

Efficient anomaly detection in a quantum network

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A quantum network enables the execution of advanced protocols, such as distributed quantum computing, quantum metrology, quantum cryptography, and long-distance quantum communication. Anomalous nodes pose significant threats to these protocols, as they are notoriously difficult to detect and can cause severe disruption. In this work, we introduce a quantum-walk-based protocol for detecting such anomalies. The protocol exploits the sensitivity of coherent quantum transport to local perturbations of the network Hamiltonian. In particular, anomalous on-site potentials or modified hopping amplitudes alter the quantum walker's interference pattern, thereby shifting the time-averaged arrival probability between two trusted end points and offering the detection signal. We develop the protocol for one-dimensional quantum networks, derive spectral criteria for detectability, and discuss how the resulting ideas may extend to more general graph topologies.

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I. INTRODUCTION

Quantum network enables interesting advanced protocols such as distributed quantum computing, quantum metrology, quantum cryptography, and long-distance quantum communication [1–11]. However, real-world quantum protocols often deviate from their idealized behavior due to adversarial nodes, device malfunctions, or environmental noise. We refer to such deviations as anomalies. These anomalies are the villains of any ideal quantum protocol as they are notoriously hard to detect and can cause severe disruption [12–14]. This raises a critical question: Can we reliably detect anomalous behavior and build a quantum network that is genuinely trustworthy? In quantum cryptography, entanglement verification techniques are commonly employed to assess the trustworthiness of quantum devices [15–18]. However, these approaches often do not generalize well to arbitrary protocols, particularly those lacking clearly defined entanglement structures.

We use the quantum-walk techniques [19,20] to detect anomalous nodes in a quantum network. Similar techniques are also used in classical methods, where suspicious messages and nodes are detected based on signal strength [21]. If all is well, the expected and actual signal strengths should match; any discrepancy indicates an attack. Inspired by this idea, we compare the predicted end-point transport under an honest Hamiltonian with the empirically observed transport in the implemented network. The quantum version is sensitive not only to local transition strengths,

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but also to phase coherence and interference. A local perturbation, such as an anomalous on-site potential or a modified hopping amplitude, alters the walker's global interference pattern and thus the probability of detecting it at a trusted end point. The main operational quantity in our protocol is the time-averaged arrival probability between two trusted nodes (A) and (B). This quantity is estimated by repeatedly preparing the walker at (A), choosing a measurement time uniformly at random from a fixed time window, and performing a binary detection measurement at (B).

Unlike other quantum algorithms, quantum-walk algorithms have an inherent graph structure, making them well suited for quantum networks. Moreover, many transport properties of both classical and quantum walks are governed by the spectra of the corresponding transition matrices or Hamiltonians. In this paper, we focus primarily on one-dimensional quantum networks, where the spectral analysis can be carried out explicitly. We analyze both anomalous nodes, modeled by on-site perturbations, and anomalous edges, modeled by modified hopping amplitudes. We then use the resulting spectral information to derive detectability criteria and polynomial-resource bounds for the protocol. Although the one-dimensional setting is analytically convenient, the same spectral perspective also suggests extensions to more general graph networks.

The paper is organized as follows. In Sec. II, we review continuous-time quantum walks and introduce the time-averaged arrival probability used in the protocol. In Sec. III, we state the network assumptions required for the anomaly-detection task. In Sec. IV, we formulate the general end-point detection protocol and derive a spectral criterion for detectability. In Sec. V, we analyze the criterion explicitly for one-dimensional networks with anomalous nodes and edges. In Sec. VI, we extend the result to more graph topologies. In Sec. VII, we discuss the time and sample complexity of the protocol. Finally, in Sec. VIII, we summarize the results and outline open problems.

II. QUANTUM WALK

Classical random walks describe a walker that moves stochastically on a graph $G = (V, E)$, with transition probabilities determined by the edges of the graph [22–24]. A continuous-time quantum walk replaces these stochastic transitions with a unitary time evolution generated by a Hamiltonian H acting on the Hilbert space spanned by the vertex basis $\{|a\rangle\}_{a \in V}$.

Throughout this work, the dynamics of the quantum-network protocol are modeled by $U(t) = e^{-iHt}$, where H is a graph-structured Hamiltonian acting on an N -dimensional Hilbert space spanned by $\{|a\rangle\}_{a=1}^N$. Quantum-walk Hamiltonians naturally encode the connectivity of the underlying graph and have been used to describe spin networks, photonic circuits, and quantum state-transfer algorithms [20, 25–27]. We formalize this as follows.

Definition 1 (Continuous-time quantum walk). A continuous-time quantum walk on a graph G for a time t is given by the unitary operator $U(t) = \exp(-iHt)$, where H is a Hermitian matrix whose matrix elements $H_{ab} = \langle a|H|b\rangle$ respect the graph structure of G .

Various quantities can be used to characterize quantum walks, such as mixing time, transport rate, and different notions of hitting time. The classical first-hitting-time picture is often used for intuition, and we recall it here for completeness. However, due to the oscillatory nature of quantum amplitudes, it is not operationally robust in our setting. Our protocol will instead be based on time-averaged arrival probabilities, which admit clean spectral bounds and are directly measurable.

Definition 2 (Quantum hitting time). For a given probability threshold p , the hitting time T^{ij} of a quantum walk between two nodes i and j on a graph G is defined as

$$T^{ij} = \min\{t : P_{ij}(t) \geq p\}, \quad (1)$$

where $P_{ij}(t) = |\langle j|U(t)|i\rangle|^2$ is the probability that a walker initially prepared at node i is found at node j after time t . For a given pair (i, j) , we denote the corresponding parameters by (T^{ij}, p) .

Unfortunately, due to the oscillatory nature of $P_{ij}(t)$, this quantity is not operationally robust. So, the operationally meaningful quantity is the time-averaged arrival probability defined as follows.

Definition 3 (Time-averaged arrival probability). For nodes i and j and evolution time $T > 0$, the time-averaged arrival probability is

$$\bar{P}_{i \rightarrow j}(T) = \frac{1}{T} \int_0^T P_{ij}(t) dt = \frac{1}{T} \int_0^T |\langle j | e^{-iHt} | i \rangle|^2 dt. \quad (2)$$

Operationally, $\bar{P}_{i \rightarrow j}(T)$ corresponds to the empirical detection rate at node j obtained by repeating the experiment over an observation window of duration T . In the limit of large observation times, the oscillatory contributions average out and one obtains a stationary transfer rate determined solely by the spectral decomposition of H .

Definition 4 (Asymptotic transfer probability). The long-time, or asymptotic, transfer probability between nodes i and j is defined as

$$\Pi_{i \rightarrow j} = \lim_{T \rightarrow \infty} \bar{P}_{i \rightarrow j}(T) = \sum_{\lambda} |\langle j | \lambda \rangle|^2 |\langle \lambda | i \rangle|^2, \quad (3)$$

where the sum runs over the eigenvalues λ of H .

These quantities play a central role in distinguishing the honest Hamiltonian H_g from a perturbed Hamiltonian H_d , since local modifications of H lead to detectable changes in $\bar{P}_{i \rightarrow j}(T)$ and $\Pi_{i \rightarrow j}$.

III. PROBLEM STATEMENT, ASSUMPTIONS, AND MAIN IDEA

A. Problem statement

Consider a quantum network described by a graph $G = (V, E)$, with vertices $V = \{P_1, P_2, \dots, P_N\}$ representing N nodes, such as quantum repeaters or small-scale quantum processors. Some of these nodes may behave incorrectly and thereby disrupt the intended network protocol. Our goal is to detect such irregularities efficiently, ideally in time polynomial in the network size.

B. Network dynamics model

We model the evolution of the network using a continuous-time quantum walk

$$U(t) = e^{-iHt}, \quad (4)$$

acting on an N -dimensional Hilbert space spanned by $\{|a\rangle\}_{a=1}^N$, where $|a\rangle$ represents an excitation localized at node P_a . Quantum-walk Hamiltonians naturally encode the graph structure and decompose into local and interaction terms,

$$H = \sum_{i \in V} H_i + \sum_{(i,j) \in E} H_{ij}, \quad (5)$$

where $H_i = \alpha_i |i\rangle\langle i|$ is an on-site potential and $H_{ij} = \beta_{ij} |i\rangle\langle j| + \kappa_{ji} |j\rangle\langle i|$ generates coherent hopping between adjacent nodes. Such Hamiltonians arise in spin networks, photonic lattices, and state-transfer architectures [28–30].

Definition 5 (Anomalous node). A node P_i is called *anomalous* if the Hamiltonian terms H_i or any interaction terms H_{ij} involving i deviate from their prescribed honest values. These local perturbations alter the global spectrum of H and thereby modify the transport behavior of the quantum walk.

C. Main idea

The key idea is that the propagation profile of a quantum walker is highly sensitive to such local perturbations. The probability distribution of the walker differs significantly in the presence of anomalous nodes. This change is reflected in operational quantities such as the time-averaged arrival probability $\bar{P}_{A \rightarrow B}(T)$ between two trusted nodes A and B . If the observed behavior deviates from that predicted by the honest Hamiltonian H_g , the network is flagged as suspicious.

In the photonic implementation, this abstract formalism has a physical meaning. The on-site terms $H_i = \beta_i |i\rangle\langle i|$ correspond to the propagation constants of individual waveguides, while the interaction terms $H_{ij} = \kappa_{ij} |i\rangle\langle j| + \kappa_{ji} |j\rangle\langle i|$ arise from evanescent coupling between neighboring waveguides. An anomalous node P_i therefore manifests physically as a waveguide whose propagation constant β_i or coupling rates κ_{ij} differ from their designed values, for example, due to fabrication imperfections, refractive-index fluctuations, or intentional tampering. Such local modifications to β_i or κ_{ij} translate into a perturbation of the global Hamiltonian H , which in turn induces measurable changes in the quantum-walk interference pattern. In the next section, we will formulate this intuition as an algorithm.

IV. GENERAL ANOMALY-DETECTION PROTOCOL

We assume that there are two trusted nodes, A and B , in the network. In the honest case, the evolution is generated by a known Hamiltonian H_g , which encodes a network without anomalous nodes. In the presence of anomalous nodes, some on-site or coupling terms are modified, leading to a perturbed Hamiltonian

$$H_d = H_g + H_{\text{anom}}, \quad (6)$$

where H_{anom} is supported only on the anomalous nodes and their neighboring edges. The goal of the protocol is to distinguish, efficiently and using only local measurements at A and B , whether the implemented dynamics is governed by H_g or by some perturbed H_d .

Our test relies on comparing the *time-averaged arrival probability* between A and B under the two dynamics.

