

Persistence of quantum triality relations in open n -dimensional systems

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We examine the complementarity among coherence (visibility), predictability, and entanglement for qubit and qutrit systems subjected to noisy quantum channels. Using the system-path entanglement framework, analytical expressions for all three quantities are derived for two-, three-, and n -slit interferometric setups. The study first establishes the validity of the triality relation in ideal conditions and then investigates its behavior under amplitude and phase damping. We find that amplitude damping redistributes coherence and population imbalance without violating complementarity, while phase damping reduces coherence but leaves predictability unchanged. We further extend the study to non-Markovian amplitude damping, where environmental memory effects lead to nonmonotonic evolution and revivals of coherence and entanglement. These results demonstrate that the complementarity relation remains preserved even in open quantum systems, highlighting its robustness against decoherence for both Markovian and non-Markovian open-system dynamics and providing a unified analytical understanding of noisy quantum interferometry in low-dimensional systems.

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I. INTRODUCTION

Quantum entanglement is one of the cornerstone resources of quantum information theory [1–3]. A composite system is said to be entangled when it cannot be represented as a direct product of its constituent subsystems. This inseparability is uniquely quantum in nature, with no counterpart in classical physics. Surprisingly, however, correlations resembling entanglement can also emerge in structured optical fields [4,5]. In such cases, the notion of classical entanglement or intrasystem entanglement has been introduced to describe nonseparable couplings between different internal degrees of freedom of a single optical beam, for instance, its polarization and spatial modes. Although rooted in classical physics, the mathematical formalism of classical entanglement [6] mirrors that of its quantum counterpart. This striking similarity has opened the door to employing

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classical entanglement as a test bed for simulating quantum-inspired tasks, ranging from the implementation of quantum algorithms [7], quantum walks [8], reconstruction of quantum channels [9], quantum tomography [10], and many more. Wave-particle duality is the age-old interest in quantum mechanics [11]. The term *quanton* is defined for the quantum system that shows this dual nature. Among various approaches, two-path interference has served as a central testing ground for the study of this duality. Greenberger and Yasin introduced a simplified framework to address the issue of wave-particle duality. In their description, the notion of “particlelike” behavior is inferred only when the quanton has a higher probability of traveling through one path than the other. For such a case, they derived the following inequality [12]:

$$\mathcal{P}^2 + \mathcal{V}^2 \leq 1, \quad (1)$$

where $\mathcal{P}$ represents the *path predictability* (or simply, predictability) and $\mathcal{V}$ denotes the *interference visibility*, which quantifies the wavelike nature of the quanton. In a typical two-path interference experiment, these two quantities capture the trade-off between the particle and wave characteristics. Visibility can be increased using conditional visibility or eraser visibility by subensemble sorting [13].

Recent studies have unfolded the presence of different nonseparable degrees of freedom in complementarity relations in optical systems [14,15]. The connection between two-particle visibility and concurrence has been drawn [16]. Concurrence is one of the most important measures of entanglement, positive for entangled states and vanishing for separable states [17–19]. Complementarity exists between the degree of polarization and concurrence [20]. In recent studies, the triality relation has been derived including the polarization of the multislit interference experiment as follows [21]:

$$\mathcal{P}^2 + \mathcal{V}^2 + \varepsilon^2 = 1, \quad (2)$$

where $\mathcal{P}$, $\mathcal{V}$, and ε are predictability, interference visibility (coherence), and entanglement. An exact triality relation has been established between which path, visibility, and I concurrence following the conclusion reached by Ref. [22].

Qian and Eberly showed that classical nondeterministic fields give a natural basis for entanglement and Bell analyses in their work “Entanglement is sometimes enough” [23]. In Ref. [24], classical beam versions of well-known quantum entangled states (Greenberger-Horne-Zeilinger, W, Noon) were obtained.

In this work, we investigate whether the triality relation among visibility, predictability, and entanglement remains valid under two common noise models—*amplitude damping* and *phase damping*—for both qubit and qutrit systems. In Sec. II, we describe the general framework of the *system-path combination*, which enables the definition of entanglement even for a single quanton through its interaction with a path detector. Section III introduces the quantitative measures of *coherence (visibility)*, *predictability*, and *entanglement*, which form the basis of the complementarity relation. Sections IV and V present detailed derivations of the triality condition for *two-slit (qubit)* and *three-slit (qutrit)* interferometric systems, respectively. Section VI generalizes the triality condition for *n-slit* interferometric system. In Sec. VII, we analyze the influence of *amplitude damping*, deriving analytical expressions for the modified visibility, predictability, and entanglement, and demonstrate that the triality relation persists. Section VIII studies *phase damping*, where coherence decay alone is considered, again confirming the robustness of the complementarity relation. Section IX extends this framework to non-Markovian amplitude damping, demonstrating that the relation remains valid even in the presence of environmental memory effects. Section X discusses the distribution of the quantities under noise. Finally, Sec. XI summarizes our findings and provides a concluding discussion on the persistence of complementarity under noisy quantum channels.

II. SYSTEM-PATH COMBINATION AND ENTANGLEMENT

In realistic interferometric scenarios, the quanton is not an isolated system but interacts with an ancillary degree of freedom, such as a path marker or detector. This composite description allows one to meaningfully discuss entanglement, even for single-particle systems. Following Ref. [25], the joint system-detector state can be written as

$$|\Psi\rangle = \sum_{i=1}^n c_i |i\rangle_{\text{path}} \otimes |\phi_i\rangle_{\text{det}}, \quad (3)$$

where $|i\rangle_{\text{path}}$ denotes the path basis of the quanton and $|\phi_i\rangle_{\text{det}}$ represents the correlated detector (or environment) states. The reduced density matrix of the quanton is obtained by tracing out the detector states:

$$\rho = \text{Tr}_{\text{det}}(|\Psi\rangle\langle\Psi|). \quad (4)$$

In this representation, the off-diagonal elements ρ_{ij} encode the coherence between paths i and j , while the diagonal elements ρ_{ii} give the path populations. The entanglement measure ε^2 thus quantifies the correlations between the quanton and the path-detector system, consistent with the tripartite complementarity framework introduced in Refs. [21,25].

