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Practical and efficient verification of entanglement with incomplete measurement settings

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In this work, we present a practical and efficient framework for verifying entangled states when only a tomographically incomplete measurement setting is available—specifically, when access to observables is severely limited. We show how the experimental estimation of a small number of observables can be directly exploited to construct a large family of entanglement witnesses, enabling the efficient identification of entangled states. Moreover, we introduce an optimization approach, formulated as a semidefinite program, that systematically searches for those witnesses best suited to reveal entanglement under the given measurement constraints. We demonstrate the practicality of the approach in a proof-of-principle experiment with photon-polarization qubits, where entanglement is certified using only a fraction of the full measurement data. These results reveal the maximal usefulness of incomplete measurement settings for entanglement verification in realistic scenarios.

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I. INTRODUCTION

Entanglement is a resource that enables quantum information processing, such as computing [1–3], communication [4–7], and metrology [8], and its verification is therefore of both fundamental and practical interest. The verification of entanglement may depend on whether the

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state of interest is identified and thus known in advance. When a state is known in advance after quantum state tomography, deciding whether it is entangled or separable is the well-defined problem of quantum state separability [9,10]. Although the problem is computationally intractable in full generality, as it is NP-hard [11,12], theoretical tools have been developed, such as positive but not completely positive maps [13–15], the realignment criteria [16,17], entropic separability [18], semidefinite programs, etc. [19–21].

Entangled states can be experimentally verified even before quantum states are identified [22]. For example, an experimental violation of Bell inequalities reveals the presence of entanglement [23]. In general, entanglement witnesses (EWs) can detect entanglement of unknown states. Let us observe that for each entangled state there exists an EW that verifies the state, or one can also construct EWs for the purpose [15,24]. In experimental realizations, EWs can be realized as local observables, which then give a recipe for a measurement setting to detect entangled states (see, e.g., Ref. [25]). Proof-of-principle demonstrations of EWs and the verification of entangled states have been presented (see, e.g., Refs. [26–31]), along which EWs are implemented and also exploited for various purposes for estimation.

Here, we approach the verification of entanglement in the opposite direction by asking how large the set of entangled states can be verified by a given measurement setting. Specifically, taking EWs as a tool for detecting entangled states, the question aims to characterize the set of EWs that can be realized by a single measurement setting; then, entangled states detected by those EWs can be verified by this measurement setting. One can expect that more measurement settings would verify a larger set of entangled states (see, e.g., Refs. [32,33]); a maximal measurement setting will realize quantum state tomography.

II. SUMMARY OF RESULTS

In this work, we present a practical, efficient method for verifying entangled states by constructing multiple EWs once a tomographically incomplete measurement setting is provided. Given measurement results that cannot reconstruct a state, we show how to verify entangled states by constructing multiple EWs from the results of that measurement setting. The construction includes both mirrored EWs and numerical optimization with measurement results, with the optimization formulated as a semidefinite program. It is also fairly straightforward to apply the results to multipartite and high-dimensional systems. We present a proof-of-principle demonstration of verifying two-qubit entangled states with polarization qubits.

We stress that our results are practically useful, as constraints on measurement settings often arise in realistic scenarios. Instances may include quantum communication between ground and moving stations, such as a satellite or an aircraft, where the simpler the measurement setup, the more practical it is (see, e.g., Ref. [34]). In fact, the full tomographic measurement setting may not be available. Furthermore, for quantum systems, the Hilbert space dimension increases exponentially according to the number of systems. From a fundamental point of view, our results resolve the measurement settings for entanglement detection. The more measurements are applied, the larger the set of entangled states that can be verified; the construction of multiple EWs from a measurement setting is interpolated with quantum state tomography, applying a maximal setting for state identification.