Definition 6 (Time-averaged arrival probability). For a Hamiltonian H , nodes A and B , and an evolution time $T > 0$, the time-averaged arrival probability is

$$\bar{P}_{A \rightarrow B}^H(T) = \frac{1}{T} \int_0^T |\langle B | e^{-iHt} | A \rangle|^2 dt. \quad (7)$$

Operationally, $\bar{P}_{A \rightarrow B}^H(T)$ is the detection rate at node B that would be obtained by repeating the experiment many times over an observation window of duration T , with the walker initially prepared at A .

We now formalize the conditions under which a quantum-walk-based test is efficient.

Definition 7 (Efficient quantum-walk test). Let $G = (V, E)$ be a quantum network with diameter $d(G)$ and trusted nodes $A, B \in V$. A test based on time-averaged arrival probabilities is called *efficient* if the following conditions are met:

(1) *Efficient honest propagation:* For the honest Hamiltonian H_g , there exists a sampling time $T_g = \text{poly}(d(G))$ such that

$$\bar{P}_{A \rightarrow B}^{H_g}(T_g) \geq \frac{1}{\text{poly}(d(G))}. \quad (8)$$

With $\text{poly}(d(G))$ independent repetitions, node B then observes at least one arrival event with constant probability.

(2) *Detectable deviation under anomalies*: For any perturbed Hamiltonian H_d , the corresponding time-averaged arrival probability at the same sampling time satisfies

$$|\bar{P}_{A \rightarrow B}^{H_d}(T_g) - \bar{P}_{A \rightarrow B}^{H_g}(T_g)| > \epsilon_{d(G)}, \quad \epsilon_{d(G)} \geq \frac{1}{\text{poly}(d(G))}. \quad (9)$$

Estimating a Bernoulli mean to accuracy $\epsilon_{d(G)}$ requires $O(1/\epsilon_{d(G)}^2)$ samples, which remains polynomial in $d(G)$.

While Definition 6 defines $\bar{P}_{A \rightarrow B}^H(T)$ as a time average, it cannot be obtained by continuously monitoring node B , since frequent measurements would project the state and suppress coherent evolution (quantum Zeno effect). Instead, the time-averaged arrival probability is realized experimentally via randomized temporal sampling. First, one fixes an observation window of duration T . Then, M independent experimental trials are performed. In trial k , one draws a measurement time $t_k \sim \text{Uniform}(0, T)$. The walker starts at A , evolves under the (unknown) network Hamiltonian for time t_k , and a projective measurement is performed at node B .

Let $X_k \in \{0, 1\}$ denote the detection outcome at B in trial k . The empirical estimator

$$\hat{P}_{A \rightarrow B} = \frac{1}{M} \sum_{k=1}^M X_k \quad (10)$$

then satisfies

$$\hat{P}_{A \rightarrow B} \xrightarrow{M \rightarrow \infty} \frac{1}{T} \int_0^T |\langle B | e^{-iHt} | A \rangle|^2 dt = \bar{P}_{A \rightarrow B}^H(T), \quad (11)$$

providing an operational realization of the time-averaged arrival probability without disturbing the unitary dynamics. An explicit decision procedure can then be stated as follows.

Algorithm: Quantum-walk anomaly test.

1. Node A prepares the initial state $|A\rangle$ and fixes a sampling time T_g and number of trials $M = \text{poly}(d(G))$.
2. For each trial $k = 1, \dots, M$, node A (or an external referee) samples a time $t_k \sim \text{Uniform}(0, T_g)$, prepares the walker at A , and lets the network evolve for time t_k .
3. At time t_k , node B performs a projective measurement and records the binary outcome $X_k \in \{0, 1\}$ (detection/no detection).
4. After M trials, node B estimates the empirical arrival frequency $\hat{P}_{A \rightarrow B}$ and communicates this value to node A .
5. Node A checks whether

$$|\hat{P}_{A \rightarrow B} - \bar{P}_{A \rightarrow B}^{H_g}(T_g)| \begin{cases} < \epsilon, & \text{declare } \textit{Honest}, \\ \geq \epsilon, & \text{declare } \textit{Anomalous}. \end{cases}$$

If the deviation exceeds ϵ , the network is declared suspicious and additional diagnostic tests may be performed to localize anomalous nodes.

To analyze when this procedure can successfully distinguish H_g from H_d , we relate the time-averaged arrival probability to spectral properties of the underlying Hamiltonian. A central quantity in our analysis is the asymptotic transfer probability

$$\Pi_{A \rightarrow B}(H) = \sum_{\lambda} |\langle \lambda | A \rangle|^2 |\langle B | \lambda \rangle|^2, \quad (12)$$

where the sum runs over all eigenstates $|\lambda\rangle$ of H . The following lemma shows that the finite-time averaged arrival probability converges to $\Pi_{A \rightarrow B}(H)$ at a rate governed by the spectral gap.

Lemma 1. Let H be a Hamiltonian with smallest nonzero spectral gap $\Delta(H) = \min_{\lambda \neq \lambda'} |\lambda - \lambda'|$. Then, for any $T > 0$,

$$\bar{P}_{A \rightarrow B}^H(T) \geq \Pi_{A \rightarrow B}(H) - \frac{2}{\Delta(H)T}. \quad (13)$$

Proof. By definition,

$$\bar{P}_{A \rightarrow B}^H(T) = \frac{1}{T} \int_0^T |\langle B | e^{-iHt} | A \rangle|^2 dt = \frac{1}{T} \int_0^T \langle B | e^{-iHt} | A \rangle \langle A | e^{iHt} | B \rangle dt. \quad (14)$$

Using the spectral decomposition $H = \sum_{\lambda} \lambda |\lambda\rangle \langle \lambda|$, $e^{-iHt} = \sum_{\lambda} e^{-i\lambda t} |\lambda\rangle \langle \lambda|$. Substituting gives

$$\bar{P}_{A \rightarrow B}^H(T) = \frac{1}{T} \sum_{\lambda, \lambda'} \langle B | \lambda \rangle \langle \lambda | A \rangle \langle A | \lambda' \rangle \langle \lambda' | B \rangle \int_0^T e^{-i(\lambda - \lambda')t} dt. \quad (15)$$

The integral evaluates to

$$\frac{1}{T} \int_0^T e^{-i(\lambda - \lambda')t} dt = \begin{cases} 1, & \lambda = \lambda', \\ \frac{1 - e^{-i(\lambda - \lambda')T}}{i(\lambda - \lambda')T}, & \lambda \neq \lambda'. \end{cases}$$

Thus,

$$\bar{P}_{A \rightarrow B}^H(T) = \Pi_{A \rightarrow B}(H) + \sum_{\lambda \neq \lambda'} \frac{1 - e^{-i(\lambda - \lambda')T}}{i(\lambda - \lambda')T} \langle B | \lambda \rangle \langle \lambda | A \rangle \langle A | \lambda' \rangle \langle \lambda' | B \rangle. \quad (16)$$

For the oscillatory term, we have

$$\left| \frac{1 - e^{-i(\lambda - \lambda')T}}{i(\lambda - \lambda')T} \right| \leq \frac{2}{|\lambda - \lambda'|T} \leq \frac{2}{\Delta(H)T}.$$

Using Cauchy-Schwarz and completeness, we get $\sum_{\lambda} |\langle B | \lambda \rangle \langle \lambda | A \rangle| \leq 1$. Therefore, $\sum_{\lambda, \lambda'} |\langle B | \lambda \rangle \langle \lambda | A \rangle| |\langle A | \lambda' \rangle \langle \lambda' | B \rangle| \leq 1$. Combining these bounds yields

$$\bar{P}_{A \rightarrow B}^H(T) \geq \Pi_{A \rightarrow B}(H) - \frac{2}{\Delta(H)T},$$

which completes the proof. $\blacksquare$

We now use Lemma 1 to quantify how much the time-averaged arrival probability must differ under H_g and H_d for the test to succeed.

Theorem 1 (Spectral criterion for detection). Let H_g and H_d be the honest and perturbed Hamiltonians with spectral gaps Δ_g and Δ_d , and asymptotic transfer probabilities $\Pi_{A \rightarrow B}(H_g)$ and $\Pi_{A \rightarrow B}(H_d)$. Then, for any sampling time $T_g > 0$,

$$|\bar{P}_{A \rightarrow B}^{H_d}(T_g) - \bar{P}_{A \rightarrow B}^{H_g}(T_g)| \geq |\Pi_{A \rightarrow B}(H_d) - \Pi_{A \rightarrow B}(H_g)| - \left(\frac{2}{\Delta_d T_g} + \frac{2}{\Delta_g T_g} \right). \quad (17)$$

If the right-hand side of Eq. (17) exceeds the threshold ϵ , the test in Definition 7 detects the presence of anomalous nodes.

Proof. From Lemma 1, we may write

$$\bar{P}_{A \rightarrow B}^{H_g}(T_g) = \Pi_{A \rightarrow B}(H_g) + \delta_g, \quad |\delta_g| \leq \frac{2}{\Delta_g T_g}, \quad (18)$$

$$\bar{P}_{A \rightarrow B}^{H_d}(T_g) = \Pi_{A \rightarrow B}(H_d) + \delta_d, \quad |\delta_d| \leq \frac{2}{\Delta_d T_g}. \quad (19)$$

Thus,

$$\bar{P}_{A \rightarrow B}^{H_d}(T_g) - \bar{P}_{A \rightarrow B}^{H_g}(T_g) = (\Pi_{A \rightarrow B}(H_d) - \Pi_{A \rightarrow B}(H_g)) + (\delta_d - \delta_g).$$

Using the reverse triangle inequality,

$$|\bar{P}_{A \rightarrow B}^{H_d}(T_g) - \bar{P}_{A \rightarrow B}^{H_g}(T_g)| \geq |\Pi_{A \rightarrow B}(H_d) - \Pi_{A \rightarrow B}(H_g)| - |\delta_d - \delta_g|.$$

Finally, the triangle inequality gives

$$|\delta_d - \delta_g| \leq |\delta_d| + |\delta_g| \leq \frac{2}{\Delta_d T_g} + \frac{2}{\Delta_g T_g}.$$

Substituting this bound yields inequality (17). ■

Theorem 1 provides a sufficient condition for the success of the protocol. One can interpret inequality (17) as a competition between the structural distinguishability of two networks and the finite-time corrections. The structural signal $|\Pi_d - \Pi_g|$ depends only on the Hamiltonians H_d and H_g , while the term $(\frac{2}{\Delta_d T_g} + \frac{2}{\Delta_g T_g})$ can be viewed as a finite-time correction resulting from the fact that protocol samples arrival events over a finite window of duration T_g . Thus, the test is efficient whenever the sampling time T_g is large enough that the transient noise is smaller than the structural signal. Quantitatively, this corresponds to choosing $T_g \gg \max\{\Delta_g^{-1}, \Delta_d^{-1}\}$ so that $|\Pi_d - \Pi_g|$ dominates the finite-time corrections.

In the following sections, we explicitly construct an efficient test Hamiltonian H_g for a one-dimensional quantum network and generalize for general graphs, analyze how local modifications associated with anomalous nodes modify its spectrum, and compute the spectral quantities appearing in Theorem 1.