III. COHERENCE, PREDICTABILITY, AND ENTANGLEMENT

The measure of wave nature of light is visibility, which is evidenced by a picture of interference pattern observed. More clarity of interference pattern leads to more defined visibility or coherence. The mathematical framework of coherence or visibility for a quanton passing through a multipath interferometer is given by [26]

$$V^2 = \frac{n}{n-1} \sum_{i \neq j} |\rho_{ij}|^2. \quad (5)$$

Similarly, the measurement of predictability demonstrates the particle nature of waves, which in the context of a multislit interferometer enables the prediction of the path through which the photon is allowed to pass. Basically, predictability overviews comprehensive idea of which path information and is formulated as [26]

$$P^2 = \sum_{i=1}^n \rho_{ii}^2 - \frac{1}{n-1} \sum_{i \neq j} \rho_{ii} \rho_{jj}. \quad (6)$$

Finally, a normalized entanglement measure that emanates from the additive effect of both diagonal and off-diagonal elements of the density matrix is given as [26]

$$\varepsilon^2 = \frac{n}{2(n-1)} \sum_{\text{pairs}} E_{ij}^2, \quad (7)$$

where E_{ij} denotes the concurrence between the quanton and the path detector.

IV. TWO-SLIT INTERFEROMETER (QUBIT SYSTEM) AND THE TRIALITY RELATION

For a *pure path state*, the quanton can be expressed as a coherent superposition of two paths,

$$|\psi\rangle = c_1 |\psi_1\rangle + c_2 |\psi_2\rangle. \quad (8)$$

The corresponding density matrix is

$$\rho = \begin{pmatrix} |c_1|^2 & c_2 c_1^* \\ c_1 c_2^* & |c_2|^2 \end{pmatrix}. \quad (9)$$

The visibility (or coherence) and predictability for this pure qubit state are obtained as

$$V^2 = 4|c_1|^2|c_2|^2, \quad P^2 = 1 - 4|c_1|^2|c_2|^2, \quad (10)$$

which clearly satisfy

$$V^2 + P^2 = 1. \quad (11)$$

This equality demonstrates that the state is completely pure and that there is no entanglement with any detector degree of freedom. Hence, the path state can be written as a direct product with the detector, implying that the total system is *separable* and nonentangled.

When the quanton interacts with a path detector, the combined state is given by

$$|\Psi\rangle = c_1|\psi_1\rangle|d_1\rangle + c_2|\psi_2\rangle|d_2\rangle, \quad (12)$$

where $|d_1\rangle$ and $|d_2\rangle$ represent the detector states correlated with each path. When the detector states are *not identical*, i.e., $\langle d_1|d_2\rangle \neq 1$, the total state cannot be written as a simple product and becomes *entangled*.

Tracing out the detector subsystem yields the reduced density matrix of the path degree of freedom,

$$\rho_r = \begin{pmatrix} |c_1|^2 & c_2 c_1^* \langle d_1|d_2\rangle \\ c_1 c_2^* \langle d_2|d_1\rangle & |c_2|^2 \end{pmatrix}. \quad (13)$$

The off-diagonal elements are suppressed by the detector-state overlap:

$$\rho_{ij}^{(r)} = \rho_{ij} \langle d_i|d_j\rangle. \quad (14)$$

The corresponding visibility and predictability are given by

$$V^2 = 4|c_1|^2|c_2|^2|\langle d_1|d_2\rangle|^2, \quad P^2 = 1 - 4|c_1|^2|c_2|^2. \quad (15)$$

The entanglement between the path and detector subsystems can then be expressed as

$$\varepsilon^2 = 4|c_1|^2|c_2|^2(1 - |\langle d_1|d_2\rangle|^2). \quad (16)$$

When the detector states are *parallel*, $|d_1\rangle = |d_2\rangle$, we have $|\langle d_1|d_2\rangle| = 1$, giving $\varepsilon^2 = 0$. In this limit, the combined system factorizes into a direct product of the path and detector parts, confirming that a pure path state corresponds to a separable (nonentangled) configuration.

If the detector states are not parallel, the reduced density matrix ρ_r becomes a *mixed state*, indicating the partial loss of coherence due to path-detector correlations.

On the other hand, when the detector states are *orthogonal*, $\langle d_1|d_2\rangle = 0$, the system becomes *maximally entangled*. In this case, the composite state

$$|\Psi\rangle = c_1|\psi_1\rangle|d_1\rangle + c_2|\psi_2\rangle|d_2\rangle \quad (17)$$

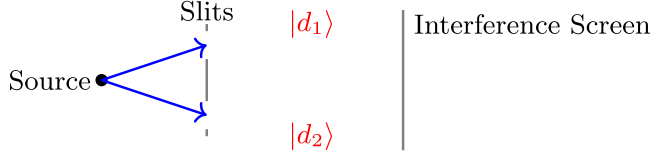
reduces to the form of a *Bell-type entangled state*. For equal superposition amplitudes, $|c_1| = |c_2| = \frac{1}{\sqrt{2}}$, we obtain

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|\psi_1\rangle|d_1\rangle + |\psi_2\rangle|d_2\rangle), \quad (18)$$

which represents a *maximally entangled Bell state* between the path and detector subsystems. In this situation, the entanglement measure becomes $\varepsilon^2 = 1$, and both visibility and predictability vanish, $V^2 = P^2 = 0$. A schematic two-slit interferometer with detector correlations is shown in Fig. 1.

Therefore, the entanglement term ε^2 in the complementarity relation captures the degree of correlation between the path and detector states, extending the conventional wave-particle duality,

$$V^2 + P^2 = 1, \quad (19)$$


 FIG. 1. Two-slit interferometer with path detector states $|d_1\rangle$ and $|d_2\rangle$.

to the complete *triality condition*,

$$V^2 + P^2 + \varepsilon^2 = 1. \quad (20)$$

Alternative Pauli-matrix representation

For completeness, the qubit density operator can also be expressed in the Pauli-matrix representation as

$$\rho = \rho_0 I + \boldsymbol{\rho} \cdot \boldsymbol{\sigma} = \begin{pmatrix} \rho_0 + \rho_3 & \rho_1 - i\rho_2 \\ \rho_1 + i\rho_2 & \rho_0 - \rho_3 \end{pmatrix}, \quad (21)$$

where $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ are the Pauli matrices and $\rho_0 = \frac{1}{2}$ ensures $\text{Tr}(\rho) = 1$. In this form, the off-diagonal elements encode the coherence terms ρ_{12} and ρ_{21} , while the population imbalance appears through ρ_3 .

Using this representation, the visibility and predictability can be rewritten as

$$V^2 = 4|\rho_{12}|^2 = 4(\rho_1^2 + \rho_2^2), \quad P^2 = (\rho_{11} - \rho_{22})^2 = 4\rho_3^2. \quad (22)$$

The entanglement measure, obtained from the reduced density matrix after tracing out the detector degrees of freedom, becomes

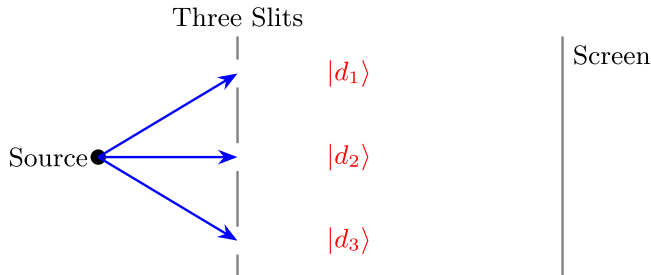
$$\varepsilon^2 = 4\rho_0^2 - V^2 - P^2. \quad (23)$$

Since $\rho_0 = \frac{1}{2}$ for a normalized two-level system, we obtain

$$V^2 + P^2 + \varepsilon^2 = 1, \quad (24)$$

which is identical to the triality condition derived previously.

