

Interaction-range-controlled crossover from first-order to continuous melting in two-dimensional diplon crystals

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Molecular dynamics simulations of two-dimensional (2D) diplon crystals (electrons on helium films) reveal a melting crossover from first order to continuous as film thickness d grows relative to lattice constant a . At small d/a , hysteresis persists at $N = 40\,000$ alongside a strong Binder-cumulant anomaly. At large d/a , hysteresis vanishes and orientational correlations decay algebraically with $\eta_6 = 0.31 \pm 0.04$, near the Kosterlitz-Thouless-Halperin-Nelson-Young bound of $1/4$, consistent with continuous melting within the investigated system sizes and numerical resolution; disclination unbinding limits any hexatic phase to $\Delta T/T_m < 0.5\%$. The diplon system thus tunes 2D melting, relevant to electrons-on-helium quantum computing.

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Two-dimensional (2D) melting remains a fundamental problem in condensed matter physics. The Kosterlitz-Thouless-Halperin-Nelson-Young (KTHNY) theory [1–4] predicts that 2D solids melt through two continuous transitions mediated by topological defects, with an intermediate hexatic phase exhibiting quasi-long-range orientational order. This scenario has been confirmed in colloidal systems [5–7] and hard-disk simulations [8,9]. However, for sufficiently long-range interactions, 2D melting can be first order [10,11], bypassing the hexatic phase entirely. The conditions controlling this crossover remain poorly understood. While the melting mechanism has been studied as a function of particle softness [12], density [13], confinement [14], and screening length [15–17], few prior works have continuously tuned the melting mechanism within a single physical system through interaction range alone. Beyond fundamental interest, understanding the stability and melting of 2D electron crystals is directly relevant to electrons-on-helium platforms proposed for quantum computing [18,19], where maintaining crystalline order is essential for qubit coherence and addressability.

Electrons confined at the surface of a helium film of thickness d , deposited on a substrate, form charge complexes known as diplons [20–25]. The diplon-diplon interaction,

$$V(r) = \frac{4e^2}{\varepsilon_s + 1} \left[\frac{1}{r} - \frac{1}{\sqrt{r^2 + d^2}} \right] + \frac{e^2(\varepsilon_s - 1)}{\varepsilon_s + 1} \left[\frac{1}{r} - \frac{1}{\sqrt{r^2 + (2d)^2}} \right], \quad (1)$$

where ε_s is the substrate dielectric constant, crosses over from Coulombic ($\sim 1/r$) at $r \ll d$ to dipolar ($\sim 1/r^3$) at $r \gg d$. The first term represents the direct electron-electron and electron–image-charge

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interactions through the helium film, while the second accounts for substrate polarization. Both terms share the same functional form $[1/r - 1/\sqrt{r^2 + X^2}]$ but differ in coupling strength and screening length (d versus $2d$). The ratio d/a , where a is the triangular-lattice nearest-neighbor distance, controls the effective interaction range. In previous studies of $N = 1600$ particles using either the explicit dipion potential [26] or a single-parameter screened Wigner-crystal model [17], first-order melting was reported, but those works employed limited system sizes, a coarse temperature grid, and lacked modern KTHNY diagnostics. Here, we revisit the dipion system with complete Ewald summation [27,28], system sizes up to $N = 40\,000$, and comprehensive KTHNY analysis to show that the melting mechanism evolves from first order at small d/a to behavior consistent with continuous melting as d/a increases.