V. ONE-DIMENSIONAL QUANTUM NETWORK

In this section, we study a one-dimensional quantum network modeled by a continuous-time quantum walk [20,26,31]. This simple model provides us with an analytical solution. Let $L = N + 2$ be the total number of sites labeled by $j = 1, \dots, L$. We place the two trusted nodes at the end points, $A = 1$ and $B = L$. The honest network is the finite path graph P_L with adjacency Hamiltonian:

$$H_g = \sum_{j=1}^{L-1} (|j\rangle\langle j+1| + |j+1\rangle\langle j|). \quad (20)$$

Equivalently, one may use the discrete Laplacian convention $H_g^\Delta = -2I + H_g$. Since the term $-2I$ only contributes a global phase e^{-iHt} , it does not change transition probabilities. We therefore use the adjacency convention in the main text.

The honest eigenvalue equation is $H_g|\lambda_m\rangle = \lambda_m|\lambda_m\rangle$, for $m = 1, \dots, L$. Writing the wavefunction component as $\psi_m(j) := \langle j|\lambda_m\rangle$, the bulk equation for $1 < j < L$ is $\psi_m(j-1) + \psi_m(j+1) = \lambda_m\psi_m(j)$. The open boundary conditions are $\psi_m(0) = 0$ and $\psi_m(L+1) = 0$ with solution

$$\psi_m(j) = \sqrt{\frac{2}{L+1}} \sin(p_m j), \quad p_m = \frac{m\pi}{L+1}. \quad (21)$$

Therefore, the eigenvalues of the honest Hamiltonian are $\lambda_m = 2 \cos(p_m) = 2 \cos(\frac{m\pi}{L+1})$ as we have explicitly

$$\begin{aligned} \psi_m(j-1) + \psi_m(j+1) &= \sqrt{\frac{2}{L+1}} [\sin(p_m(j-1)) + \sin(p_m(j+1))] \\ &= \sqrt{\frac{2}{L+1}} [2 \cos(p_m) \sin(p_m j)] \\ &= 2 \cos(p_m) \psi_m(j). \end{aligned} \quad (22)$$



FIG. 1. One-dimensional quantum network with trusted end-point nodes A and B , and an anomalous node at position q .

We consider two local anomalies. First, an anomalous node, as depicted in Fig. 1, at position q is modeled by an on-site perturbation,

$$H_q = H_g + \alpha_q |q\rangle\langle q|, \quad 1 \leq q \leq L, \quad (23)$$

where α_q is the anomalous on-site strength. Such an anomalous node could correspond to a node that holds the walker for too long. The corresponding transmission probability is shown in Fig. 2. Second, an anomalous edge, as depicted in Fig. 3 between sites d and $d + 1$, is modeled by modifying the hopping amplitude on that edge:

$$H_d = H_g + \beta_d (|d\rangle\langle d+1| + |d+1\rangle\langle d|), \quad 1 \leq d \leq L-1. \quad (24)$$

The honest hopping amplitude is 1, while the anomalous edge has a hopping amplitude $\tau_d := 1 + \beta_d$. Physically, it could represent biased or asymmetric transitions. Although these anomalies are local, they can affect the global interference and modify the time-averaged arrival probability at the end point B .

A. Anomalous node

We first analyze the on-site anomalous Hamiltonian H_q in Eq. (23). The chain in Hamiltonian equation (23) is finite. Before treating this finite chain, let us consider an infinite-line scattering process, which provides intuition on why such an on-site anomaly suppresses the honest transport.

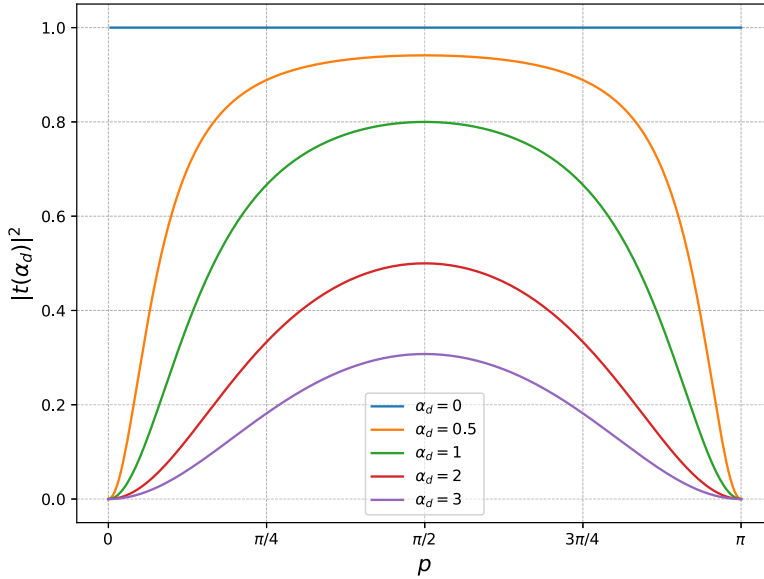


FIG. 2. Transmission probability of the quantum walker as a function of anomalous parameter α_d . The case $\alpha_d = 0$ represents the absence of an anomalous node, and higher values of α_d indicate higher malfunction.

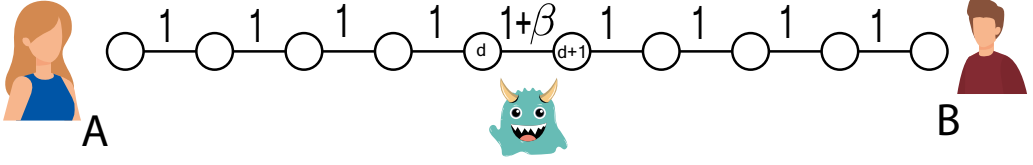


FIG. 3. An anomalous edge between sites d and $d + 1$, modeled by a modified hopping amplitude $\tau_d = 1 + \beta_d$.

1. Infinite-line scattering intuition

The honest Hamiltonian has momentum eigenstates $|p\rangle$ with eigenvalues $2 \cos p$, i.e., $H_g|p\rangle = 2 \cos p|p\rangle$, and the components are $\langle j|p\rangle = \frac{1}{\sqrt{2\pi}} e^{ipj}$, $-\pi \leq p \leq \pi$. We introduce an anomalous node at position d through an on-site potential

$$H_d = H_g + \alpha_d |d\rangle\langle d|, \quad (25)$$

which acts as a scattering center for waves propagating through the chain. Using scattering theory, we will see that, in the presence of such an anomalous node, the transmission probability from node A to B changes. A momentum eigenstate incident from the left is scattered as

$$|\psi_s(p)\rangle = \begin{cases} |p\rangle + r(p)|-p\rangle, & j \leq d, \\ t(p)|p\rangle, & j \geq d + 1, \end{cases}$$

where $t(p)$ and $r(p)$ are, respectively, transmission and reflection amplitudes. Matching the wavefunction at $j = d$ yields the result (a derivation is given in Appendix A)

$$t(p) = \frac{2i \sin p}{\alpha_d + 2i \sin p}, \quad |t(p)|^2 = \frac{4 \sin^2 p}{\alpha_d^2 + 4 \sin^2 p}. \quad (26)$$

For $\alpha_d = 0$, we recover $|t(p)|^2 = 1$, while for $\alpha_d > 0$ the transmission is strictly reduced. In particular, low-momentum modes ($p \approx 0$), which dominate long-distance transport, satisfy

$$|t(p)|^2 \approx \frac{4p^2}{\alpha_d^2 + 4p^2},$$

showing strong suppression when $p \ll \alpha_d$. In Fig. 2, we have plotted transmission probability for different values of α_d , where increasing α_d indicates increasing malfunction. We observe that for fixed momentum p , $|t(\alpha_{d_1})|^2 < |t(\alpha_{d_2})|^2$ for $\alpha_{d_1} > \alpha_{d_2}$. The case $\alpha_d = 0$ represents the absence of an anomalous node and corresponds to the test Hamiltonian H_g , while higher values of α_d represent the anomalous Hamiltonian H_d .

2. Finite chain with defect at arbitrary position

Let $L = N + 2$ be the total number of nodes, with trusted end points $A = 1$ and $B = L$. We impose open boundary conditions, $\psi_0 = \psi_{L+1} = 0$. The honest finite-chain Hamiltonian is

$$H_g = \sum_{j=1}^L (-2) |j\rangle\langle j| + \sum_{j=1}^{L-1} (|j\rangle\langle j+1| + |j+1\rangle\langle j|). \quad (27)$$

An anomalous node at an arbitrary position $q \in \{1, \dots, L\}$ is modeled by an on-site perturbation $H_q = H_g + \alpha_q |q\rangle\langle q|$.

Away from the anomalous site q , the eigenvalue equation is $\psi_{j-1} + \psi_{j+1} - 2\psi_j = \lambda\psi_j$. Taking $\lambda = 2 \cos p - 2$, this becomes $\psi_{j-1} + \psi_{j+1} = 2 \cos p \psi_j$. Therefore, the solution is a superposition

of sine waves. The open boundary conditions $\psi_0 = \psi_{L+1} = 0$ imply the form

$$\psi_j = \begin{cases} A_L \sin(pj), & 1 \leq j \leq q, \\ A_R \sin(p(L+1-j)), & q \leq j \leq L. \end{cases} \quad (28)$$

The wavefunction must be single valued at $j = q$, so $A_L \sin(qp) = A_R \sin((L+1-q)p) = \psi_q$. The only modified eigenvalue equation occurs at $j = q$, where $\psi_{q-1} + \psi_{q+1} - 2\psi_q + \alpha_q \psi_q = (2 \cos p - 2)\psi_q$. Equivalently,

$$\psi_{q-1} + \psi_{q+1} = (2 \cos p - \alpha_q) \psi_q. \quad (29)$$

We may write $A_L = \frac{\psi_q}{\sin(qp)}$ and $A_R = \frac{\psi_q}{\sin((L+1-q)p)}$. Hence, $\psi_{q-1} = \psi_q \frac{\sin((q-1)p)}{\sin(qp)}$ and $\psi_{q+1} = \psi_q \frac{\sin((L-q)p)}{\sin((L+1-q)p)}$.

Substituting these expressions into Eq. (29) gives

$$\frac{\sin((q-1)p)}{\sin(qp)} + \frac{\sin((L-q)p)}{\sin((L+1-q)p)} = 2 \cos p - \alpha_q. \quad (30)$$

Rearranging this expression and using the trigonometric identity, the allowed momenta p are now obtained via the secular equation

$$\sin p \sin((L+1)p) = \alpha_q \sin(qp) \sin((L+1-q)p). \quad (31)$$

For $\alpha_q = 0$, this reduces to the honest open-chain condition $\sin((L+1)p) = 0$ and therefore $p_m = \frac{m\pi}{L+1}$, $m = 1, \dots, L$. For $\alpha_q \neq 0$, the allowed momenta are shifted away from the honest values $p_m = m\pi/(L+1)$. Since the eigenvalues are $\lambda(p) = 2 \cos p$, this momentum shift changes the spectrum of the anomalous Hamiltonian and, in general, also changes its smallest nonzero spectral gap Δ_q . The size of this spectral shift depends both on the anomaly strength α_q and on the defect position q , through the factors $\sin(qp)$ and $\sin((L+1-q)p)$.