III. PRELIMINARIES

Local measurements can be expressed in terms of correlators of canonical observables. For qubits, the Pauli observables $\mathbb{I}$, X, Y, and Z form a basis, and thus estimating them provides tomographically complete measurement results (see Fig. 1). For bipartite systems, the collection of all correlators

$$\langle A \otimes B \rangle_{\text{est}} = \text{tr}[A \otimes B \rho] \quad \text{for } A, B \in \{\mathbb{I}, X, Y, Z\} \tag{1}$$

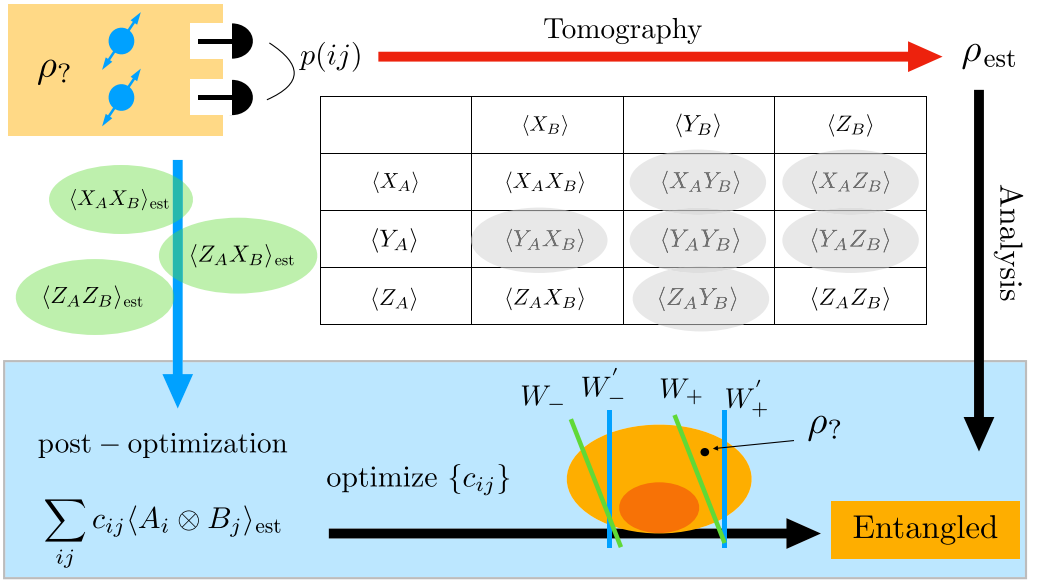


FIG. 1. (Red-colored arrow) For an unknown state, denoted by $\rho_?$, measurement results of all correlators can characterize a state ρ_{est} through state tomography. Theoretical tools such as the partial transpose can be used to determine whether a state ρ_{est} is entangled. (Blue-colored box) In a realistic scenario such as long-distance quantum communication, all experimental data may not be available (gray ellipsoid). Given a fraction of measurement results (green ellipsoid) that cannot reconstruct a state, the present work (blue box) introduces a postoptimization protocol with incomplete measurement results. We collect experimental data, $\langle A_i \otimes B_j\rangle_{\text{est}}$, and consider their linear combination with coefficients $\{c_{ij}\}$, corresponding to an observable S in Eq. (2). Note that coefficients $\{c_{ij}\}$ are optimized such that EWs constructed from experimental data can verify the largest set of entangled states; for instance, one can optimize an EW W_+' to achieve W_+ that may detect an even larger set of entangled states. Each set $\{c_{ij}\}$ defines a pair of EWs [see Eq. (4)]; hence, numerous EWs are generated, including mirrored EWs [see Eq. (4)]. This work shows that a fraction of measurement results suffices to construct various EWs to verify the entanglement of an unknown state.

can reconstruct a two-qubit state ρ. As shown in Fig. 1, a measurement setting may generally be used to obtain a collection of correlators, which we exploit for the verification of entangled states.

IV. RESULTS

Throughout, we suppose that two-qubit observables $A_i \otimes B_j$ for $A_i, B_j \in \{X, Y, Z\}$ are experimentally estimated and provided for an unknown state. Our goal is to construct EWs that may verify whether the state is entangled. To this end, let S denote a linear combination of observables that are experimentally available and accessible:

$$S = \sum_{ij} c_{ij} A_i \otimes B_j, \tag{2}$$

with real parameters c_{ij}, which are fully under manipulation to detect entangled states.