We simulate N dipions on a 2D triangular lattice with periodic boundary conditions using velocity-Verlet (VV) molecular dynamics with periodic velocity rescaling. Each dipion is modeled as a single classical particle of mass $m = 3.82 \times 10^{-23}$ g (the Na^+ ion mass, following Ref. [26]); this corresponds to the picture of an electron rigidly coupled to its substrate image charge, the pair translating as a single complex during melting, and equilibrium structural observables are independent of this choice. The substrate dielectric constant is $\epsilon_s = 17$ and the areal density is $n = 1.477 \times 10^8 \text{ cm}^{-2}$, giving the nearest-neighbor distance $a = \sqrt{2/(n\sqrt{3})} = 8842 \text{ \AA}$. Six film thicknesses are studied: $d = 100, 500, 1000, 2000, 3000$, and $5000 \text{ \AA}$, corresponding to $d/a = 0.011\text{--}0.565$. Long-range interactions are computed using complete Ewald summation [27,28] with parameters scaled to system size. The real-space cutoff is $r_{\text{cut}} = 25a$. Each temperature point consists of 5×10^4 equilibration steps followed by 2×10^5 production steps (time step 10^{-13} s). Configurations are saved every 500 steps (400 snapshots per temperature). Both heating and cooling protocols are employed. Three system sizes are studied: $N = 2500, 10\,000$, and $40\,000$, with finite-size scaling (FSS) carried out at three representative thicknesses $d = 100, 1000$, and $5000 \text{ \AA}$. To obtain canonical energy fluctuations at the largest size, we reran these three thicknesses at $N = 40\,000$ using a BAOAB Langevin integrator [29] with friction coefficient $\gamma = 10^{10} \text{ s}^{-1}$. In the solid phase ($T < 0.8 T_m$), the hexatic order parameter $|\psi_6|$ agrees to within 0.01 between VV+rescaling and Langevin thermostats [root-mean-square deviation (RMSD) < 0.011], confirming that configurational sampling is thermostat independent in equilibrium [30]. Near the transition, the overdamped Langevin dynamics produces broader metastable regions and apparent T_m shifts of +9% ($d = 5000 \text{ \AA}$), +21% ($d = 1000 \text{ \AA}$), and +57% ($d = 100 \text{ \AA}$) relative to VV+rescaling; we therefore restrict all energy-fluctuation claims to the canonically sampled $N = 40\,000$ Langevin data, where these shifts are quantified [30]. Structural observables (ψ_6 , $g_6(r)$, defect counts) are based on VV+rescaling trajectories throughout. We have verified that the integrated energy autocorrelation time satisfies $\tau_{\text{int}}/N_{\text{samples}} < 0.4$ at every temperature, that the block-averaged energy over the four equal sub-blocks of the 2×10^5 production trajectory differs by less than two standard errors of the block mean while the block-averaged $|\psi_6|$ is stable to within 0.02, and that a finite heating-cooling hysteresis at $d = 100 \text{ \AA}$, $N = 10\,000$ is present in each of three independent random-seed replicas [30].

For each saved configuration, we compute (1) the global hexatic order parameter

$$|\psi_6| = \left| \frac{1}{N} \sum_j \frac{1}{n_j} \sum_k e^{6i\theta_{jk}} \right|, \quad (2)$$

where the sum runs over Delaunay neighbors k of particle j ; (2) the orientational correlation function $g_6(r) = \langle \psi_6^*(r') \psi_6(r' + r) \rangle$, fit to algebraic [$g_6(r) \sim r^{-\eta_6}$] or exponential [$g_6(r) \sim \exp(-r/\xi)$] models using Akaike information criterion (AIC) selection; (3) topological defects from Delaunay triangulation (coordination numbers, 5–7 dislocation pairs, and free disclinations); and (4) the hexatic susceptibility $\chi_6 = N(\langle |\psi_6|^2 \rangle - \langle |\psi_6| \rangle^2)$. Melting temperatures T_m are determined from the interpolated $|\psi_6| = 0.3$ crossing. Error bars on η_6 are obtained from block bootstrap resampling (20 blocks of 20 snapshots).