This simple analysis reveals that the disorder anomaly induces a detectable shift in the time-averaged arrival probability due to a reduced transmission probability, leading to $\Pi_{A \rightarrow B}(H_d) \neq \Pi_{A \rightarrow B}(H_g)$.

B. Anomalous edge

We now consider another type of anomaly: a local modification of the hopping amplitudes between two adjacent sites. This corresponds to a “biased coin” in the language of quantum walks, and thus can produce a measurable deviation in the time-averaged arrival probability used in our protocol.

The honest test Hamiltonian is the finite translation-invariant adjacency operator

$$H_g = |1\rangle\langle A| + |A\rangle\langle 1| + \sum_{j=1}^{N-1} (|j+1\rangle\langle j| + |j\rangle\langle j+1|) + |B\rangle\langle N| + |N\rangle\langle B|, \quad (32)$$

which generates uniform propagation on the line. At the boundaries, $H_g|A\rangle = |1\rangle$ and $H_g|B\rangle = |N\rangle$, and for each interior node j , $H_g|j\rangle = |j-1\rangle + |j+1\rangle$. Thus, all edges have equal strength.

We next consider an anomalous edge between sites d and $d+1$. The Hamiltonian is

$$H_d = H_g + \beta_d(|d\rangle\langle d+1| + |d+1\rangle\langle d|), \quad \tau_d = 1 + \beta_d. \quad (33)$$

Because the quantum-walk detection algorithm relies on the quantity

$$\bar{P}_{A \rightarrow B}^H(T) = \frac{1}{T} \int_0^T |\langle B|e^{-iHt}|A\rangle|^2 dt,$$

any perturbation that modifies the spectral properties of the Hamiltonian can lead to an observable difference between the honest and anomalous values. To apply the spectral criterion of Theorem 1,

we therefore compute how the biased edge affects the smallest spectral gap of the system. This is accomplished in Lemma 2 below with proof in Appendix B.

Lemma 2 (Spectral-gap bound for a finite network with an arbitrary position of anomalous node). Let $\Delta_g = \min_{\lambda_g \neq \lambda'_g} |\lambda_g - \lambda'_g|$ and $\Delta_d = \min_{\lambda_d \neq \lambda'_d} |\lambda_d - \lambda'_d|$ denote the smallest nonzero spectral gaps of H_g and H_d , respectively. For fixed defect strength τ_d , bounded away from zero and infinity, there exist constants $c_g > 0$ and $c_d(\tau_d) > 0$, independent of the network size N and of the defect position d , such that

$$\Delta_g \geq \frac{c_g}{N^2}, \quad \Delta_d \geq \frac{c_d(\tau_d)}{N^3}. \quad (34)$$

If the anomaly node is exactly at the center of the network, one can also obtain the constants as

$$\Delta_g \geq \frac{\pi^2}{((N+2)/2)^3}, \quad \Delta_d \geq \frac{2\pi^2}{(2+\beta_d)((N+2)/2)^3}, \quad (35)$$

where $N+2$ is the number of nodes in a one-dimensional quantum network. The proof of this inequality is in Appendix B, Lemma B1.

C. Detection criteria

First, let us establish that the test Hamiltonian H_g satisfies the efficiency conditions of Definition 7.

Lemma 3. The test Hamiltonian H_g in Eq. (32) is efficient in the sense of Definition 7.

Proof. For any Hamiltonian H , Lemma 1 yields $\bar{P}_{A \rightarrow B}^H(T) \geq \Pi_{A \rightarrow B}(H) - \frac{2}{\Delta(H)T}$. For the honest Hamiltonian H_g , reflection symmetry implies $\langle A | \lambda_g \rangle = \pm \langle B | \lambda_g \rangle$, and therefore

$$\Pi_{A \rightarrow B}(H_g) = \sum_{\lambda_g} |\langle A | \lambda_g \rangle|^2 |\langle B | \lambda_g \rangle|^2 = \sum_{\lambda_g} |\langle A | \lambda_g \rangle|^4.$$

By Cauchy-Schwarz, $\sum_{\lambda_g} |\langle A | \lambda_g \rangle|^4 \geq \frac{1}{N+2}$, so the asymptotic transfer probability satisfies $\Pi_{A \rightarrow B}(H_g) = \Theta(1/N)$. From Lemma 2, $\Delta_g \geq \Theta(N^{-2})$. Choosing $T_g = \Theta(N^3)$ ensures that $\frac{2}{\Delta_g T_g} = O(1/N)$ and hence $\bar{P}_{A \rightarrow B}^{H_g}(T_g) = \Theta(1/N)$, satisfying the first efficiency requirement. ■

Now, let us obtain the conditions under which anomalous nodes can be efficiently detected. Let

$$\eta_s := |\Pi_{A \rightarrow B}(H_s) - \Pi_{A \rightarrow B}(H_g)|, \quad (36)$$

where $s = q$ for an on-site anomaly and $s = d$ for an edge anomaly. Applying Theorem 1 to the pair (H_g, H_s) gives

$$\begin{aligned} |\bar{P}_{A \rightarrow B}^{H_s}(T_g) - \bar{P}_{A \rightarrow B}^{H_g}(T_g)| &\geq |\Pi_{A \rightarrow B}(H_s) - \Pi_{A \rightarrow B}(H_g)| - \left(\frac{2}{\Delta_s T_g} + \frac{2}{\Delta_g T_g} \right) \\ &= \eta_s - \left(\frac{2}{\Delta_s T_g} + \frac{2}{\Delta_g T_g} \right). \end{aligned} \quad (37)$$

Once η_s is known, the sampling time and number of experimental trials can be chosen to resolve it. We make this precise by defining the detectable anomaly class.

Definition 8 (Detectable one-dimensional anomaly class). For constants $c > 0$ and $k > 0$ independent of the network size $L = N + 2$, define

$$\mathcal{A}_{\text{det}}(c, k) := \left\{ H_s : \eta_s = |\Pi_{A \rightarrow B}(H_s) - \Pi_{A \rightarrow B}(H_g)| \geq \frac{c}{L^k} \right\}. \quad (38)$$

An anomaly in this class produces an inverse-polynomial structural signal at the trusted end points. The dependence of the structural signal on the defect position is illustrated in Fig. 4.

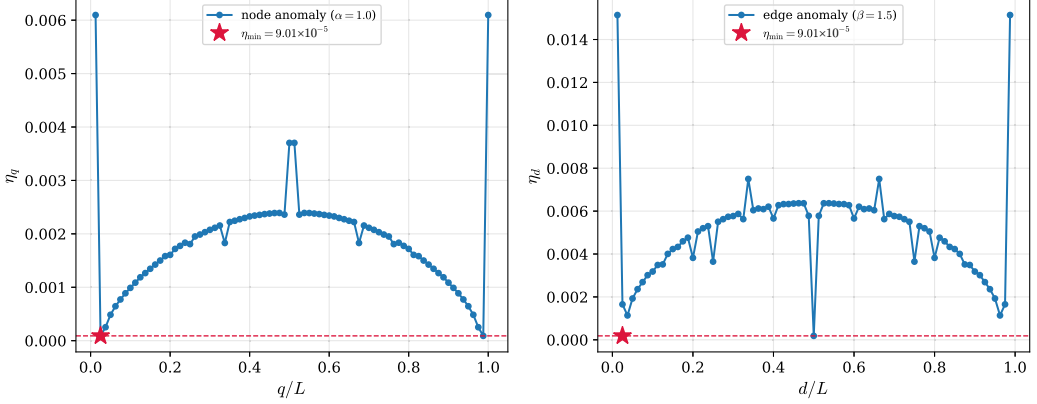


FIG. 4. Structural anomaly signal $\eta_s = |\Pi_{A \rightarrow B}(H_s) - \Pi_{A \rightarrow B}(H_g)|$ as a function of the normalized defect position q/L for $L = 80$. The left panel is a node anomaly with $\alpha = 1.0$, while the right panel is an edge anomaly with $\beta = 1.5$. The red star and dashed line indicate the worst-case signal $\eta_{\min} = 9.01 \times 10^{-5}$.

Theorem 2 (Detectability criteria). Choose the sampling time T_g such that $T_g \geq \frac{8}{\eta_s} (\frac{1}{\Delta_s} + \frac{1}{\Delta_g})$. If $M \geq \frac{8}{\eta_s^2} \ln(\frac{2}{\delta})$ samples, then, with probability at least $1 - \delta$, the decision rule

$$|\hat{P}_{A \rightarrow B} - \bar{P}_{A \rightarrow B}^{H_g}(T_g)| \geq \frac{\eta_s}{2} \implies \text{declare anomaly} \quad (39)$$

detects the anomaly.

Proof. Since $T_g \geq \frac{8}{\eta_s} (\frac{1}{\Delta_s} + \frac{1}{\Delta_g})$, we get

$$\frac{2}{\Delta_s T_g} + \frac{2}{\Delta_g T_g} = \frac{2}{T_g} \left(\frac{1}{\Delta_s} + \frac{1}{\Delta_g} \right) \leq \frac{\eta_s}{4}. \quad (40)$$

Combined with Eq. (37), this provides the signal bound:

$$|\bar{P}_{A \rightarrow B}^{H_s}(T_g) - \bar{P}_{A \rightarrow B}^{H_g}(T_g)| \geq \frac{3\eta_s}{4}. \quad (41)$$

Suppose the empirical estimator $\hat{P}_{A \rightarrow B} = \frac{1}{M} \sum_{r=1}^M X_r$ is obtained from M independent randomized-time samples, where each $X_r \in \{0, 1\}$ records detection or no detection at B . Hoeffding's inequality gives

$$\Pr \left[|\hat{P}_{A \rightarrow B} - \bar{P}_{A \rightarrow B}^{H_s}(T_g)| \geq \frac{\eta_s}{4} \right] \leq 2 \exp \left(-\frac{M\eta_s^2}{8} \right). \quad (42)$$

Choosing $M \geq \frac{8}{\eta_s^2} \ln(\frac{2}{\delta})$ makes the right-hand side at most δ . Therefore, with probability at least $1 - \delta$,

$$|\hat{P}_{A \rightarrow B} - \bar{P}_{A \rightarrow B}^{H_g}(T_g)| \geq |\bar{P}_{A \rightarrow B}^{H_s}(T_g) - \bar{P}_{A \rightarrow B}^{H_g}(T_g)| - |\hat{P}_{A \rightarrow B} - \bar{P}_{A \rightarrow B}^{H_s}(T_g)| \geq \frac{3\eta_s}{4} - \frac{\eta_s}{4} = \frac{\eta_s}{2}. \quad (43)$$

Hence, the decision rule in Eq. (39) detects the anomaly with probability at least $1 - \delta$. Moreover, if $\eta_s \geq c/L^k$ and Δ_g, Δ_s are lower bounded by inverse polynomials in L , then the sampling requirement is also polynomial in time, $M = O(L^{2k} \ln(1/\delta))$. ■

The protocol for anomaly detection is outlined in Fig. 5. In practice, one chooses T_g using the worst-case inverse-polynomial bounds derived for the admissible anomaly class. Let us evaluate η_s for some special cases. In the extreme case, an edge gets totally broken, $\tau = 0$, and the Hamiltonian decomposes into two disconnected chains, $H_d(0) = H_{\text{left}} \oplus H_{\text{right}}$. There is no transition amplitude

Algorithm: Unified anomaly detection on a finite one-dimensional chain

Require: Honest Hamiltonian H_g , unknown Hamiltonian H , endpoints $A = 1$ and $B = L$, sampling time T_g , number of trials M , threshold ϵ .