This alternative formulation demonstrates that the same complementarity relation holds naturally in the Bloch (Pauli) representation of the qubit state, confirming the consistency of the approach across both path-detector and algebraic frameworks. The corresponding three-slit interferometric configuration is illustrated in Fig. 2.


 FIG. 2. Three-slit (qutrit) interferometer showing path-detector correlations $|d_1\rangle$, $|d_2\rangle$, and $|d_3\rangle$.

V. THREE-SLIT (QUTRIT) INTERFEROMETER AND THE TRIALITY RELATION

Having established the triality relation for the two-path (qubit) interferometer and verified its consistency in both the path-detector and Pauli-matrix representations, we now extend the analysis to a three-path (qutrit) system. The inclusion of a third path enriches the interference structure and provides a natural framework for exploring complementarity in higher-dimensional Hilbert spaces, where coherence, predictability, and entanglement coexist in more intricate forms.

For a pure qutrit state, the quanton can be expressed as

$$|\psi\rangle = c_1|\psi_1\rangle + c_2|\psi_2\rangle + c_3|\psi_3\rangle, \quad (25)$$

with the corresponding density matrix

$$\rho = \begin{pmatrix} |c_1|^2 & c_1 c_2^* & c_1 c_3^* \\ c_2 c_1^* & |c_2|^2 & c_2 c_3^* \\ c_3 c_1^* & c_3 c_2^* & |c_3|^2 \end{pmatrix}. \quad (26)$$

From this, the visibility and predictability for the pure qutrit state are obtained as

$$V^2 = 3(|c_1|^2|c_2|^2 + |c_2|^2|c_3|^2 + |c_3|^2|c_1|^2), \quad (27)$$

$$P^2 = 1 - 3(|c_1|^2|c_2|^2 + |c_2|^2|c_3|^2 + |c_3|^2|c_1|^2), \quad (28)$$

which satisfy the duality relation $V^2 + P^2 = 1$ in the absence of entanglement.

When a path-detector correlation is introduced, the combined state becomes

$$|\Psi\rangle = c_1|\psi_1\rangle|d_1\rangle + c_2|\psi_2\rangle|d_2\rangle + c_3|\psi_3\rangle|d_3\rangle, \quad (29)$$

where $|d_i\rangle$ represent the detector states associated with each path. Tracing out the detector degrees of freedom yields the reduced density matrix

$$\rho_r = \begin{pmatrix} |c_1|^2 & c_1 c_2^* \langle d_1 | d_2 \rangle & c_1 c_3^* \langle d_1 | d_3 \rangle \\ c_2 c_1^* \langle d_2 | d_1 \rangle & |c_2|^2 & c_2 c_3^* \langle d_2 | d_3 \rangle \\ c_3 c_1^* \langle d_3 | d_1 \rangle & c_3 c_2^* \langle d_3 | d_2 \rangle & |c_3|^2 \end{pmatrix}. \quad (30)$$

The corresponding measures of visibility, predictability, and entanglement become

$$V^2 = 3(|c_1|^2|c_2|^2|\langle d_1 | d_2 \rangle|^2 + |c_2|^2|c_3|^2|\langle d_2 | d_3 \rangle|^2 + |c_3|^2|c_1|^2|\langle d_3 | d_1 \rangle|^2), \quad (31)$$

$$P^2 = 1 - 3(|c_1|^2|c_2|^2 + |c_2|^2|c_3|^2 + |c_3|^2|c_1|^2), \quad (32)$$

$$\varepsilon^2 = 3|c_1|^2|c_2|^2(1 - |\langle d_1 | d_2 \rangle|^2) + 3|c_2|^2|c_3|^2(1 - |\langle d_2 | d_3 \rangle|^2) + 3|c_3|^2|c_1|^2(1 - |\langle d_3 | d_1 \rangle|^2). \quad (33)$$

These expressions together satisfy the generalized triality condition

$$V^2 + P^2 + \varepsilon^2 = 1, \quad (34)$$

confirming that the complementarity relation extends naturally to a three-path interferometer.

To connect with the algebraic (Bloch-like) representation, the density operator for the qutrit can be expanded in terms of the Gell-Mann matrices λ_i as

$$\rho = \frac{I_3}{3} + \frac{1}{\sqrt{3}} \sum_{i=1}^8 S_i \lambda_i, \quad (35)$$

leading to

$$V^2 = S_1^2 + S_2^2 + S_4^2 + S_5^2 + S_6^2 + S_7^2, \quad (36)$$

$$P^2 = S_3^2 + S_8^2. \quad (37)$$

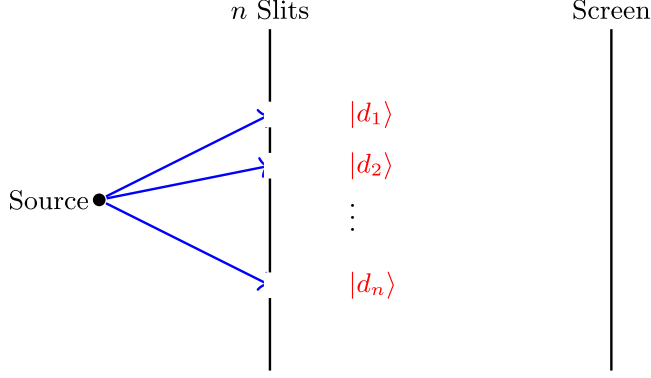


FIG. 3. n -slit interferometer showing path-detector correlations $|d_1\rangle, |d_2\rangle, \dots, |d_n\rangle$.

Consequently, the entanglement can be written as

$$\varepsilon^2 = 1 - (S_1^2 + S_2^2 + S_3^2 + S_4^2 + S_5^2 + S_6^2 + S_7^2 + S_8^2). \quad (38)$$

Substituting the expressions for V^2 and P^2 immediately yields

$$V^2 + P^2 + \varepsilon^2 = 1.$$

Thus, the three-slit (qutrit) interferometer satisfies the same fundamental complementarity condition as the qubit case. The introduction of the third path enhances the richness of the interference and entanglement structure while preserving the unity of the triality relation, confirming its universality across different Hilbert-space dimensions. The generalized n -slit interferometric setup is shown in Fig. 3.