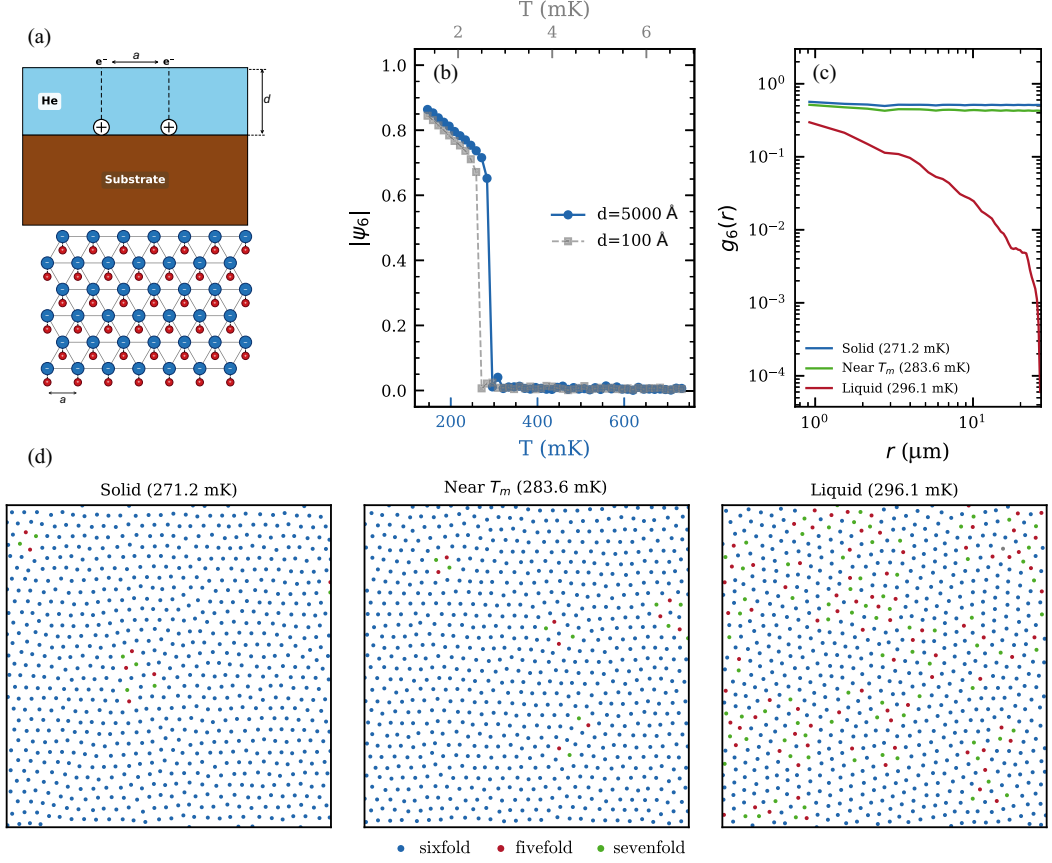


FIG. 1. System setup and continuous melting at large d/a (Coulomb-leaning regime). (b)–(d) $d = 5000 \text{ \AA}$, $d/a = 0.565$, and $N = 40\,000$; lattice constant $a = 8842 \text{ \AA}$. Panels (b)–(d) use VV+rescaling structural trajectories. (a) Schematic of the dipion system (illustrative, not to scale). Top: Cross section showing electrons (e^-) on the helium surface bound to image charges (+) in the substrate, separated by the helium film thickness d . Bottom: Top-down view of the resulting triangular lattice of dipions with nearest-neighbor distance a . (b) Hexatic order parameter $|\psi_6|(T)$ for $d = 5000 \text{ \AA}$ (blue circles, bottom axis) and $d = 100 \text{ \AA}$ (gray squares, top axis). Both order parameters drop at the respective melting temperatures; the first-order vs continuous character of the transition is established quantitatively in the text from the hysteresis (vs d/a) and the orientational decay exponent. (c) Orientational correlation function $g_6(r)$ at three temperatures: solid (271.2 mK, blue), near T_m (283.6 mK, green), and liquid (296.1 mK, red). The algebraic decay near T_m has exponent $\eta_6 = 0.31 \pm 0.04$. (d) Voronoi snapshots colored by coordination number (blue = sixfold, red = fivefold, and green = sevenfold) at the same three temperatures, showing progressive defect proliferation from solid to liquid.