Ensure: Decision: HONEST or ANOMALOUS.

```

1: Node  $A = 1$  prepares the walker in the state  $|A\rangle$ .
2: Choose  $T_g$  such that  $T_g \geq \frac{8}{\eta_s} \left( \frac{1}{\Delta_s} + \frac{1}{\Delta_g} \right)$ .
3: for  $r = 1, \dots, M$  do
4:   Sample  $t_r \sim \text{Unif}[0, T_g]$ .
5:   Evolve  $|\psi(t_r)\rangle = e^{-iH t_r} |A\rangle$ .
6:   Measure with  $\{|B\rangle\langle B|, \mathbb{I} - |B\rangle\langle B|\}$ .
7:   Set  $X_r = 1$  if the walker is detected at  $B$ , and set  $X_r = 0$  otherwise.
8: end for
9: Compute  $\hat{P}_{A \rightarrow B} = \frac{1}{M} \sum_{r=1}^M X_r$ .
10: Compute  $\tilde{P}_{A \rightarrow B}^{H_g}(T_g) = \frac{1}{T_g} \int_0^{T_g} |\langle B | e^{-iH_g t} | A \rangle|^2 dt$ .
11: if  $|\hat{P}_{A \rightarrow B} - \tilde{P}_{A \rightarrow B}^{H_g}(T_g)| \geq \epsilon$  then
12:   return ANOMALOUS
13: else
14:   return HONEST
15: end if
    
```

Threshold choice: In Theorem 5, one may take $\epsilon = \eta_s/2$.

FIG. 5. Protocol for detecting anomalous behavior on a finite one-dimensional quantum network via randomized-time empirical arrival probabilities.

between the two end points, $\Pi_{A \rightarrow B}(H_d(0)) = 0$. In this case, the structural difference is simply

$$\eta_s = |\Pi_{A \rightarrow B}(H_d(0)) - \Pi_{A \rightarrow B}(H_g)| = \frac{3}{2(2N+1)} = \Theta\left(\frac{1}{L}\right).$$

When the anomaly is exactly at the center, the standard first-order perturbation calculation shows that the anomaly shifts the allowed momenta by

$$p \longrightarrow p - \frac{\beta p}{(2 + \beta)N} + O(N^{-2}),$$

which gives the structural difference as $\eta_s = \Pi_{A \rightarrow B}(H_d) - \Pi_{A \rightarrow B}(H_g) = \Theta(\beta/N)$.

The value of η_s depends on the location of the anomaly. To quantify the effect of anomaly location, we numerically compute η_s as a function of the position, as shown in Fig. 4. Since the sample complexity scales as $M = O(\frac{1}{\eta_s^2})$, larger values of η_s correspond to anomalies that are easier to detect, requiring fewer samples to distinguish from the honest network. Conversely, smaller values of η_s indicate weaker perturbations of the transfer probability and therefore lead to a larger sample complexity. The position sweep thus identifies both the most and least detectable defect locations, with the worst-case performance governed by the minimum signal $\eta_{\min} = \min_s \eta_s$.

D. Extension to multiple anomalies

So far, we have only considered a single anomalous node or a single anomalous edge. In realistic quantum networks, however, several nodes or edges may deviate from their prescribed values simultaneously. The honest Hamiltonian then modifies as follows.

Let $S_q \subset \{2, \dots, L-1\}$ denote a set of anomalous internal nodes, with $|S_q| = m$. The corresponding multinode anomalous Hamiltonian is

$$H_{S_q} = H_g + \sum_{q \in S_q} \alpha_q |q\rangle\langle q|, \quad (44)$$

where α_q is the on-site anomaly strength. Similarly, for a set $S_d \subset \{1, \dots, L-1\}$ of anomalous edges, with $|S_d| = m$, we define

$$H_{S_d} = H_g + \sum_{d \in S_d} \beta_d (|d\rangle\langle d+1| + |d+1\rangle\langle d|), \quad (45)$$

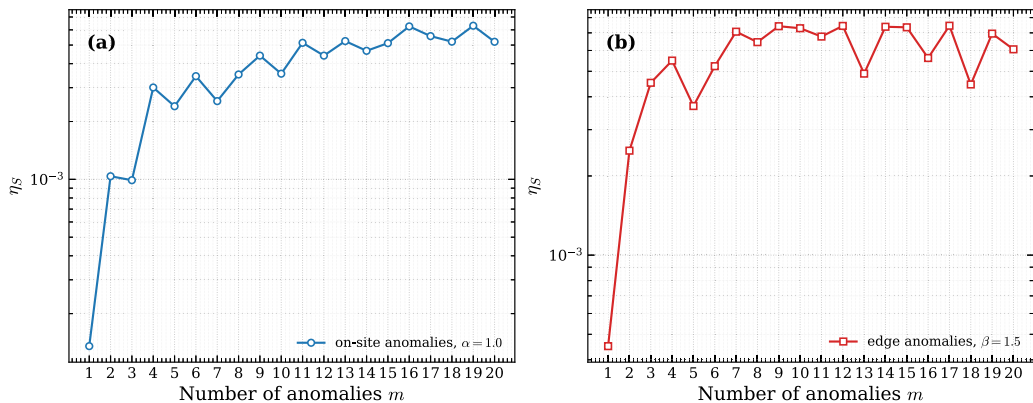


FIG. 6. For each value of m , a set S of m locations is chosen randomly once, and the corresponding structural signal $\eta_S = |\Pi_{A \rightarrow B}(H_S) - \Pi_{A \rightarrow B}(H_g)|$ is plotted. The left panel belongs to on-site anomalies, while the right panel belongs to edge anomalies.

where β_d modifies the hopping amplitude on each anomalous edge. For a given defect set S , we define the structural anomaly signal as

$$\eta_S = |\Pi_{A \rightarrow B}(H_S) - \Pi_{A \rightarrow B}(H_g)|, \quad (46)$$

where H_S denotes either H_{S_q} or H_{S_d} . η_S measures how strongly the simultaneous presence of several local anomalies changes the asymptotic transfer probability between the trusted end points.

To illustrate this, we consider randomly placed sets of local anomalies of different size m , compute η_S , and plot in Fig. 6. As m increases, η_S increases, making anomalies easier to detect. However, the growth of η_S is not monotonous. This is because different anomalies can interfere constructively or destructively with each other in their effects on the end-point transfer probability. Adding an additional defect can enhance the structural signal, but it can also partially cancel the effect of existing defects. Therefore, the multiple-anomaly case is significantly richer than the single-anomaly case. We leave a systematic analysis of this worst-case multidefect problem for future work.

VI. EXTENSION TO GENERAL GRAPHS

In Sec. V, we relied on the special symmetries of the one-dimensional line, such as translation invariance and analytically tractable eigenvalue formulas, to obtain explicit expressions for transfer probabilities. Realistic quantum networks, however, may have arbitrary topology. Even though closed-form spectral decompositions or scattering analyses are no longer available, the detection protocol of Sec. IV and the spectral criterion of Theorem 1 remain fully valid on *all* connected graphs. This follows from the fact that the time-averaged arrival probability depends only on the spectral projectors of the Hamiltonian, which exist for any Hermitian operator.

Let $G = (V, E)$ be a connected graph with $|V| = N$. The trusted nodes $A, B \in V$ are fixed. The honest Hamiltonian is taken to be a weighted Laplacian

$$H_g = \sum_{(u,v) \in E} w_{uv} |u\rangle\langle v| + \sum_{u \in V} D_u |u\rangle\langle u|, \quad (47)$$

with symmetric weights $w_{uv} = w_{vu} > 0$ and on-site potentials D_u . A anomalous node or corrupted edge introduces a localized perturbation

$$H_d = H_g + H_{\text{anomalous}}, \quad H_{\text{anomalous}} = \sum_{(u,v) \in S} \delta w_{uv} |u\rangle\langle v| + \sum_{u \in S} \delta D_u |u\rangle\langle u|, \quad (48)$$

where $S \subseteq V \cup E$ is the anomalous region and $\|H_{\text{anomalous}}\| = O(1)$.

Diagonalizing H_g explicitly is not feasible in general. Nevertheless, standard first-order perturbation theory yields the deformation of eigenstates:

$$\delta|\lambda_k\rangle = \sum_{\ell \neq k} \frac{\langle \lambda_\ell | H_{\text{anomalous}} | \lambda_k \rangle}{\lambda_k - \lambda_\ell} |\lambda_\ell\rangle. \quad (49)$$

A. A detectability theorem for delocalized graph families

First, let us formalize the notion of an eigenvector delocalization relevant for transport between A and B .

Definition 9 (Polynomial delocalization). We say that H_g has *polynomially delocalized eigenvectors* with respect to A and B if there exists a constant $c > 0$ such that

$$|\langle A | \lambda_k \rangle|^2, |\langle B | \lambda_k \rangle|^2 \geq \frac{c}{N}, \quad \text{for at least } \Theta(N) \text{ eigenmodes } k.$$

This condition holds for many important graph families: d -dimensional grids, random regular graphs, and expander networks, among others. Under this assumption, local perturbations produce a macroscopic change in the asymptotic transfer probability.

Suppose that H_g has spectral gap $\Delta_g \geq N^{-\alpha}$ for some constant $\alpha > 0$, and that its eigenvectors are polynomially delocalized with respect to A and B in the sense of Definition 9. A perturbation of strength $\|H_{\text{anomalous}}\| = O(1)$ produces a collective shift

$$\Pi_{A \rightarrow B}(H_d) - \Pi_{A \rightarrow B}(H_g) = \Theta\left(\frac{\|H_{\text{anomalous}}\|}{N}\right). \quad (50)$$

Consequently, if the sampling time satisfies $T_g \gg 1/\Delta_g$, then

$$|\bar{P}_{A \rightarrow B}^{H_d}(T_g) - \bar{P}_{A \rightarrow B}^{H_g}(T_g)| \geq \Theta\left(\frac{1}{N}\right) - O\left(\frac{1}{\Delta_g T_g}\right), \quad (51)$$

and choosing $T_g = \text{poly}(N)$ ensures that the finite-time correction is negligible.