VI. n -SLIT INTERFEROMETER AND TRIALITY RELATION

Consider a pure n -path quantum system in an n -slit interferometer. The quantum state can be written as

$$|\psi\rangle = \sum_{i=1}^n c_i |\psi_i\rangle, \quad \sum_{i=1}^n |c_i|^2 = 1. \quad (39)$$

The corresponding density matrix in the path basis is

$$\rho_{ij} = c_i c_j^*, \quad i, j = 1, \dots, n. \quad (40)$$

For a pure n -slit state, the wave and particle measures—visibility V and predictability P —are defined as

$$V^2 = \frac{2n}{n-1} \sum_{i < j} |c_i|^2 |c_j|^2, \quad (41)$$

$$P^2 = 1 - \frac{2n}{n-1} \sum_{i < j} |c_i|^2 |c_j|^2. \quad (42)$$

These quantities satisfy the standard duality relation

$$V^2 + P^2 = 1, \quad (43)$$

indicating the absence of entanglement.

When the quanton becomes correlated with a path detector, the combined state takes the form

$$|\Psi\rangle = \sum_{i=1}^n c_i |\psi_i\rangle |d_i\rangle, \quad (44)$$

where $\{|d_i\rangle\}$ are (generally nonorthogonal) detector states.

Tracing over the detector degrees of freedom yields the reduced density matrix of the quanton,

$$\rho_{ij}^{(r)} = c_i c_j^* \langle d_j | d_i \rangle. \quad (45)$$

The corresponding visibility is now reduced to

$$V^2 = \frac{2n}{n-1} \sum_{i < j} |c_i|^2 |c_j|^2 |\langle d_i | d_j \rangle|^2, \quad (46)$$

while the predictability remains

$$P^2 = 1 - \frac{2n}{n-1} \sum_{i < j} |c_i|^2 |c_j|^2. \quad (47)$$

The loss of coherence due to path-detector correlations is quantified by an entanglement measure ε , defined as

$$\varepsilon^2 = \frac{2n}{n-1} \sum_{i < j} |c_i|^2 |c_j|^2 (1 - |\langle d_i | d_j \rangle|^2). \quad (48)$$

These three quantities satisfy the *triality relation*

$$V^2 + P^2 + \varepsilon^2 = 1, \quad (49)$$

which generalizes wave-particle duality in the presence of path-detector entanglement.

VII. COMPLEMENTARITY UNDER AMPLITUDE-DAMPING CHANNELS

A. Qubit system under amplitude damping

The amplitude-damping channel for a qubit is represented by the Kraus operators

$$E_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma_A} \end{pmatrix}, \quad E_1 = \begin{pmatrix} 0 & \sqrt{\gamma_A} \\ 0 & 0 \end{pmatrix}, \quad (50)$$

where $0 \leq \gamma_A \leq 1$ is the damping parameter. The evolved density matrix is

$$\rho' = \sum_{k=0}^1 E_k \rho E_k^\dagger = \begin{pmatrix} \rho_{11} + \gamma_A \rho_{22} & \sqrt{1-\gamma_A} \rho_{12} \\ \sqrt{1-\gamma_A} \rho_{21} & (1-\gamma_A) \rho_{22} \end{pmatrix}. \quad (51)$$

The predictability becomes

$$P'^2 = (\rho'_{11})^2 + (\rho'_{22})^2 - 2\rho'_{11}\rho'_{22} = (\rho_{11} - \rho_{22} + 2\gamma_A \rho_{22})^2, \quad (52)$$

and the visibility reduces to

$$V'^2 = 4|\rho'_{12}|^2 = (1-\gamma_A)V^2. \quad (53)$$

Entanglement in this case is

$$\begin{aligned} \epsilon'^2 &= E_{12}'^2 = 4(1-\gamma_A)(\rho_{11}\rho_{22} - |\rho_{12}|^2) + 4\gamma_A(1-\gamma_A)\rho_{22}^2 \\ &= (1-\gamma_A)E_{12}^2 + 4\gamma_A(1-\gamma_A)\rho_{22}^2, \end{aligned} \quad (54)$$

$$P'^2 + V'^2 + \epsilon'^2 = (\rho_{11} + \rho_{22})^2 = 1. \quad (55)$$

We found that triality relation holds for the amplitude-damping case.

B. Qutrit system under amplitude damping

For the qutrit case, we consider a cascade decay $|2\rangle \rightarrow |1\rangle \rightarrow |0\rangle$ with rates γ_2 and γ_1 . The Kraus operators are

$$\begin{aligned} E_0 &= |0\rangle\langle 0| + \sqrt{1-\gamma_1}|1\rangle\langle 1| + \sqrt{1-\gamma_2}|2\rangle\langle 2|, \\ E_1 &= \sqrt{\gamma_1}|0\rangle\langle 1|, \\ E_2 &= \sqrt{\gamma_2}|1\rangle\langle 2|. \end{aligned} \quad (56)$$

The evolved density matrix is $\rho' = \sum_{k=0}^2 E_k \rho E_k^\dagger$ with updated populations

$$\rho'_{00} = \rho_{00} + \gamma_1 \rho_{11} + \gamma_1 \gamma_2 \rho_{22}, \quad (57)$$

$$\rho'_{11} = (1-\gamma_1)\rho_{11} + \gamma_2(1-\gamma_1)\rho_{22}, \quad (58)$$

$$\rho'_{22} = (1-\gamma_2)\rho_{22}. \quad (59)$$

Off-diagonal elements are damped as

$$\rho'_{ij} = \sqrt{(1-\gamma_i)(1-\gamma_j)} \rho_{ij} \quad (i \neq j). \quad (60)$$

$$s'_{1,2} = \sqrt{1-\gamma_1} s_{1,2},$$

$$s'_{4,5} = \sqrt{1-\gamma_2} s_{4,5},$$

$$s'_{6,7} = \sqrt{(1-\gamma_1)(1-\gamma_2)} s_{6,7},$$

$$s'_8 = (1-\gamma_2) s_8 + \frac{\gamma_2}{2},$$

$$s'_3 = (1-\gamma_1) s_3 + \frac{\sqrt{3}}{3} \gamma_1 (1-\gamma_2) s_8. \quad (61)$$

Visibility is

$$V'^2 = (s'_1)^2 + (s'_2)^2 + (s'_4)^2 + (s'_5)^2 + (s'_6)^2 + (s'_7)^2,$$

predictability is

$$P'^2 = (s'_3)^2 + (s'_8)^2,$$

and entanglement is

$$\epsilon'^2 = 1 - [(s'_1)^2 + (s'_2)^2 + (s'_3)^2 + (s'_4)^2 + (s'_5)^2 + (s'_6)^2 + (s'_7)^2 + (s'_8)^2]. \quad (62)$$

We also found that

$$V'^2 + P'^2 + \epsilon'^2 = 1.$$

C. Persistence of triality

Notably, the triality relation $V'^2 + P'^2 + \epsilon'^2 = 1$ remains valid under amplitude damping. The channel redistributes the contributions of coherence, predictability, and entanglement without breaking their sum rule.