Let $C = [c_{ij}]$ denote a matrix with elements c_{ij}. We recall the operator norm of a matrix, $\|C\|_\infty = \max_k \lambda_k$, where $\{\lambda_k\}$ are singular values of a matrix C. For all separable states σ_{sep} [see Eq. (1)], it holds that

$$-\|C\|_\infty \leqslant \sum_{ij} c_{ij}\langle A_i \otimes B_j\rangle_{\text{est}} \leqslant \|C\|_\infty. \tag{3}$$

See the Supplemental Material for the proof [35], and see also related work, e.g., Ref. [36].

In fact, the inequalities above define two EWs:

$$W_{\pm} = \|C\|_{\infty}\mathbb{I} \pm S. \tag{4}$$

Note that they satisfy the relation $W_{+} + W_{-} = 2\|C\|_{\infty}\mathbb{I}$ and hence define a pair of so-called mirrored EWs [37]. One can see that Eq. (4) provides a construction of the entire class of EWs parametrized by matrix elements $C = [c_{ij}]$. Hence, given a fraction of experimentally accessible observables $A_i \otimes B_j$, we aim to search for a matrix C to identify instances in which the inequalities in Eq. (3) are violated, or equivalently, to construct EWs detecting entangled states as in Eq. (4). For this purpose, we need to optimize a matrix C (see Fig. 1).

We now formulate the optimization problem on a matrix C for verifying entangled states via the detection criteria in Eq. (3). We introduce a normalized expectation, denoted by $\mathbb{NE}$: Given a set of qubit observables, denoted by $\mathcal{O} = \{A_i \otimes B_j\}$, and their experimental estimations for an unknown state ρ, a normalized estimation parameter $\mathbb{NE}_{\rho}[\mathcal{O}]$ is defined as

$$\begin{aligned} \mathbb{NE}_{\rho}[\mathcal{O}] &= \max \left| \sum_{ij} c_{ij} \langle A_i \otimes B_j \rangle_{\text{est}} \right| \\ \text{subject to} \quad & \|C\|_{\infty} = 1 \end{aligned} \tag{5}$$

where maximization runs over real parameters $\{c_{ij}\}$. It is clear that $\mathbb{NE}_{\sigma_{\text{sep}}}[\mathcal{O}] \leqslant 1$ for all separable states σ_{sep}, and a state ρ is entangled if $\mathbb{NE}_{\rho}[\mathcal{O}] > 1$. The optimization problem in Eq. (5) can be solved numerically via the semidefinite program (see the Supplemental Material for the derivation [35]). Note also that the semidefinite programming (SDP) is computable since it scales polynomially with the number of observables.

V. ILLUSTRATIONS WITH TWO CORRELATORS

Let us illustrate constructions of multiple EWs with an example of two correlators XX and ZZ, where $XX := X \otimes X$ and $ZZ := Z \otimes Z$. In this case, we have $C = \text{diag}[\alpha, 0, \beta]$,

$$\|C\|_{\infty} = \max(|\alpha|, |\beta|).$$

Once the observables are estimated, it holds that for all separable states

$$-\|C\|_{\infty} \leqslant \alpha \langle XX \rangle_{\text{est}} + \beta \langle ZZ \rangle_{\text{est}} \leqslant \|C\|_{\infty}. \tag{6}$$

Therefore, violations of either inequality for an unknown state imply that the state must be entangled.

Our framework for verifying entangled states works by searching for an optimal matrix C in Eq. (6). The normalized estimation, from the inequalities above, is computed as follows:

$$\mathbb{NE}_{\rho}[\mathcal{O}] = \max_{\alpha,\beta} \frac{|\alpha \langle XX \rangle_{\text{est}} + \beta \langle ZZ \rangle_{\text{est}}|}{\|C\|_{\infty}}, \tag{7}$$

where $\mathcal{O} = \{XX, ZZ\}$ denotes a collection of observables. Note that for all separable states σ_{sep}, we have $\mathbb{NE}_{\sigma_{\text{sep}}}[\mathcal{O}] \leqslant 1$ [see Eq. (6)]. If the maximal value in Eq. (7) exceeds 1, the unknown state must be entangled. Interestingly, one can analytically solve the optimization,

$$\mathbb{NE}_{\rho}[\mathcal{O}] = |\langle XX \rangle_{\text{est}}| + |\langle ZZ \rangle_{\text{est}}|.$$

Hence, we conclude that a two-qubit state must be entangled if it is found that $|\langle XX \rangle_{\text{est}}| + |\langle ZZ \rangle_{\text{est}}| > 1$.