This result subsumes the one-dimensional case and shows that *any* graph with sufficient spectral gap and delocalization exhibits polynomially detectable anomalies.

B. Conditions on arbitrary networks

The preceding analysis shows that detectability on general graphs hinges on two structural properties: a polynomial spectral gap and delocalized eigenvectors. These conditions fail for many natural graph families.

First, graphs such as lollipop and barbell graphs satisfy $\Delta_g = \exp(-\Theta(N))$, implying that the required sampling time $T_g \geq 1/\Delta_g$ is exponential in N . Second, many graphs exhibit strong eigenvector localization. In such cases,

$$\Pi_{A \rightarrow B}(H_g) = \sum_k |\langle A | \lambda_k \rangle|^2 |\langle B | \lambda_k \rangle|^2$$

may decay exponentially in N , even in the absence of anomalies. Then, the shift $\Pi_{A \rightarrow B}(H_d) - \Pi_{A \rightarrow B}(H_g)$ becomes dominated by finite-time oscillations, making any efficient detection protocol impossible. Third, an anomalous node placed in a region that couples weakly to the transport modes may influence at most $O(1)$ eigenvectors, resulting in a perturbation that scales as $o(1/N)$ and becomes undetectable within polynomial time.

The discussion above shows that efficient anomaly detection cannot be guaranteed on general networks without imposing additional assumptions. A sufficient set of conditions is as follows:

- (1) a polynomial spectral gap for the honest Hamiltonian,

$$\Delta_g \geq 1/\text{poly}(|V|),$$

- (2) delocalized eigenvectors with respect to A and B , ensuring

$$\Pi_{A \rightarrow B}(H_g) \geq 1/\text{poly}(|V|),$$

- (3) and an anomaly that perturbs a polynomial fraction of the transport modes.

The conditions described above are expected to hold for graph families with strong transport properties. The one-dimensional network studied in Sec. V is the simplest example. Higher-dimensional grids are a natural generalization of the one-dimensional case. These graphs usually do not have constant spectral gaps, but their gaps shrink inverse polynomially as the network size increases. This makes them good candidates for polynomial-time anomaly detection. Expander graphs and random regular graphs are also strong candidates, as they rarely have narrow bottlenecks. This also suggests that the graph families with bottlenecks may not work well. This discussion hints at a possible approach to handling an arbitrary graph network: decompose the graph into smaller subgraphs on which the quantum-walk test can be analyzed directly. In this sense, the one-dimensional analysis provides a building block for more complicated network topologies.

C. Subgraph decomposition strategy

We will now present a decomposition strategy for handling a complex quantum network. Let $G = (V, E)$ be the network graph with a collection of connected subgraphs $\mathcal{C} = \{G_r = (V_r, E_r)\}_{r=1}^R$ such that each G_r contains a pair of trusted vertices A_r, B_r , and the anomalous region is contained in at least one member of the cover. The subgraphs are chosen so that each can efficiently detect anomalies. Pathlike subgraphs provide a natural class of efficiently analyzable components.

For each subgraph G_r , define the projector $P_r = \sum_{v \in V_r} |v\rangle\langle v|$. The corresponding restricted honest Hamiltonian is $H_g^{(r)} = P_r H_g P_r$. Similarly, if an anomaly is present, the restricted anomalous Hamiltonian is $H_d^{(r)} = P_r H_d P_r$. On the r th subgraph, the structural signal is $\eta_r = |\Pi_{A_r \rightarrow B_r}(H_d^{(r)}) - \Pi_{A_r \rightarrow B_r}(H_g^{(r)})|$. By the same argument as in Theorem 1, the finite-time signals obey

$$|\bar{P}_{A_r \rightarrow B_r}^{H_d^{(r)}}(T_r) - \bar{P}_{A_r \rightarrow B_r}^{H_g^{(r)}}(T_r)| \geq \eta_r - \left(\frac{2}{\Delta(H_d^{(r)})T_r} + \frac{2}{\Delta(H_g^{(r)})T_r} \right). \quad (52)$$

Therefore, the r th subgraph test is efficient whenever $\Delta(H_g^{(r)}), \Delta(H_d^{(r)}) \geq \frac{1}{\text{poly}(|V_r|)}$ and $\eta_r \geq \frac{1}{\text{poly}(|V_r|)}$.

The full graph is declared anomalous if any subgraph test declares an anomaly. If the number of tested subgraphs is polynomial in N , $R = \text{poly}(N)$, then the statistical error can be controlled by a union bound. To ensure total failure probability at most δ , it is sufficient to run each subgraph test with failure probability at most δ/R . If M_r denotes the number of randomized-time trials on subgraph G_r , Hoeffding's inequality gives the scaling

$$M_r = O\left(\frac{1}{\eta_r^2} \ln \frac{R}{\delta}\right). \quad (53)$$

Thus, if $R = \text{poly}(N)$ and η_r is polynomial, the total sample complexity remains polynomial.

VII. PERFORMANCE ANALYSIS

Previous research on anomalous node detection primarily focuses on verifying entanglement between nodes, particularly in the context of device-independent Quantum Key Distribution (QKD) protocols [15–17]. In contrast, our approach exploits other quantum phenomena, namely, superposition and interference, making our proposal novel in the field. The performance of network protocols is commonly measured through time or memory complexity, expressed using Big O notation, which characterizes the upper bound of an algorithm's growth rate with respect to the input size. We find that while we achieve efficient time complexity, the same cannot be said for memory complexity.

For a quantum-network graph, $G = (V, E)$, we will characterize time and memory complexity measures in terms of the graph's diameter $d(G)$, defined as the minimum number of edges required to traverse between any two nodes in the graph or, equivalently, the length of the shortest path between nodes i and j . For a one-dimensional network, this corresponds to the number of nodes N between the trusted nodes A and B .

A. Time and resource complexity

We demand that all test Hamiltonians exhibit a sampling time at most polynomial in the network's diameter $d(G)$. Specifically, for a one-dimensional network's test protocol, we showed that sampling time scales explicitly as N^4 , where N represents the number of nodes. Therefore, the protocol's time complexity is polynomial and efficient.

Compared with classical walk protocols, the quantum-walk protocol has greater memory and resource requirements. This is because quantum devices and memories at each node must remain active throughout the protocol to maintain a global quantum state, enabling quantum effects such as superposition and interference. Therefore, the memory complexity of the quantum-walk protocol is of order $O(N)$, where N is the number of nodes in the network. This poses a significant challenge due to the inherent difficulty in maintaining quantum memories [32].

B. Comparison with classical baselines

We now compare with the classical approach, where the anomaly is detected via classical signal strength or classical random walk test on the same graph. One potential advantage is that, even in the honest network, quantum walks can exhibit a quadratic improvement in hitting time over their classical counterparts for suitable graph families. As the classical hitting-time distribution changes in the presence of anomalies, we do not claim that the quantum-walk test universally outperforms classical transport diagnostics for all anomaly models. However, a crucial point to note is that such a classical approach may not even be accessible in a fully quantum network running distributed quantum computing or QKD with no classical signal to measure. Furthermore, quantum walks probe coherent interference, producing much richer signatures than classical transport observables.

Another natural baseline is an end-to-end fidelity test, which can be highly effective when the full output state is accessible. However, such a test requires substantially stronger measurement capabilities than the binary end-point detection assumed in our protocol. Similarly, local process tomography could be used to characterize a suspected region of the network, but its resource cost grows rapidly with the size of that region. This makes tomography unsuitable as a scalable screening procedure under the polynomial-resource assumptions adopted here.

VIII. CONCLUSION

We developed a continuous-time quantum-walk protocol for detecting anomalous nodes and edges in a quantum network. The core idea exploits the sensitivity of quantum interference patterns to local perturbations of the network Hamiltonian: The presence of an anomalous node or edge shifts the time-averaged arrival probability between trusted end points A and B by a detectable amount η_s , which serves as the detection signal. We formalized this idea through a spectral criterion and showed that, for one-dimensional networks, both on-site and edge anomalies can be detected with polynomial sampling time and polynomial repetition complexity whenever η_s is inverse polynomial in the system size. We also discussed extensions to general graphs satisfying suitable spectral-gap and eigenvector-delocalization conditions.

Several directions remain open. First, one should analyze resistance attacks in which anomalous nodes try to deceive the test, for example, by intercepting the walker and forwarding a replacement state. Since the quantum-walk distribution depends sensitively on the initial state chosen by (A) , randomizing over initial states, phases, or test Hamiltonians may provide countermeasures. Second, the implementation of the required Hamiltonians in realistic quantum-network hardware needs

further study. Platforms such as spin networks and integrated photonic circuits can realize related quantum-walk dynamics [28–30], but noise, loss, and hardware constraints must be incorporated carefully [33]. Third, it is important to extend the analysis beyond one-dimensional chains to arbitrary graph networks. Hierarchical graphs and Krylov-subspace methods provide possible routes for mapping more general dynamics to effective one-dimensional walks [26,34–36]. Finally, future work could compare the single-walker protocol studied here with tests based on coherent states, squeezed states, or multiphoton inputs.

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX A: SCATTERING ANALYSIS OF ANOMALOUS NODES AS A DISORDER

The test Hamiltonian H_g is

$$H_g = \sum_j -2|j\rangle\langle j| + |j\rangle\langle j-1| + |j\rangle\langle j+1|, \quad (\text{A1})$$

and the anomalous Hamiltonian H_d is modeled by

$$H_d = \left(\sum_j -2|j\rangle\langle j| + |j\rangle\langle j-1| + |j\rangle\langle j+1| \right) + \alpha_d |d\rangle\langle d|. \quad (\text{A2})$$

The eigenstates of H_g are the momentum eigenstates $|p\rangle$ with eigenvalues $2 \cos p$, i.e., $H_g|p\rangle = 2 \cos p|p\rangle$, and the components are $\langle j|p\rangle = \frac{1}{\sqrt{2\pi}} e^{ikj}$, $-\pi \leq p \leq \pi$. A momentum eigenstate p , coming from the node A , incident on anomalous node d has the scattering property

$$|\psi_s(p)\rangle = \begin{cases} |p\rangle + r(p)|-p\rangle, & \text{for } j \leq d, \\ t(p)|p\rangle, & \text{for } j \geq d+1, \end{cases}$$

where $t(p)$ and $r(p)$ are the transmission and reflection coefficients, respectively.