VIII. COMPLEMENTARITY UNDER PHASE-DAMPING (DEPHASING) CHANNELS

A. Qubit system under phase damping (dephasing)

The phase-damping channel for a qubit is represented by the Kraus operators

$$E_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma_p} \end{pmatrix}, \quad E_1 = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{\gamma_p} \end{pmatrix}, \quad (63)$$

where $0 \leq \gamma_p \leq 1$ is the dephasing parameter. The evolved density matrix is

$$\rho' = \sum_{k=0}^1 E_k \rho E_k^\dagger = \begin{pmatrix} \rho_{11} & \sqrt{1-\gamma_p} \rho_{12} \\ \sqrt{1-\gamma_p} \rho_{21} & \rho_{22} \end{pmatrix}. \quad (64)$$

The predictability is unaffected by phase damping,

$$P'^2 = (\rho_{11} - \rho_{22})^2 = P^2, \quad (65)$$

while the visibility reduces according to

$$V'^2 = 4|\rho'_{12}|^2 = (1-\gamma_p)V^2, \quad (66)$$

$$\epsilon'^2 = E_{12}'^2 = E_{12}^2 + 4\gamma_p|\rho_{12}|^2. \quad (67)$$

Here also we see

$$P'^2 + V'^2 + \epsilon'^2 = 1. \quad (68)$$

Thus, phase damping diminishes only the coherence, leaving the population imbalance unchanged, and redistributes the balance between visibility and entanglement without violating the complementarity relation.

B. Qutrit system under phase damping (dephasing)

The phase-damping (or pure dephasing) channel for a three-level system is represented by the following set of Kraus operators [27–29]:

$$\begin{aligned} E_0 &= \sqrt{\frac{1+2\sqrt{1-\gamma_p}}{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\ E_1 &= \sqrt{\frac{1-\sqrt{1-\gamma_p}}{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \\ E_2 &= \sqrt{\frac{1-\sqrt{1-\gamma_p}}{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}, \end{aligned} \quad (69)$$

where $\omega = e^{2\pi i/3}$ and $0 \leq \gamma_p \leq 1$ denotes the dephasing parameter. These operators satisfy the trace-preserving completeness relation

$$E_0^\dagger E_0 + E_1^\dagger E_1 + E_2^\dagger E_2 = I_3, \quad (70)$$

ensuring a completely positive, trace-preserving map.

Under this channel, the diagonal elements of the density matrix remain unchanged, while the off-diagonal elements are attenuated:

$$\rho'_{ii} = \rho_{ii}, \quad \rho'_{ij} = \sqrt{1-\gamma_p} \rho_{ij} \quad (i \neq j). \quad (71)$$

In the Gell-Mann representation, this leads to the transformation

$$s'_{1,2,4,5,6,7} = \sqrt{1 - \gamma_p} s_{1,2,4,5,6,7}, \quad s'_3 = s_3, \quad s'_8 = s_8. \quad (72)$$

Consequently, the modified quantities become

$$V'^2 = (1 - \gamma_p)V^2, \quad P'^2 = P^2, \quad \varepsilon'^2 = 1 - P^2 - (1 - \gamma_p)V^2. \quad (73)$$

Therefore, the complementarity (trality) relation remains valid under qutrit phase damping:

$$V'^2 + P'^2 + \varepsilon'^2 = 1.$$

IX. COMPLEMENTARITY UNDER NON-MARKOVIAN AMPLITUDE-DAMPING CHANNEL

We next consider non-Markovian amplitude damping, where the system-environment interaction exhibits memory effects arising from a structured reservoir or finite environmental correlation time [30,31]. In contrast to the Markovian regime, the resulting time-dependent decay rates can produce nonmonotonic dynamics in coherence and population imbalance [32].

A. Qubit system under non-Markovian amplitude damping

For a qubit subjected to non-Markovian amplitude damping, the reduced density matrix evolves as

$$\rho'(t) = \begin{pmatrix} \rho_{11} + [1 - |G(t)|^2]\rho_{22} & G(t)\rho_{12} \\ G^*(t)\rho_{21} & |G(t)|^2\rho_{22} \end{pmatrix}, \quad (74)$$

where $G(t)$ is the non-Markovian decoherence function that incorporates environmental memory effects.

For a Lorentzian spectral density, $G(t)$ is given by [30]

$$G(t) = e^{-\lambda t/2} \left[\cosh\left(\frac{\Delta t}{2}\right) + \frac{\lambda}{\Delta} \sinh\left(\frac{\Delta t}{2}\right) \right], \quad (75)$$

where $\Delta = \sqrt{\lambda^2 - 4g^2}$, g denotes the system-environment coupling strength, and λ is the spectral width of the reservoir. The dynamics becomes non-Markovian for $2g > \lambda$, in which case $G(t)$ exhibits oscillatory behavior and information backflow.

The visibility under non-Markovian noise is

$$V'^2(t) = |G(t)|^2 V^2, \quad (76)$$

showing that the exponential decay factor of the Markovian case is replaced by the non-Markovian survival probability $|G(t)|^2$.

The predictability for this case is

$$\rho'_{11} - \rho'_{22} = \rho_{11} - \rho_{22} + 2[1 - |G(t)|^2]\rho_{22}, \quad (77)$$

which leads to

$$P'^2(t) = [P + 2(1 - |G(t)|^2)\rho_{22}]^2. \quad (78)$$

The entanglement between the system and the path-detector for this case is

$$\varepsilon'^2(t) = |G(t)|^2 \varepsilon^2 + 4|G(t)|^2 [1 - |G(t)|^2] \rho_{22}^2. \quad (79)$$

Combining Eqs. (76), (78), and (79), we arrive at

$$V'^2(t) + P'^2(t) + \varepsilon'^2(t) = 1, \quad (80)$$

confirming that the complementarity (trality) relation is exactly preserved even in the presence of non-Markovian amplitude damping.

B. Qutrit system under non-Markovian amplitude damping

We now analyze a qutrit system coupled to a structured reservoir that induces non-Markovian amplitude damping.

