-range level correlations in quantum systems with finite Hilbert space dimension, [Phys. Rev. E](#) **103**, 012208 (2021).
- [105] M. C. Gutzwiller, *Chaos in Classical and Quantum Mechanics* (Springer, New York, 1990).