The dipion system is sketched in Fig. 1(a): An electron on the helium surface binds to an image charge in the underlying substrate, and the resulting electron–image-charge complexes arrange on a triangular lattice. We first examine the large- d/a (Coulomb-leaning) end of the parameter range. At $d = 5000 \text{ \AA}$ ($d/a = 0.565$), the hexatic order parameter $|\psi_6|(T)$ [Fig. 1(b)] drops over $\sim 10 \text{ mK}$; while the relative drop magnitude is comparable to the $d = 100 \text{ \AA}$ case (gray squares, top axis), the $d = 5000 \text{ \AA}$ curve carries no significant heating-cooling hysteresis (in marked contrast to $d = 100 \text{ \AA}$, Fig. 2) and $g_6(r)$ decays algebraically with $\eta_6 = 0.31 \pm 0.04$, consistent with a continuous transition within the system sizes and temperature resolution investigated. A fine temperature grid (1.27 mK spacing, 80 points around T_m) reveals $T_m \approx 285 \pm 0.6 \text{ mK}$, identical for heating and cooling within this resolution at both $N = 10\,000$ and $40\,000$, while the 12.5 mK

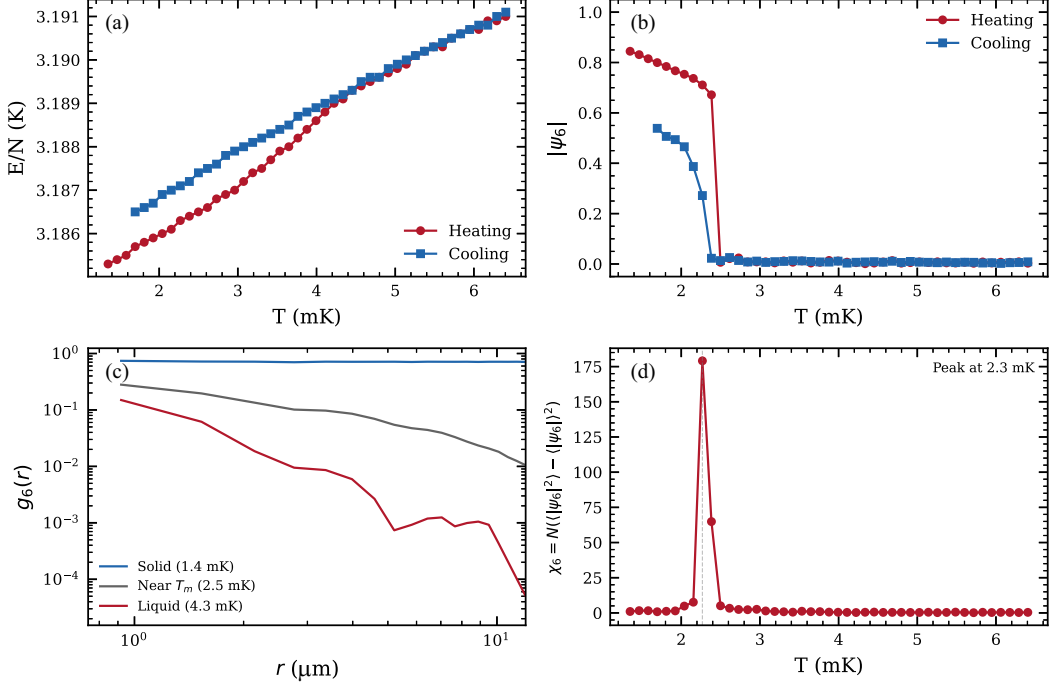


FIG. 2. First-order melting at small d/a (dipolar limit) ($d = 100 \text{ \AA}$, $d/a = 0.011$, $N = 40\,000$). Panel (a) uses the BAOAB Langevin thermostat (canonical energy fluctuations); panels (b)–(d) use VV+rescaling structural trajectories. (a) Energy per particle E/N vs temperature for heating (red) and cooling (blue) protocols. (b) $|\psi_6|(T)$ for heating (red) and cooling (blue), with sharp discontinuity and visible thermal hysteresis. (c) $g_6(r)$ at solid, near- T_m , and liquid temperatures. (d) Susceptibility $\chi_6 = N(\langle |\psi_6|^2 \rangle - \langle |\psi_6| \rangle^2)$ showing a sharp peak at the melting transition.