The time-dependent Kraus operators are

$$\begin{aligned} E_0(t) &= |0\rangle\langle 0| + G_1(t) |1\rangle\langle 1| + G_2(t) |2\rangle\langle 2|, \\ E_1(t) &= \sqrt{1 - |G_1(t)|^2} |0\rangle\langle 1|, \\ E_2(t) &= G_1(t)\sqrt{1 - |G_2(t)|^2} |1\rangle\langle 2|. \end{aligned} \quad (81)$$

Here, $G_1(t)$ and $G_2(t)$ denote the non-Markovian survival amplitudes associated with the excited states $|1\rangle$ and $|2\rangle$, respectively. For a Lorentzian spectral density, these functions are given by

$$\begin{aligned} G_j(t) &= e^{-\lambda t/2} \left[\cosh\left(\frac{\Delta_j t}{2}\right) + \frac{\lambda}{\Delta_j} \sinh\left(\frac{\Delta_j t}{2}\right) \right], \\ \Delta_j &= \sqrt{\lambda^2 - 4g_j^2}, \end{aligned} \quad (82)$$

where g_j denotes the system-environment coupling strength and λ characterizes the reservoir spectral width. The condition $2g_j > \lambda$ corresponds to the non-Markovian regime.

Under this evolution, the diagonal elements of the density matrix transform as

$$\begin{aligned} \rho_{00}(t) &= \rho_{00} + [1 - |G_1(t)|^2]\rho_{11} + [1 - |G_2(t)|^2]\rho_{22}, \\ \rho_{11}(t) &= |G_1(t)|^2\rho_{11} + |G_1(t)|^2[1 - |G_2(t)|^2]\rho_{22}, \\ \rho_{22}(t) &= |G_2(t)|^2\rho_{22}. \end{aligned} \quad (83)$$

The off-diagonal elements decay multiplicatively as

$$\rho_{ij}(t) = G_i(t)G_j^*(t)\rho_{ij}, \quad i \neq j, \quad (84)$$

with $G_0(t) \equiv 1$.

In the Gell-Mann representation, the coherence-related components evolve as

$$s'_{1,2}(t) = |G_1(t)| s_{1,2}, \quad s'_{4,5}(t) = |G_2(t)| s_{4,5}, \quad s'_{6,7}(t) = |G_1(t)G_2(t)| s_{6,7}, \quad (85)$$

while the population-related components s_3 and s_8 evolve due to the redistribution of diagonal populations.

The non-Markovian visibility is therefore

$$\begin{aligned} V'^2(t) &= (s'_1)^2 + (s'_2)^2 + (s'_4)^2 + (s'_5)^2 + (s'_6)^2 + (s'_7)^2 \\ &= |G_1(t)|^2(s_1^2 + s_2^2) + |G_2(t)|^2(s_4^2 + s_5^2) + |G_1(t)G_2(t)|^2(s_6^2 + s_7^2), \end{aligned} \quad (86)$$

which explicitly reveals the possibility of coherence revivals induced by environmental memory effects. The predictability is given by

$$P'^2(t) = (s'_3)^2 + (s'_8)^2, \quad (87)$$

and the entanglement between the qutrit and the path-detector degrees of freedom follows as

$$\epsilon'^2(t) = 1 - \sum_{i=1}^8 (s'_i)^2. \quad (88)$$

Combining Eqs. (86)–(88), we obtain

$$V'^2(t) + P'^2(t) + \epsilon'^2(t) = 1, \quad (89)$$

establishing that the complementarity (triality) relation remains exactly valid for a qutrit system under non-Markovian amplitude damping. While memory effects induce a time-dependent redistribution of coherence, predictability, and entanglement, their fundamental complementarity remains structurally robust.

X. DISCUSSION

The action of environmental noise on a qubit induces a structured redistribution of visibility V , predictability P , and system-environment entanglement ε , whose form depends sensitively on the nature of the noise channel. For Markovian amplitude damping, energy exchange with the environment simultaneously suppresses coherence and redistributes populations, leading to a reduction of visibility accompanied by an increase in predictability. A portion of the lost coherence is irreversibly converted into system-environment entanglement, reflecting the leakage of information into environmental degrees of freedom. The resulting redistribution therefore involves all three quantities and arises from the competing effects of population transfer and coherence decay, rather than from a single monotonic process.

In contrast, non-Markovian amplitude damping introduces environmental memory effects that qualitatively modify this redistribution. The time-dependent survival amplitude permits partial information backflow from the environment, giving rise to the nonmonotonic evolution of both visibility and entanglement. Revivals of coherence are accompanied by corresponding reductions in predictability and entanglement, indicating a temporary reversal of information flow. This behavior demonstrates that system-environment entanglement does not merely act as a sink for coherence but actively participates in its redistribution under memory-assisted dynamics.

Phase damping leads to a simpler yet conceptually distinct redistribution pathway. In the Markovian regime, populations remain unchanged and predictability therefore remains invariant. The decay of visibility arises solely from dephasing, with the lost coherence being transferred entirely into system-environment entanglement. When phase damping becomes non-Markovian, this redistribution between visibility and entanglement becomes time dependent: Coherence revivals are accompanied by a reduction in entanglement, while predictability remains fixed throughout the evolution.

Across all these cases, although the detailed dynamical behavior of V , P , and ε is strongly channel dependent, their redistribution is not arbitrary. Instead, it is tightly constrained by complementarity, which enforces the exact conservation of the triality relation

$$V^2 + P^2 + \varepsilon^2 = 1.$$

This identifies complementarity as a structural principle governing information flow in open quantum systems, valid not only for irreversible decoherence but also in regimes characterized by environmental memory and information backflow.

XI. CONCLUSIONS

In this work, we have carried out a detailed analytical investigation of the complementarity relation among visibility, predictability, and entanglement for two- and three-path interferometric systems. Using the system-path framework, we derived closed-form expressions for these quantities in both qubit and qutrit settings and confirmed that the triality condition $V^2 + P^2 + \varepsilon^2 = 1$ holds exactly for pure, isolated systems. We examined how this relation behaves under two widely studied noisy quantum channels: amplitude damping and phase damping. For both qubit and qutrit systems, our results show that although decoherence redistributes coherence, population imbalance, and system-environment correlations in distinct ways, the triality relation remains strictly preserved. Amplitude damping affects both coherence and populations, while phase damping suppresses only the off-diagonal elements; yet, in both cases, the sum of visibility, predictability, and entanglement

continues to satisfy the complementarity equality. We also verified the preservation of triality relations for non-Markovian amplitude-damping noise.

These findings demonstrate that the triality relation is structurally robust, not only in closed interferometric systems but also under realistic open-system dynamics. The unified treatment in two- and three-dimensional Hilbert spaces provides a consistent framework for understanding how coherence and entanglement coevolve under noise. The analysis presented here may support future investigations of complementarity in higher-dimensional systems, non-Markovian evolution, and experimentally relevant photonic, atomic, or classical-quantum analog platforms.

DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX A: ILLUSTRATIVE EXAMPLE— EVOLUTION OF V'^2 , P'^2 , AND ε'^2 UNDER AMPLITUDE DAMPING

To visualize the effect of amplitude damping, we consider simple maximally coherent initial states for qubit and qutrit systems and track the evolution of coherence, predictability, and entanglement.

1. Qubit example

We take the initial qubit state as the pure, equal superposition

$$\rho = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \quad (\text{A1})$$

for which the initial values are $V^2 = 1$, $P^2 = 0$, and $\varepsilon^2 = 0$. After amplitude damping, the evolved state becomes

$$\rho' = \begin{pmatrix} \frac{1}{2} + \frac{\gamma}{2} & \frac{1}{2}\sqrt{1-\gamma} \\ \frac{1}{2}\sqrt{1-\gamma} & \frac{1}{2}(1-\gamma) \end{pmatrix}. \quad (\text{A2})$$

Now, we obtain

$$V'^2 = 1 - \gamma, \quad P'^2 = \gamma^2, \quad \varepsilon'^2 = 1 - V'^2 - P'^2 = \gamma - \gamma^2. \quad (\text{A3})$$

2. Qutrit example

For the qutrit, we take the maximally coherent state

$$\rho = \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad (\text{A4})$$

which yields $V^2 = 1$, $P^2 = 0$, and $\varepsilon^2 = 0$. Under cascade amplitude damping with equal decay rates $\gamma_1 = \gamma_2 = \gamma$, the diagonal elements evolve as

$$\rho'_{00} = \frac{1}{3}(1 + \gamma + \gamma^2), \quad \rho'_{11} = \frac{1}{3}(1 - \gamma^2), \quad \rho'_{22} = \frac{1}{3}(1 - \gamma), \quad (\text{A5})$$

while all off-diagonal coherences are multiplied by $(1 - \gamma)$. Thus,

$$V'^2 = (1 - \gamma)^2. \quad (\text{A6})$$

The updated predictability, using Eq. (2), simplifies to

$$P'^2 = \frac{1}{2}[(\rho'_{00} - \rho'_{11})^2 + (\rho'_{11} - \rho'_{22})^2 + (\rho'_{22} - \rho'_{00})^2], \quad (\text{A7})$$

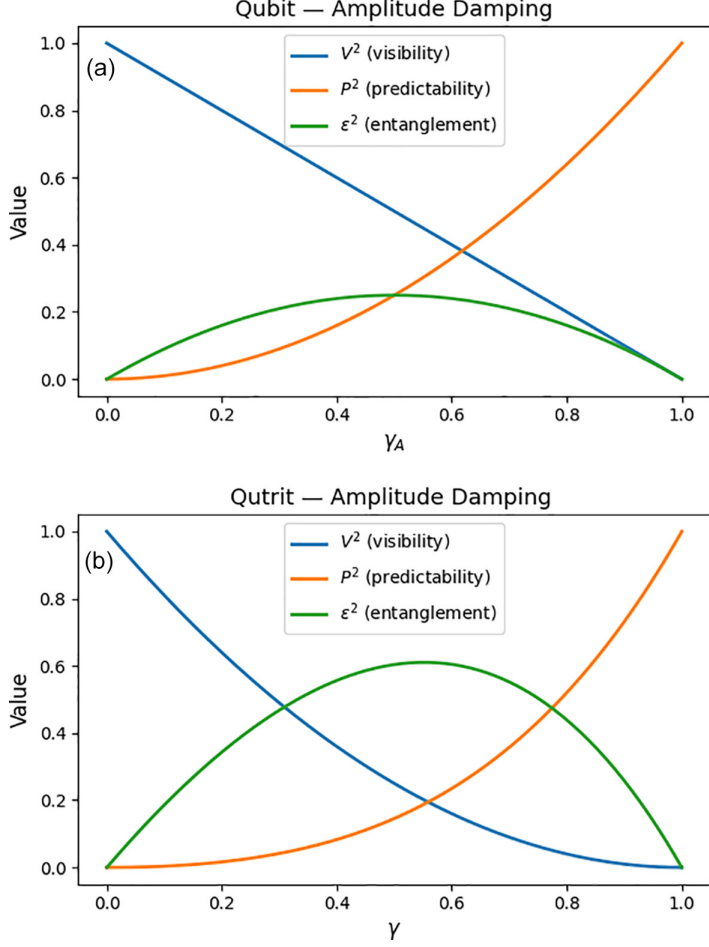


FIG. 4. (a) Qubit and (b) qutrit visibility, predictability, and entanglement under amplitude damping.

and entanglement again follows from the triality condition,

$$\varepsilon'^2 = 1 - V'^2 - P'^2. \quad (\text{A8})$$

APPENDIX B: ILLUSTRATIVE EXAMPLES—EVOLUTION OF V'^2 , P'^2 , AND ε'^2 UNDER PHASE DAMPING

We now analyze the effect of phase damping (pure dephasing) on the same qubit and qutrit systems. The relevant Kraus operators and transformations correspond to Eqs. (52)–(62). The behavior under amplitude damping is presented in Fig. 4. The phase-damping behavior is shown in Fig. 5. The non-Markovian amplitude-damping dynamics are illustrated in Fig. 6.

1. Qubit example

We use phase-damping channel for a qubit to the maximally coherent state (A1); the evolved density matrix becomes

$$\rho' = \begin{pmatrix} \frac{1}{2} & \frac{1}{2}\sqrt{1-\gamma_p} \\ \frac{1}{2}\sqrt{1-\gamma_p} & \frac{1}{2} \end{pmatrix}. \quad (\text{B1})$$