We remark that Eq. (7) obtained from the estimation of two observables enables one to construct pairs of EWs,

$$W_{\pm} = \|C\|_{\infty}\mathbb{I} \pm (\alpha XX + \beta ZZ).$$

TABLE I. A collection of two-qubit correlators are estimated. Given $\langle A_i \otimes B_j \rangle_{\text{est}}$, for a state $|\chi_3\rangle$ (see Fig. 3), the normalized estimation parameter in Eq. (5) is computed by optimizing a matrix C. In all cases, the maximum exceeds 1, and more observables can exhibit larger violations; hence, more observables can detect a larger set of entangled states.

Normalized estimation	Max.
$1.0\langle XX\rangle_{\text{est}} + 1.0\langle ZZ\rangle_{\text{est}}$	1.36
$1.0\langle XX\rangle_{\text{est}} + 0.67\langle ZY\rangle_{\text{est}} + 0.74\langle ZZ\rangle_{\text{est}}$	1.60
$0.76\langle XX\rangle_{\text{est}} - 0.65\langle YX\rangle_{\text{est}} + 0.67\langle ZY\rangle_{\text{est}} + 0.74\langle ZZ\rangle_{\text{est}}$	1.82

Various EWs can be obtained by choosing parameters (α, β); see the Supplemental Material for the list of EWs [35].

Example

Suppose a two-qubit entangled state in a chosen basis, $|\psi_\theta\rangle = \cos\theta\,|00\rangle + \sin\theta\,|11\rangle$, for $\theta \neq 0, \pi$, for which the estimation of two observables XX and ZZ is available. We may have experimental estimation: $\langle XX\rangle_{\text{est}} = \sin 2\theta$ and $\langle ZZ\rangle_{\text{est}} = 1$. Without loss of generality, we assume $|\beta| \geqslant |\alpha|$ so that the normalized estimation in Eq. (7)

$$\frac{|\alpha\langle XX\rangle_{\text{est}} + \beta\langle ZZ\rangle_{\text{est}}|}{|\beta|} = \left|1 + \frac{\alpha}{\beta}\sin 2\theta\right|.$$

Taking $\alpha = \pm\beta$, one finds $1 + |\sin 2\theta| > 1$. We have shown that a two-qubit entangled state $|\psi_\theta\rangle$, while the basis is known, can be detected.

Note also that if local bases are not known in advance, two observables may not be sufficient to verify all entangled states. For instance, observables XX and ZZ vanish with respect to a maximally entangled state

$$|\varphi\rangle = \frac{1}{\sqrt{2}}(|0\rangle|+\rangle + |1\rangle|-\rangle)/\sqrt{2},$$

i.e.,

$$\text{tr}[XX|\varphi\rangle\langle\varphi|] = 0 \quad \text{and} \quad \text{tr}[ZZ|\varphi\rangle\langle\varphi|] = 0,$$

where $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$. Clearly, two observables XX and ZZ do not suffice to determine whether a state $|\varphi\rangle$ is entangled. An additional observable is required to verify the presence of entanglement, which will be demonstrated experimentally in what follows.

VI. EXPERIMENTAL DEMONSTRATIONS

We here consider a fraction of measurement results from two-photon experiments with polarization qubits and demonstrate the verification of entanglement via the optimization in Eq. (5). Horizontal and vertical polarizations are denoted as the computational basis. We also demonstrate constructions of multiple EWs from tomographically incomplete measurements. In particular, we increase the number of accessible observables from two to four (see Table I). Note also that more measurements can detect a larger set of entangled states.