standard grid yields a residual $\sim 1.3\%$ gap consistent with grid-resolution limits rather than true thermodynamic hysteresis. At the last solidlike temperature ($T = 283.6 \text{ mK}$, $N = 40\,000$), $g_6(r)$ decays algebraically with $\eta_6 = 0.31 \pm 0.04$ [Fig. 1(c)], exceeding the KTHNY hexatic-liquid boundary of $1/4$. However, Voronoi analysis already reveals 136 free disclinations alongside 1013 free dislocations at this temperature [Fig. 1(d)], rising to 490 free disclinations and 2076 dislocations at the last algebraic fine-grid temperature ($T = 285.3 \text{ mK}$, $\eta_6 = 0.33$), indicating that dislocation and disclination unbinding occur nearly simultaneously. No temperature exhibits a pure hexatic signature. If a hexatic phase exists, its window is narrower than 1.3 mK ($\Delta T/T_m < 0.5\%$), consistent with barely resolvable hexatic phases in hard disks [9] and soft particles [12,31].

In contrast, at $d = 100 \text{ \AA}$ ($d/a = 0.011$, small d/a , dipolar limit), melting is clearly first order (Fig. 2). The order parameter $|\psi_6|$ drops sharply from ~ 0.7 to < 0.1 within a single temperature step ($\sim 0.1 \text{ mK}$). The correlation function $g_6(r)$ transitions directly from algebraic to exponential decay with no intermediate regime, and $\eta_6 \sim 1.2$ at the last solidlike temperature far exceeds the KTHNY bound of $1/4$. Both E/N and $|\psi_6|$ exhibit thermal hysteresis between heating and cooling, and the susceptibility χ_6 shows a sharp peak at the melting transition. FSS confirms the first-order nature: Hysteresis $\Delta T/T_m$ is 3.7% at $N = 2500$ and remains $8.8 \pm 3.0\%$ at $N = 40\,000$ (nonmonotonic N scaling reflects competing nucleation barriers). The energy Binder cumulant

$$V_L = 1 - \frac{\langle (E - \langle E \rangle)^4 \rangle}{3 \langle (E - \langle E \rangle)^2 \rangle^2} \quad (3)$$

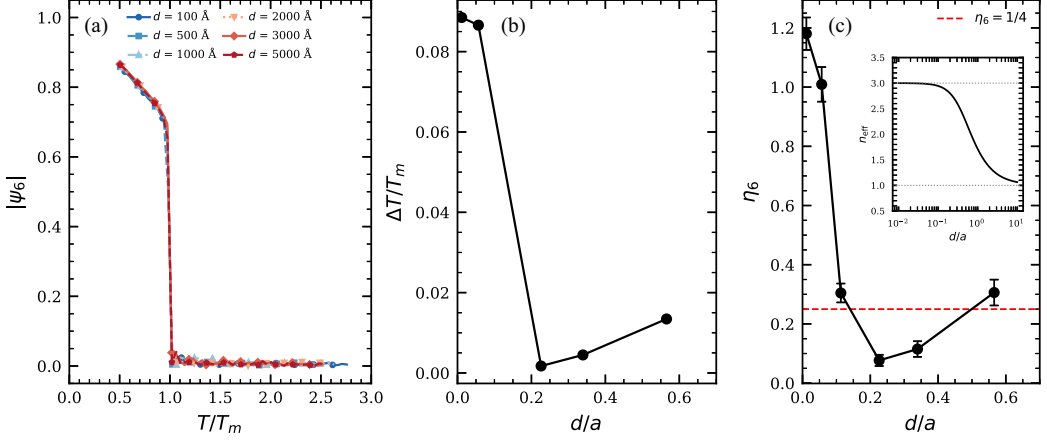


FIG. 3. Interaction-range-controlled crossover ($N = 40\,000$). (a) $|\psi_6|(T/T_m)$ for all six d values. When rescaled by T_m , the order parameter curves nearly collapse onto a common form; the crossover from first-order (small d/a) to continuous (large d/a) melting is resolved quantitatively in panels (b) and (c). (b) Thermal hysteresis $\Delta T/T_m$ vs d/a , dropping from $\sim 9\%$ at $d/a \leq 0.06$ to $\sim 0.2\%$ at $d/a = 0.23$, with a residual $\sim 1.3\%$ at $d/a = 0.57$ attributed to standard-grid resolution (fine-grid analysis at $d = 5000\text{ Å}$ gives zero hysteresis within 1.3 mK). (c) Orientational decay exponent η_6 vs d/a (filled circles with bootstrap error bars); dashed line marks the KTHNY hexatic bound $\eta_6 = 1/4$. η_6 is nonmonotonic (see main text). Inset: Effective interaction exponent n_{eff} at the nearest-neighbor distance, evolving from ≈ 3 (dipolar, small d/a) to ≈ 1 (Coulomb, large d/a).