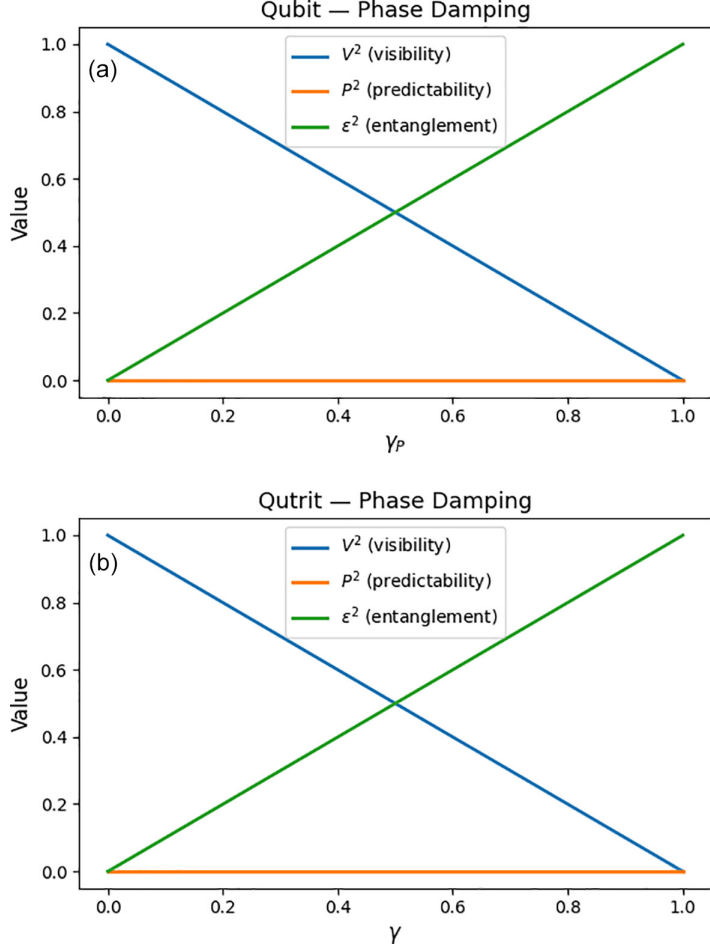


FIG. 5. (a) Qubit and (b) qutrit visibility, predictability, and entanglement under phase damping.

The off-diagonal terms decay by $\sqrt{1 - \gamma_p}$, while the populations remain constant. Thus,

$$V'^2 = 1 - \gamma_p, \quad P'^2 = 0, \quad \varepsilon'^2 = \gamma_p, \quad (\text{B2})$$

satisfying

$$V'^2 + P'^2 + \varepsilon'^2 = 1. \quad (\text{B3})$$

2. Qutrit example

For the three-level system, applying the dephasing channel to the maximally coherent qutrit state (A4) yields

$$\rho' = \frac{1}{3} \begin{pmatrix} 1 & \sqrt{1 - \gamma_p} & \sqrt{1 - \gamma_p} \\ \sqrt{1 - \gamma_p} & 1 & \sqrt{1 - \gamma_p} \\ \sqrt{1 - \gamma_p} & \sqrt{1 - \gamma_p} & 1 \end{pmatrix}. \quad (\text{B4})$$

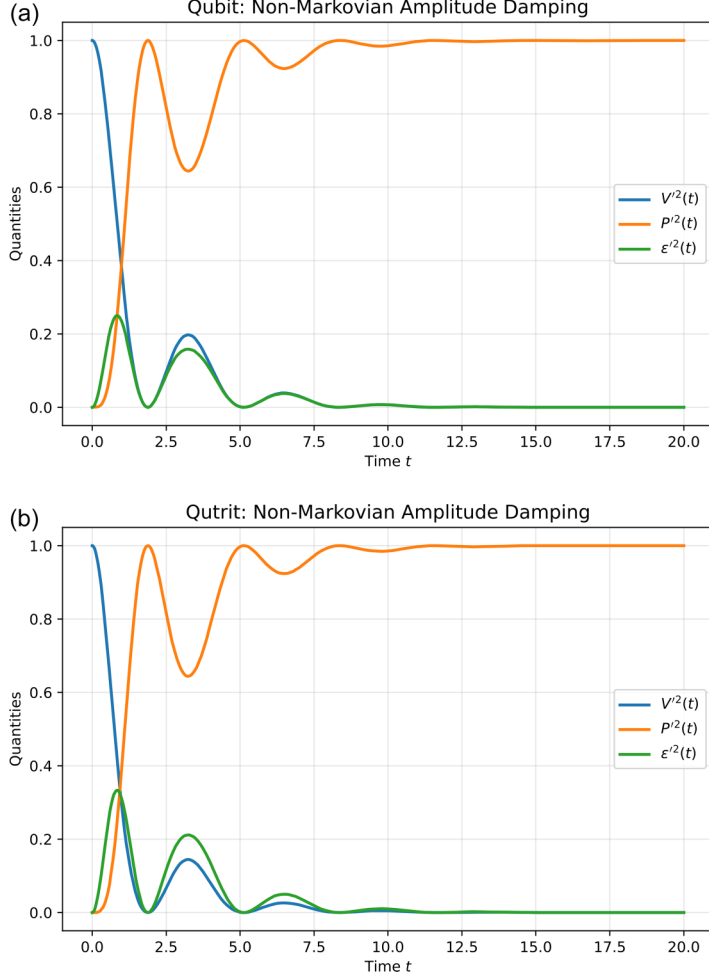


FIG. 6. (a) Qubit and (b) qutrit visibility, predictability, and entanglement under non-Markovian amplitude damping. For the qutrit case, we consider $G_1(t) = G_2(t)$.

Since only the coherences are attenuated,

$$V'^2 = 1 - \gamma_p, \quad P'^2 = 0, \quad \varepsilon'^2 = \gamma_p, \quad (\text{B5})$$

so that $V'^2 + P'^2 + \varepsilon'^2 = 1$ holds.

APPENDIX C: ILLUSTRATIVE EXAMPLES—EVOLUTION OF V'^2 , P'^2 , AND ε'^2 UNDER NON-MARKOVIAN AMPLITUDE DAMPING

1. Qubit example

For the initial state (A1), we obtain

$$V'^2(t) = |G(t)|^2, \quad P'^2(t) = [1 - |G(t)|^2]^2, \quad \varepsilon'^2(t) = |G(t)|^2[1 - |G(t)|^2], \quad (\text{C1})$$

which clearly illustrates the revival of coherence and entanglement induced by non-Markovian memory effects.

2. Qutrit example

We start from the maximally coherent initial qutrit state (A4). We obtain the following equations:

$$V^2(t) = \frac{1}{3}[|G_1(t)|^2 + |G_2(t)|^2 + |G_1(t)G_2(t)|^2], \quad (C2)$$

$$P^2(t) = \frac{1}{3}[|G_1(t)|^4 + |G_2(t)|^4 + |G_1(t)G_2(t)|^4 - |G_1(t)|^2|G_2(t)|^2], \quad (C3)$$

$$\varepsilon^2(t) = \frac{1}{3}[2|G_1(t)|^2 + 2|G_2(t)|^2 - |G_1(t)|^4 - |G_2(t)|^4 - 2|G_1(t)G_2(t)|^2]. \quad (C4)$$

Consistency checks

At the initial time $t = 0$, $G_1(0) = G_2(0) = 1$, which yields

$$\varepsilon^2(0) = 0. \quad (C5)$$

In the long-time limit $G_1(t), G_2(t) \rightarrow 0$, the entanglement vanishes,

$$\varepsilon^2(t) \rightarrow 0, \quad (C6)$$

corresponding to a classical mixture. Finally, the complementarity (triality) relation

$$V^2(t) + P^2(t) + \varepsilon^2(t) = 1 \quad (C7)$$

is exactly preserved for all times.

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