The experimental setup is shown in Fig. 2. An entangled photon pair is generated in a quasi-phase-matched periodically polarized potassium titanyl phosphate (PPKTP) crystal designed for type-II phase matching, pumped by a single-mode laser with a center wavelength of $\lambda_p = 403.6$ nm. Entangled photon pairs are initially generated in a state $|\psi^-\rangle = (|01\rangle - |10\rangle)/\sqrt{2}$, which can be rotated to other Bell states by half-wave plates and quarter-wave plates. The photon pairs are separated into distinct spatial modes and detected with single-photon avalanche diodes. In each

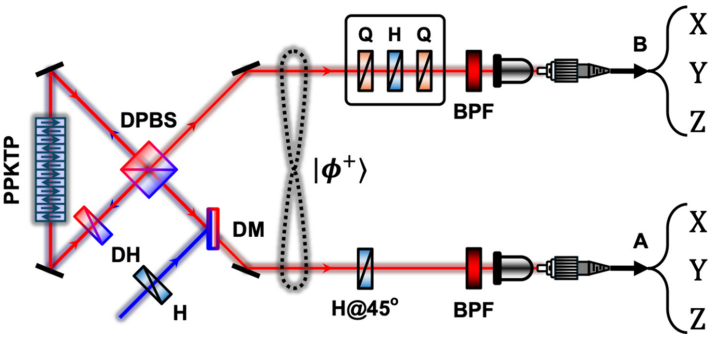


FIG. 2. An experimental setup for generating two-photon entangled states and estimating correlators is shown, where optical elements are half-wave plates (H), quarter-wave plates (Q), dichroic mirrors (DM), dual-wavelength polarizing beam splitters (DPBS), dual-wavelength half-wave plates (DH), bandpass filters (BPF), and periodically poled $KTiOPO_4$ (PPTKP). Entangled photon pairs are generated in a Sagnac interferometer, and then local operations are performed to prepare a state $|\phi^+\rangle$. A rotation of a polarization qubit can be realized with Q and H, by which states $|\chi_1\rangle$ in Eq. (8) and $|\chi_3\rangle$ in Eq. (10) are prepared. Measurements are performed in bases $\{X, Y, Z\}$, from which correlators are estimated. The results of state tomography for states $|\chi_1\rangle$ and $|\chi_3\rangle$ are shown in Fig. 3.

mode, the measurement basis can be independently chosen, and coincidence measurements are performed within a 2 ns window. This setup enables state manipulation and measurement for entanglement verification.

In the experiment, we have prepared a state in the following form, by varying an angle θ:

$$|\chi_1\rangle = (I \otimes R_Y(\theta))|\phi^+\rangle, \qquad R_Y(\theta) = \exp[-i\theta Y/2]. \tag{8}$$

For instance, for $\theta = \pi$, we have the state fidelity as $F = 98\%$ (see Fig. 3). From the nine correlators, we consider three observables XX, XY, and ZX as the accessible ones, and have estimation results experimentally,

$$\langle XX\rangle_{\text{est}} = -0.95, \quad \langle XY\rangle_{\text{est}} = 0.03, \quad \langle ZX\rangle_{\text{est}} = -0.96.$$

From these, we search for EWs for verifying entangled states by the optimization problem [see Eq. (5)],

$$\max_{\alpha,\beta,\gamma} |\alpha\langle XX\rangle_{\text{est}} + \beta\langle XY\rangle_{\text{est}} + \gamma\langle ZX\rangle_{\text{est}}|$$
$$\text{subject to} \left\| \begin{pmatrix} \alpha & \beta & 0 \\ 0 & 0 & 0 \\ \gamma & 0 & 0 \end{pmatrix} \right\|_\infty = 1. \tag{9}$$

The optimization can be approached by the semidefinite program, and we obtain the optimal parameters: $\alpha = -0.70$, $\beta = 0.04$, and $\gamma = -0.71$, i.e.,

$$\mathbb{NE}_\rho[\mathcal{O}] = \max_{\alpha,\beta,\gamma} |-0.95\alpha + 0.03\beta - 0.96\gamma| = 1.35.$$

Since $\mathbb{NE}_\rho[\mathcal{O}] > 1$, we therefore conclude that the state is entangled, without ambiguity. Thus, we have shown that estimating three observables, XX, XY, and ZX, can verify an entangled state in Eq. (8) for $\theta = \pi$.