computed from the Langevin energy time series departs from the Gaussian value ($V_L = 0$ for kurtosis 3) near T_m and rises toward the bimodal-coexistence value ($2/3$ for symmetric latent-heat splitting), independently confirming first-order character [30].

The first-order classification at small d/a is surprising in light of the asymptotic $1/r^3$ tail of Eq. (1), which formally places the dipole potential in the KTHNY-stable universality class for all d . We interpret the small- d/a result as a preemption of the KTHNY scenario by a defect-driven mechanism rather than as a change in asymptotic universality. Three lines of quantitative evidence support this picture. First, the latent-heat proxy $\Delta E/N$ at T_m , evaluated as the energy jump across the steepest segment of $E(T)$ (see Sec. S9 of the Supplemental Material [30]), is independent of N at $d = 100\text{ Å}$ ($\Delta E/N = 0.0007\text{ K}$ at $N = 2500, 10\,000$, and $40\,000$), consistent with the N -independent latent heat expected at a first-order transition. Second, the dislocation core energy E_c , extracted from the solid-side Arrhenius dependence of the free-dislocation density (see Sec. S9 of the Supplemental Material [30]), is $E_c/k_B T_m = 5.6$ (5.0 – 5.9 at 68% confidence) at $d = 100\text{ Å}$: statistically below the $d = 5000\text{ Å}$ value of 6.7 (6.3 – 6.9) and within $\sim 30\%$ of the Nelson-Halperin lower bound ($\sim 4 k_B T_m$) for KTHNY stability. Third, the canonical energy Binder cumulant V_L [Eq. (3)], computed from the $N = 40\,000$ Langevin time series, rises toward its bimodal-coexistence value in the transition region at $d = 100\text{ Å}$ while remaining near zero at $d = 5000\text{ Å}$ (see Sec. S2 of the Supplemental Material [30]). Together, these signatures are consistent with grain-boundary-mediated first-order melting at small d/a rather than with the gradual two-stage defect unbinding scenario of KTHNY [3,4], in line with long-standing theoretical scenarios for first-order preemption of KTHNY by abrupt defect proliferation [10,32,33].

Figure 3 presents the central result: the systematic evolution of melting as d/a increases from 0.011 to 0.565. Three independent metrics confirm the crossover. First, hysteresis $\Delta T/T_m$ drops by more than an order of magnitude from $\sim 8.9\%$ at small d/a ($d/a \leq 0.06$) to $\sim 0.2\%$ at $d/a = 0.23$, with a residual $\sim 1.3\%$ at $d/a = 0.57$ attributed to standard-grid resolution (the fine-grid measurement at $d/a = 0.565$ yields zero hysteresis within 1.3 mK) [Fig. 3(b)]. Second,