Outside the position $j = d$, i.e., outside the anomalous node region, we have $\langle j|H_d|\psi_s(p)\rangle = \langle j|H_g|\psi_s(p)\rangle$ as $\langle j|\alpha_d|d\rangle\langle d|d\rangle = 0$ and

$$\begin{aligned} \langle j|H_d|\psi_s(p)\rangle &= \langle j|H_g|\psi_s(p)\rangle = -2\langle j|p\rangle + \langle j-1|p\rangle + \langle j+1|p\rangle \\ &= \frac{1}{\sqrt{2\pi}} (-2e^{ipj} + e^{ip(j-1)} + e^{ip(j+1)}) = \frac{e^{ipj}}{\sqrt{2\pi}} (-2 + e^{-ip} + e^{ip}) \\ &= \frac{e^{ipj}}{\sqrt{2\pi}} (-2 + 2 \cos p) = 2\langle j|p\rangle(\cos p - 1) = 2\langle j|\psi_s(p)\rangle(\cos p - 1). \end{aligned} \quad (\text{A3})$$

At the anomalous node $j = d$, we still expect the following two conditions to hold:

$$1 + r = t, \quad (\text{A4})$$

$$\frac{\langle j|H_d|\psi_s(p)\rangle}{\langle j|\psi_s(p)\rangle} = 2(\cos p - 1). \quad (\text{A5})$$

Condition (A4) is a conservation of probability, while condition (A5) is a quantization criterion.

Without loss of generality, let us assume the position of the anomalous node at $d = 0$. We evaluate $\langle j = 0 | \psi_k \rangle$ as

$$\begin{aligned} \langle j = 0 | \psi_k \rangle &= \langle 0 | p \rangle + r \langle 0 | -p \rangle = \frac{1}{\sqrt{2\pi}} (e^{ip(j=0)} + r e^{-ik(j=0)}) = \frac{1}{\sqrt{2\pi}} (1 + r) \\ &= \frac{t}{\sqrt{2\pi}}, \quad \text{using condition (A4)}. \end{aligned} \quad (\text{A6})$$

Now, let us evaluate $\langle j | H_d | \psi_s(p) \rangle$ at the anomalous node position $j = d = 0$:

$$\begin{aligned} \langle j | H_d | \psi_s(p) \rangle|_{j=d=0} &= \langle j + 1 | p \rangle + \langle j - 1 | p \rangle - 2 \langle j | p \rangle + \alpha_d \langle j | | p \rangle \\ &= \frac{1}{\sqrt{2\pi}} (t e^{ip(j+1)} + e^{ip(j-1)} + r e^{-ip(j-1)} - 2 e^{ipj} - 2 r e^{-ipj} + \alpha_d e^{ipj} + \alpha_d r e^{-ipj})|_{j=d=0} \\ &= \frac{1}{\sqrt{2\pi}} (t e^{ip} + e^{-ip} + r e^{ip} - 2 - 2r + \alpha_d + \alpha_d r) \\ &= \frac{1}{\sqrt{2\pi}} (2 e^{ip} - 2i \sin p - 2t + \alpha_d t), \quad \text{using condition (A4)}. \end{aligned} \quad (\text{A7})$$

Using conditions (A5) and (A7), we have

$$\frac{\langle j | H_d | \psi_s(p) \rangle}{\langle j | \psi_s(p) \rangle} = 2(\cos p - 1), \quad \frac{2e^{ik} - 2i \sin p - 2t + \alpha_d t}{t} = 2 \cos p - 2. \quad (\text{A8})$$

Solving for t gives

$$t = \frac{2i \sin p}{\alpha_d + 2i \sin p}, \quad |t(\alpha_d)|^2 = \frac{4 \sin^2 p}{\alpha_d^2 + 4 \sin^2 p}, \quad (\text{A9})$$

where $|t|^2(\alpha_d)$ is the transmission probability in the presence of an anomalous node at position $j = d$ with anomalous strength α_d .

APPENDIX B: SPECTRAL-GAP CALCULATIONS FOR ANOMALOUS EDGE

The honest test Hamiltonian is the translation-invariant adjacency operator

$$H_g = \sum_j (|j\rangle \langle j-1| + |j\rangle \langle j+1|), \quad (\text{B1})$$

while the anomalous Hamiltonian is

$$H_d = H_g + \kappa_d |d+1\rangle \langle d| + \beta_{d+1} |d\rangle \langle d+1|. \quad (\text{B2})$$

In the following lemma, we now show the behavior of the smallest gap in the spectrum of H_g and H_d when an anomalous node is placed exactly at the center of the chain. Due to the reflection symmetry, a closed-form quantization condition is available.

Lemma B1 (Centered anomalous edge). Let the spectrum of the test Hamiltonian H_g be expressed as $H_g = \sum_{\lambda_g} \lambda_g |\lambda_g\rangle \langle \lambda_g|$, and the Hamiltonian with anomalous nodes H_d as $H_d = \sum_{\lambda_d} \lambda_d |\lambda_d\rangle \langle \lambda_d|$. Suppose the anomalous edge is placed exactly at the center. Then, the smallest gap in the spectrum of H_g and H_d denoted as Δ_g and Δ_d , respectively, satisfies the following conditions:

$$\Delta_g \geq \frac{\pi^2}{((N+2)/2)^3}, \quad \Delta_d \geq \frac{2\pi^2}{(2 + \beta_d)((N+2)/2)^3}, \quad (\text{B3})$$

where $N + 2$ is the number of nodes in a one-dimensional quantum network.

Proof. We start the analysis for the anomalous Hamiltonian H_d . The eigenstates of the Hamiltonian H_d are the momentum eigenstates $|p\rangle$ with eigenvalues $2 \cos p$, i.e., $H_d|p\rangle = 2 \cos p|p\rangle$ and the components are

$$\langle j|p\rangle = \frac{1}{\sqrt{2\pi}} e^{ipj}, \quad -\pi \leq p \leq \pi. \quad (\text{B4})$$

Thus, we can use the following conditions:

$$\begin{aligned} \langle j = d|H_d|\lambda_d\rangle &= 2 \cos p \langle j = d|\lambda_d\rangle, \\ \langle d|H_d|\lambda_d\rangle &= 2 \cos p \langle d|\lambda_d\rangle, \\ (1 + \beta)\langle d + 1|\lambda_d\rangle + \langle d - 1|\lambda_d\rangle &= 2 \cos p \langle d|\lambda_d\rangle. \end{aligned} \quad (\text{B5})$$

We now index the $N + 2$ nodes in the network as $j = 1$ to $j = 2n$ and assume that the anomalous nodes are at the center, i.e., at $j = n$ and $j = n + 1$ for analytical simplicity. Defects at other positions yield the same scaling laws but lead to more cumbersome boundary matching conditions.

We define a reflection operator R that acts as $R|j\rangle = |2n + 1 - j\rangle$. $R^2 = 1$, so R has eigenvalues ± 1 . $[R, H] = 0$, so R and H can be simultaneously diagonalized:

$$\langle j|\lambda_d\rangle = \begin{cases} \sin pj, & 1 \leq j \leq n, \\ \pm \sin(p(2n + 1 - j)), & n + 1 \leq j \leq 2n. \end{cases} \quad (\text{B6})$$

Now, we can continue Eq. (B5) as

$$\pm(1 + \beta) \sin np + \sin(n - 1)p = 2 \cos p \sin np, \quad (\text{B7})$$

which gives us the following quantization criteria:

$$\frac{\sin((n + 1)p)}{\sin(np)} = \pm(1 + \beta). \quad (\text{B8})$$

The roots of Eq. (B8) for different values of β are shown in Fig. 7, and the corresponding function is shown in Fig. 8. For a fixed β , it intersects with $-(1 + \beta)$ at the left of $\frac{\pi l}{n}$, i.e., zeros of $\sin np$, for $l = 1, 2, \dots, n - 1$. Suppose it intersects at $p = \frac{\pi l}{n} - \delta$,

$$-(1 + \beta) \sin(np) = \sin\left(n\delta - \frac{l\pi}{n} + \delta\right). \quad (\text{B9})$$

Expanding δ as $\delta = (c/n) + (d/n^2) + O(1/n^3)$ and taking $n \rightarrow \infty$ gives

$$d = \frac{l\pi}{2 + \beta}. \quad (\text{B10})$$

Thus, for fixed β , the roots of Eq. (B8) are of the form

$$p = \frac{l\pi}{n} - \frac{l\pi}{(2 + \beta)n^2} + O(1/n^3). \quad (\text{B11})$$

Now, we want to find the smallest spacing of p . For this, let p' and p'' be the two roots closest to p such that $p' < p < p''$, as illustrated in Fig. 9. From Figs. 8 and 9, we observe

$$p' < \frac{l\pi}{n} - \frac{l\pi}{n^2} + O(1/n^3), \quad p'' > \frac{l\pi}{n}. \quad (\text{B12})$$

This gives the smallest spacing of p to be of order $\Delta p = \frac{\pi}{(2 + \beta)n^2}$. In Fig. 7, we plot the roots of Eq. (B8) for $n = 5$ and different values of β . In the same figure, we show that the smallest spacing of p is greater than the bound $\frac{\pi}{(2 + \beta)n^2}$. For a given p , the corresponding eigenvalue is $2 \cos p$. For small

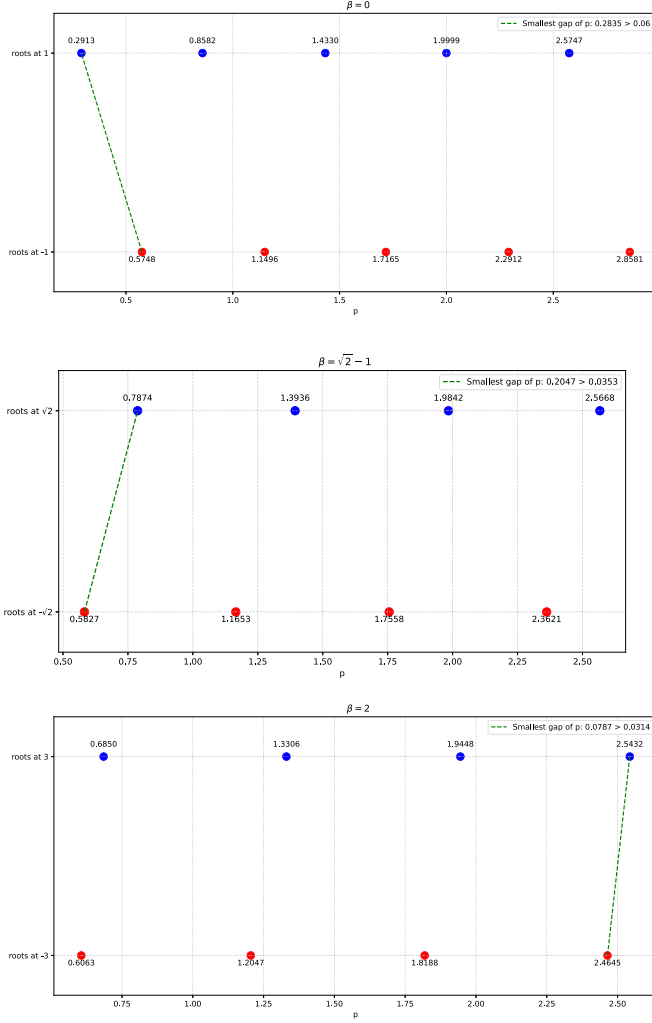


FIG. 7. Roots of Eq. (B8) for $n = 5$ and various values of β . The minimum spacing of the roots is greater than the bound $\frac{\pi}{(2+\beta_d)n^2}$.