We have also considered another instance, a maximally entangled state,

$$|\chi_3\rangle = (I \otimes V)|\phi^+\rangle, \quad V = \frac{1}{\sqrt{2}}(\mathbb{I} + i(\cos\theta X + \sin\theta Z)) \tag{10}$$

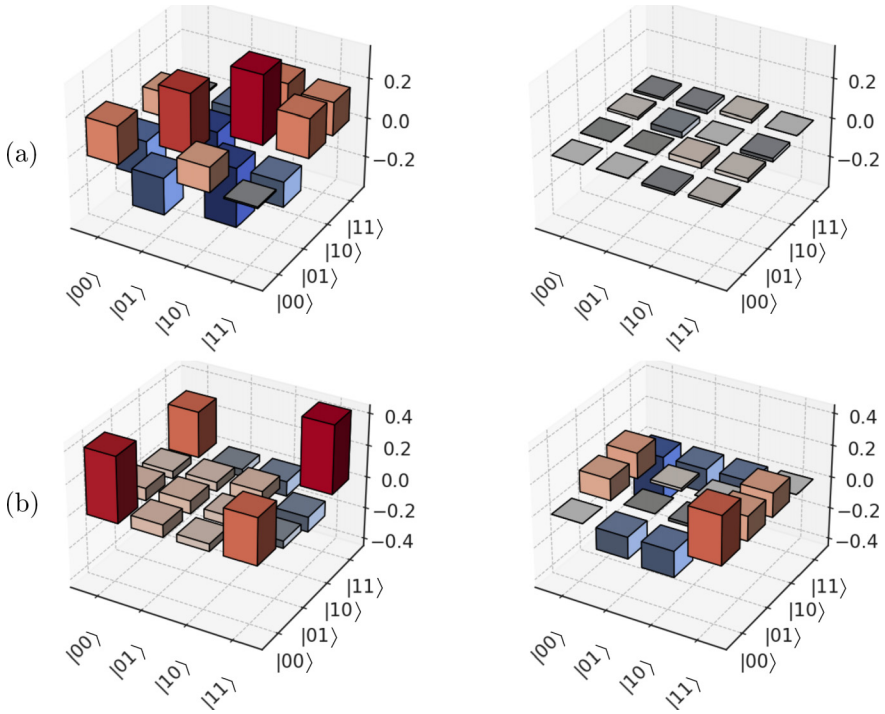


FIG. 3. The state tomography is performed for polarization-entangled two-photon states: (a) a state $|\chi_1(\theta = \pi)\rangle$ in Eq. (8) and (b) a state $|\chi_3\rangle$ in Eq. (10). The left column shows real parts, and the right column shows imaginary parts. The fidelities are obtained as 98% and 95%, respectively.

with $\theta = 7/9\,\pi$. An experimental realization of the state has a state fidelity $F = 95\%$ (see Fig. 3). From two to four observables, we compute Eq. (5) and find that more measurement settings yield larger violations. The results are summarized in Table I, see also Fig. 4. Two observables suffice to verify an entangled state, and a larger violation is shown with more observables. A larger violation implies that entanglement can be verified even when the state contains a higher noise rate.

VII. GENERALIZATIONS

The framework of detecting entangled states with a limited set of correlators [see Eq. (5)] can be generalized to high-dimensional quantum systems. In this case, local observables A_i and B_j may be chosen as d-dimensional Hermitian operators, e.g., generators of SU(d) [38]: G_i^A $(i = 1, \ldots, d_A^2)$ and G_j^B $(j = 1, \ldots, d_B^2)$ with $\mathrm{Tr}(G_k^A G_\ell^A) = d_A \delta_{kl}$ and the similarly for G_j^B. Now, for any $d_A \otimes d_B$ separable state one has [36]

$$\left| \sum_{ij} c_{ij} \langle G_i^A G_j^B \rangle_{\mathrm{est}} \right| \leqslant \sqrt{(d_A - 1)(d_B - 1)} \|C\|_\infty, \tag{11}$$

which reduces to Eq. (3) for $d_A = d_B = 2$.