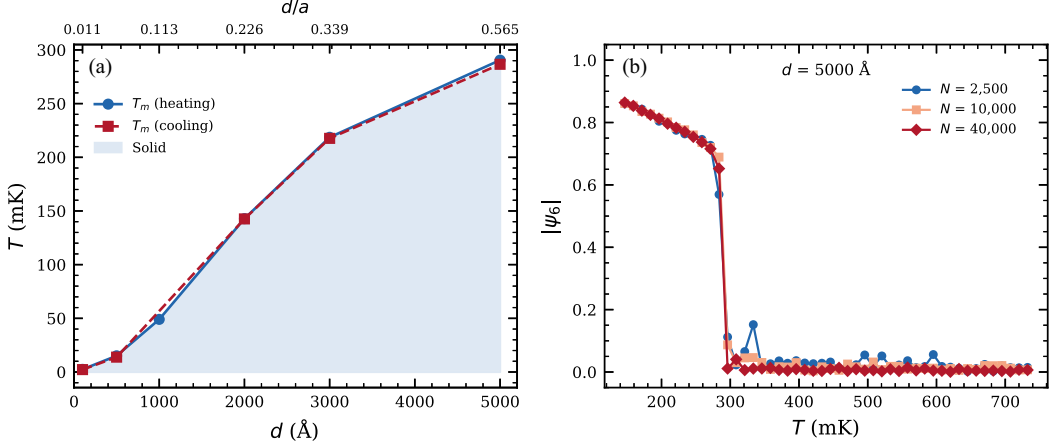


FIG. 4. Phase diagram and finite-size scaling. (a) Melting temperature T_m vs d (lower axis) and d/a (upper axis) from heating (blue circles, solid line) and cooling (red squares, dashed line) at $N = 40\,000$. Blue shading marks the solid phase; the narrow green region between the heating and cooling curves marks the hysteresis interval, which closes (within standard-grid resolution) for $d \geq 2000$ Å. (b) $|\psi_6|(T)$ at $d = 5000$ Å for $N = 2500$, $10\,000$, and $40\,000$, showing the transition sharpening with system size while T_m converges—consistent with continuous melting.

when rescaled by T_m the $|\psi_6|(T/T_m)$ curves nearly collapse onto a common form [Fig. 3(a)]. Third, η_6 varies systematically: At small d/a (≤ 0.06), $\eta_6 \sim 1.0$ – 1.2 (poor power-law fits to first-order configurations); at $d/a = 0.565$, $\eta_6 = 0.31 \pm 0.04$ approaches the KTHNY bound [Fig. 3(c)]. Notably, η_6 is nonmonotonic, falling below $1/4$ at intermediate d/a , where AIC selects algebraic fits with shallow exponents; this reflects the gradual loss of long-range orientational order rather than a true KTHNY hexatic. The effective interaction exponent $n_{\text{eff}} \equiv -d \ln V/d \ln r$, evaluated at the nearest-neighbor distance $r = a$, evolves from $n_{\text{eff}} \approx 3$ (dipolar limit, small d/a) to $n_{\text{eff}} \approx 1$ (Coulomb limit, large d/a) [Fig. 3(c), inset].

The melting temperature increases monotonically from $T_m \approx 2.45$ mK at $d = 100$ Å to $T_m \approx 285$ mK at $d = 5000$ Å (fine-grid value) [Fig. 4(a)]. The standard-grid heating-cooling gap narrows with increasing d and is statistically indistinguishable for $d \geq 2000$ Å at $N = 40\,000$ within standard-grid resolution. FSS at $d = 5000$ Å [Fig. 4(b)] shows $|\psi_6|(T)$ sharpening with N while T_m converges, consistent with continuous melting within the sizes investigated [8]. In contrast, the $d = 100$ Å hysteresis remains substantial ($\sim 8.8\%$) at $N = 40\,000$ [Fig. 2; quantified in Fig. 3(b)].

The dipion system offers a single, tunable setting for two-dimensional melting in which the mechanism is determined by the short-distance, Coulomb-like part of the interaction in Eq. (1) rather than by its long-distance dipolar ($1/r^3$) tail. The film thickness d tunes the effective interaction at the nearest-neighbor scale from dipolar (small d/a) to Coulombic (large d/a), distinct from approaches based on particle softness [12], screening [15–17], or confinement [14]. For interactions decaying asymptotically as $V(r) \sim r^{-n}$, KTHNY theory predicts continuous melting for $n \geq 3$ and first-order melting for $n < 2$ [10,11]. The dipion potential *always* has a dipolar long-range tail ($V \sim 1/r^3$ at $r \gg d$), which formally places it in the KTHNY-stable regime for all d ; the d/a dependence we observe therefore reflects the strength of the short-range Coulombic core ($V \sim 1/r$ at $r \ll d$) at the nearest-neighbor scale rather than a change in asymptotic universality class. The local n_{eff} at $r = a$ [Fig. 3(c), inset] is a short-range descriptor and should not be directly compared with the KTHNY n threshold; within this caveat, the local crossover at $1.5 < n_{\text{eff}} < 2.2$ corresponds to $d/a \sim 0.1$ – 0.3 [34]. We attribute the change of melting mechanism to the energy cost of creating dislocations: As d/a grows, the interaction at the nearest-neighbor distance becomes more Coulomb-like and the lattice stiffer, so the dislocation core energy is larger at the Coulomb-like end ($E_c \sim 6.7 k_B T_m$ at