Δp , the spacing Δ_d is related by $\Delta_d > 2\Delta p \sin p$. The factor $\sin p$ is smallest for $\sin p = \sin \pi/n$. Using $\Delta p = \frac{\pi}{(2+\beta_d)n^2}$, we thus get the expression for Δ_d as

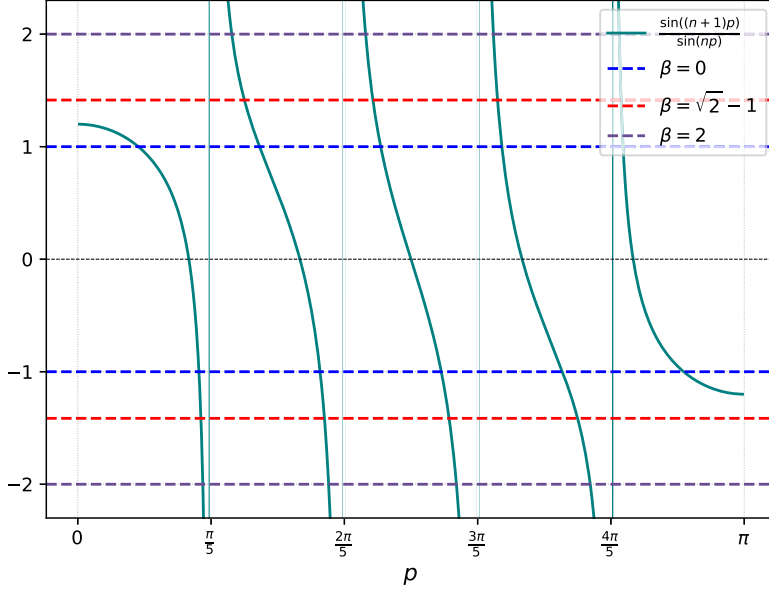
$$\Delta_d > \frac{2\pi^2}{(2+\beta)n^3} = \frac{2\pi^2}{(2+\beta)((N+2)/2)^3}. \quad (\text{B13})$$

Substituting $\beta = 0$ gives

$$\Delta_g > \frac{\pi^2}{((N+2)/2)^3}. \quad (\text{B14})$$

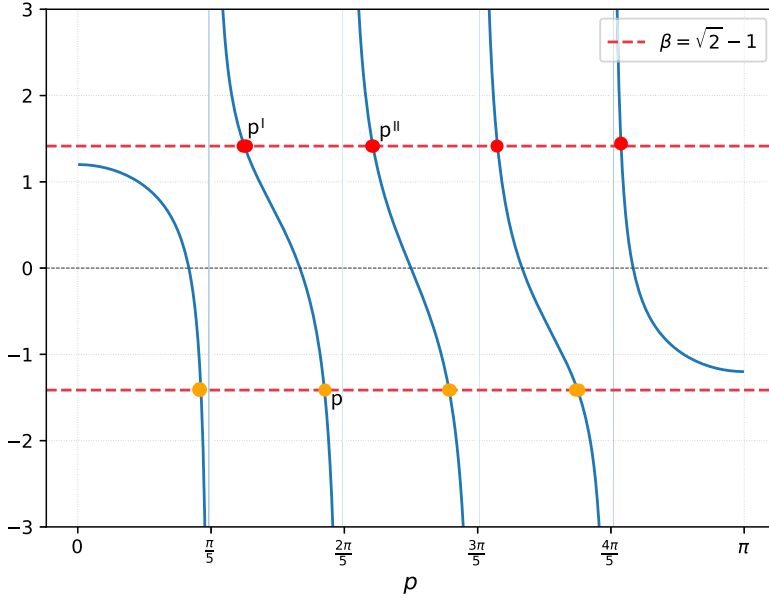
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Lemma B2 (Spectral-gap bound for a finite chain with an arbitrary edge defect). Let $L = N + 2$ be the total number of sites in the finite one-dimensional network, with end points $A = 1$ and


 FIG. 8. $\frac{\sin((n+1)p)}{\sin(np)}$ for $n = 5$.

$B = L$. Consider the honest finite-chain Hamiltonian

$$H_g = \sum_{j=1}^{L-1} (|j\rangle\langle j+1| + |j+1\rangle\langle j|). \quad (\text{B15})$$


 FIG. 9. Nearest roots p' and p'' of p for Eq. (B8) with $n = 5$ and $\beta = \sqrt{2} - 1$.

Let H_d be obtained from H_g by modifying a single edge between sites d and $d + 1$, $1 \leq d \leq L - 1$, so that

$$H_d = H_g + \beta_d(|d\rangle\langle d+1| + |d+1\rangle\langle d|), \quad \tau_d := 1 + \beta_d > 0. \quad (\text{B16})$$

Let $\Delta_g = \min_{\lambda_g \neq \lambda'_g} |\lambda_g - \lambda'_g|$ and $\Delta_d = \min_{\lambda_d \neq \lambda'_d} |\lambda_d - \lambda'_d|$ denote the smallest nonzero spectral gaps of H_g and H_d , respectively. For fixed defect strength τ_d , bounded away from zero and infinity, there exist constants $c_g > 0$ and $c_d(\tau_d) > 0$, independent of L and of the defect position d , such that

$$\Delta_g \geq \frac{c_g}{L^2}, \quad \Delta_d \geq \frac{c_d(\tau_d)}{L^3}. \quad (\text{B17})$$

In particular, both spectral gaps are inverse polynomial in the system size. Therefore the finite-time correction in the time-averaged arrival probability can be made polynomially small by choosing $T = \text{poly}(L)$.

Proof. We first treat the honest Hamiltonian. Since H_g is the adjacency matrix of the finite path graph P_L , its normalized eigenvectors are standing waves,

$$\psi_m(j) = \sqrt{\frac{2}{L+1}} \sin(p_m j), \quad p_m = \frac{m\pi}{L+1}, \quad m = 1, \dots, L, \quad (\text{B18})$$

with eigenvalues $\lambda_m = 2 \cos p_m$. Thus,

$$|\lambda_{m+1} - \lambda_m| = 2|\cos p_{m+1} - \cos p_m| = 4 \sin\left(\frac{p_{m+1} + p_m}{2}\right) \sin\left(\frac{p_{m+1} - p_m}{2}\right). \quad (\text{B19})$$

Since $p_{m+1} - p_m = \pi/(L+1)$, we have $\lambda_m - \lambda_{m+1} = 4 \sin(\frac{(2m+1)\pi}{2(L+1)}) \sin(\frac{\pi}{2(L+1)})$. The smallest spacing occurs at the band edge, for example, at $m = 1$. Therefore, $\Delta_g = 4 \sin(\frac{3\pi}{2(L+1)}) \sin(\frac{\pi}{2(L+1)})$. Using $\sin x = x + O(x^3)$, we find $\Delta_g = \frac{3\pi^2}{(L+1)^2} + O(L^{-4})$. Hence, $\Delta_g = \Theta(L^{-2})$.

We now consider the anomalous Hamiltonian H_d . Away from the anomalous edge, the eigenvalue equation is the same second-order difference equation as in the honest chain,

$$\psi_{j-1} + \psi_{j+1} = \lambda \psi_j, \quad \lambda = 2 \cos p. \quad (\text{B20})$$

Therefore, the eigenfunction is a sine wave on each side of the defect:

$$\psi_j = \begin{cases} A_L \sin(pj), & 1 \leq j \leq d, \\ A_R \sin(p(L+1-j)), & d+1 \leq j \leq L. \end{cases} \quad (\text{B21})$$

The open boundary conditions are already incorporated through $\psi_0 = \psi_{L+1} = 0$. The only nontrivial equations are the eigenvalue equations at the two sites adjacent to the anomalous edge. They are

$$\psi_{d-1} + \tau_d \psi_{d+1} = 2 \cos p \psi_d, \quad \tau_d \psi_d + \psi_{d+2} = 2 \cos p \psi_{d+1}. \quad (\text{B22})$$

Substituting Eq. (B21) into these two equations and using the trigonometric identity $2 \cos p \sin(kp) - \sin((k-1)p) = \sin((k+1)p)$, one obtains

$$A_L \sin((d+1)p) = \tau_d A_R \sin((L-d)p), \quad A_R \sin((L-d+1)p) = \tau_d A_L \sin(dp). \quad (\text{B23})$$

Multiplying these equations gives the finite-chain quantization condition

$$\sin((d+1)p) \sin((L-d+1)p) - \tau_d^2 \sin(dp) \sin((L-d)p) = 0. \quad (\text{B24})$$

This equation replaces the reflection-symmetric quantization condition used for a centered defect. The centered calculation is recovered as the special case in which the anomalous edge is placed at the middle of the chain.

For fixed $\tau_d > 0$, the Hamiltonian H_d is a real symmetric irreducible Jacobi matrix. Hence, its eigenvalues are simple and can be ordered as $\lambda_1 > \lambda_2 > \dots > \lambda_L$. Equivalently, the allowed momenta solving Eq. (B24) may be ordered as $0 < p_1 < p_2 < \dots < p_L < \pi$.

Standard Sturm oscillation implies that a single bounded local defect can shift the roots relative to the honest values $m\pi/(L+1)$, but cannot produce exponentially small level spacings. More precisely, for fixed τ_d , the root spacing obeys a position-uniform inverse-polynomial bound

$$p_{m+1} - p_m \geq \frac{c'_d(\tau_d)}{L^2}, \quad m = 1, \dots, L-1. \quad (\text{B25})$$

Since $\lambda(p) = 2 \cos p$, we have

$$|\lambda(p_{m+1}) - \lambda(p_m)| = 2|\cos p_{m+1} - \cos p_m| \geq 2 \sin(p_m)(p_{m+1} - p_m) + O((p_{m+1} - p_m)^2). \quad (\text{B26})$$

The smallest value of $\sin p_m$ occurs near the band edge and scales as $O(L^{-1})$. Therefore,

$$\Delta_d \geq \frac{c_d(\tau_d)}{L^3}, \quad (\text{B27})$$

for some constant $c_d(\tau_d) > 0$ independent of L and of the defect position d . This proves the claimed inverse-polynomial spectral-gap bound. $\blacksquare$

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