The detection method can be generalized to multipartite and high-dimensional systems. For multipartite qubit states, we formulate Eq. (5) for observables of multipartite systems,

$$\mathbb{NE}_\rho[\mathcal{O}] = \max_{c_{i_1 \cdots i_n}} \frac{1}{\lambda_{\max}} \left| \sum_{i_1 \cdots i_n} c_{i_1 \cdots i_n} \langle o_{i_1}^{[1]} \otimes \cdots \otimes o_{i_n}^{[n]} \rangle_{\mathrm{est}} \right|, \tag{12}$$

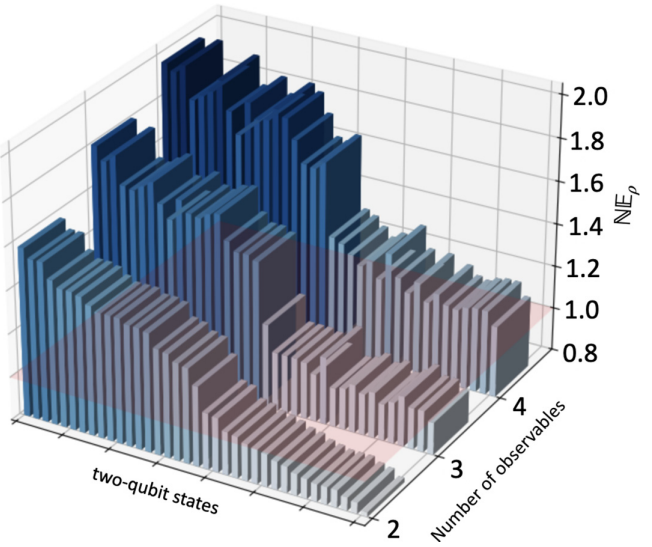


FIG. 4. Entanglement detection with two, three, and four observables is considered for two-qubit states. A larger number of observables can detect a larger set of entangled states, i.e., $\mathbb{NE}_\rho > 1$; see the Supplemental Material for all instances of two-qubit states [35]. For each state, a larger number of observables provides a higher value of the normalized estimation (see also Table I).

where $o_{i_k}^{[k]}$ denotes a local observable of the kth system for $k = 1, \ldots, n$, and

$$\lambda_{\max} = \max_{|\psi_{\text{prod}}\rangle} \langle \psi_{\text{prod}}| \sum_{i_1 \cdots i_n} c_{i_1 \cdots i_n} o_{i_1}^{[1]} \otimes \cdots \otimes o_{i_n}^{[n]} |\psi_{\text{prod}}\rangle, \tag{13}$$

where $|\psi_{\text{prod}}\rangle = |e_1 \cdots e_n\rangle$, i.e., the maximal expectation of an observable of a multipartite system over product states. Note also that the construction with EWs can be generalized to correlation tensors in multipartite systems [39]. The maximization in Eq. (13) can be computed numerically [45]. Entangled states are detected when Eq. (12) is larger than 1. Since $\lambda_{\max}$ in Eq. (13) is computed for fully separable states, the method in Eq. (12) can detect entangled states that are not fully separable. For multipartite systems, mirrored EWs are efficient as there are mirrored pairs of optimal EWs [46].

The estimation in Eq. (13) can be used to detect various structures of entangled states, such as k-separable pure states, by applying constraints on a state $|\psi\rangle$. For instance, an n-partite state in cuts $A|B|C$,

$$|\psi\rangle_{1\cdots n} = |a\rangle_A |b\rangle_B |c\rangle_C,$$

where A, B, and C are disjoint subjects with indices $\{1, \ldots, n\}$, can be used to detect those entangled states that are not separable in the cuts, called 3-separable states. In general, entangled states that are not k-separable can be detected by computing $\lambda_{\max}$ with k-separable states. The proposed method can be applied to detecting entangled states that are not k-separable.

VIII. DETECTING HIGHLY NOISY ENTANGLED STATES

The detection method works for detecting highly noisy entangled states. Let us consider a multipartite state under depolarization noise,

$$\rho(p) = (1 - p)\rho + p\mathbb{I}/d, \tag{14}$$

where d is the dimension of a Hilbert space $\mathcal{H}$ that can be for multipartite and high-dimensional systems; we have $\mathbb{NE}_{\rho(p)}[\mathcal{O}] = (1 - p)\mathbb{NE}_\rho[\mathcal{O}]$ for observables $\mathcal{O}$. Hence, to tolerate an error rate

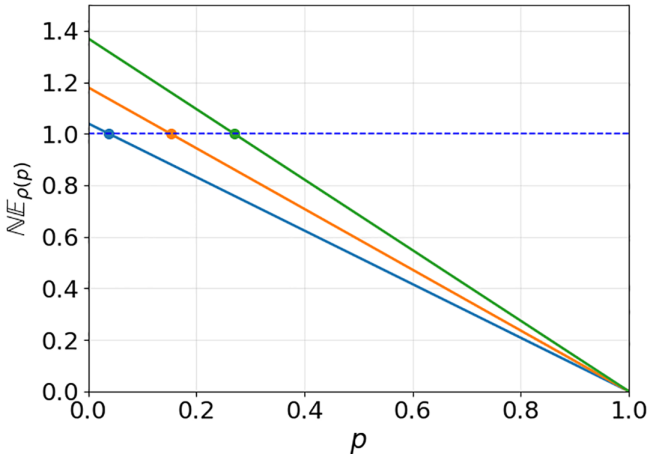


FIG. 5. When a GHZ state in Eq. (15) suffers from depolarization noise as shown in Eq. (14), a larger number of measurement settings allows the normalized estimation $\mathbb{NE}_\rho$ to tolerate a higher value of noise fraction p. By choosing Pauli observables, 27 observables are needed for state tomography; once a state is verified, theoretical tools can be used to determine whether it is entangled. Among them, four observables, which are not sufficient for state verification, allow the normalized estimation to exceed 1 for $p < 0.04$ (blue). Five observables can tolerate noise fractions $p < 0.15$ (orange). Seven observables can detect entangled states up to $p < 0.27$ (green).

p, one should have $\mathbb{NE}_\rho[\mathcal{O}] > (1-p)^{-1}$ so that $\mathbb{NE}_{\rho(p)}[\mathcal{O}] > 1$. See the Supplemental Material for the derivation [35].

It is also worth considering noise due to local basis changes, since an EW is not robust against local unitaries. For instance, an EW detecting a Bell state cannot detect the other three Bell states that are local unitarily equivalent [46]. Let us here consider a Greenberger-Horne-Zeilinger (GHZ) state

$$|\Psi\rangle = U_A \otimes U_B \otimes U_C |\psi\rangle \tag{15}$$

for randomly chosen single-qubit gates U_A, U_B, and U_C, where $|\psi\rangle = (|000\rangle + |111\rangle)/\sqrt{2}$. Then, depolarization noise may occur on a state $|\Psi\rangle$ [see Eq. (14)]. In Fig. 5, we demonstrate that, with Pauli observables for three qubits, a larger number of measurement settings enables the detection method to tolerate a higher value of noise fraction p. See the Supplemental Material for the detailed analysis [35]. The demonstration shows the usefulness of the detection method of higher-dimensional states such as $2 \otimes 4$ systems.

IX. CONCLUSION

We have presented a practical, efficient method for verifying entangled states using tomographically incomplete measurements. Our results here do not demonstrate EWs, but we present a framework for constructing multiple EWs to verify a maximal set of entangled states, provided limited access to the correlation data, i.e., a measurement setting that yields tomographically incomplete results. The construction of EW also exploits mirrored EWs, which have both upper and lower bounds satisfied by all separable states. Hence, our results provide an innovative, efficient, and practical approach to verifying entangled states in realistic scenarios. The proposed method readily applies to various realistic scenarios for verifying entangled states, particularly when the measurement setting is limited and thus tomographically incomplete, e.g., at ground and moving stations that aim to distribute entanglement, where state tomography is not practically feasible.

In future investigations, it would be interesting to deepen the characterization of the optimality of EWs by using a limited set of observables in multipartite quantum systems. We envisage that the normalized estimation can also be used to verify various entanglement structures in practical and realistic scenarios. Furthermore, as the framework of mirrored EWs applies to general convex operational theories [46], the proposed framework also applies to resource theories (see, e.g., Ref. [40]). Useful resources are detected using upper and lower bounds. Resources characterized by nonstabilizerness [41] could be one of such instances, being particularly useful for quantum computing.

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

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