$d = 5000 \text{ \AA}$) than at the dipolar end ($4.3\text{--}5.6 k_B T_m$ at $d \leq 500 \text{ \AA}$) (see Sec. S9 of the Supplemental Material [30]). At low core energy, dislocations proliferate abruptly and collectively, and a first-order transition preempts the KTHNY mechanism; at high core energy, this collective route is suppressed and the continuous, defect-unbinding melting expected from the long-distance $1/r^3$ form of the potential appears to be recovered. A complementary methodological caveat applies on the continuous side: A strict KTHNY hexatic certification additionally requires verifying the universal jump in the renormalized orientational stiffness (the Frank constant K_A), which encodes how strongly the system resists local rotations of the bond-orientational order. Using the KTHNY identity $\eta_6 = 18 k_B T / (\pi K_A)$ [3], the measured $\eta_6 = 0.31 \pm 0.04$ at $d = 5000 \text{ \AA}$ ($N = 40\,000$, at $T = 283.6 \text{ mK}$, the highest temperature at which $g_6(r)$ still decays as a power law) corresponds to $K_A / (k_B T) \approx 18.5 \pm 2$, below the universal-jump value $72/\pi \approx 22.9$. The renormalized Frank constant has therefore already fallen below the hexatic-stability threshold by the time η_6 reaches $1/4$ from below, consistent with the absence of a clean hexatic phase between the solid and liquid at $d = 5000 \text{ \AA}$. Because K_A is obtained from the same η_6 fit through the KTHNY identity, however, it restates that measurement in stiffness units rather than constituting an independent diagnostic. The present large- d/a analysis moreover does not include size scaling of the hexatic susceptibility χ_6 , of the orientational Binder cumulant, or of the orientational correlation length, and the thermostat-dependent shifts of T_m near melting counsel further caution. A very weakly first-order transition at large d/a therefore cannot be excluded, and the large- d/a results are best read as consistent with continuous melting within the investigated system sizes and numerical resolution rather than as a conclusive demonstration of it. Using the thermal de Broglie wavelength $\lambda_{dB} = h / \sqrt{2\pi m k_B T}$ with the translating-diplon mass m (electron rigidly bound to its substrate image charge, taken as $m = m(\text{Na}^+)$ per Ref. [26]), $\lambda_{dB}/a \approx 8 \times 10^{-4}$ at $d = 5000 \text{ \AA}$ ($T_m \approx 285 \text{ mK}$) and $\lambda_{dB}/a \approx 8 \times 10^{-3}$ at $d = 100 \text{ \AA}$ ($T_m \approx 2.5 \text{ mK}$), so the classical treatment is justified across the full d/a range studied. For the alternative limit of a bare electron, the relevant mass is reduced by a factor $m(\text{Na}^+)/m_e \approx 4 \times 10^4$ and λ_{dB}/a is enhanced by approximately 200, reaching a value of order unity at $d = 100 \text{ \AA}$ and approximately 0.16 at $d = 5000 \text{ \AA}$. In that bare-electron regime, quantum statistics and zero-point motion become non-negligible near T_m and a fully quantum treatment is required; the present classical results should then be read as the leading-order high-temperature description. They remain quantitatively accurate for the dressed electrons-on-helium-film diplons of the experimental literature [20–25], for which the dressing produces a heavy translating complex, and the d/a control parameter identified here, determined by the classical potential of Eq. (1), continues to govern the location of the melting transition in both regimes.

The diplon system is experimentally realizable with electrons on helium films of controlled thickness ($100\text{--}5000 \text{ \AA}$) over a semiconducting substrate [23]. We chose $\varepsilon_s = 17$ as a model parameter inherited from Ref. [26] (representing an idealized substrate); ε_s sets the overall energy scale and the relative weight of the two terms in Eq. (1), so the crossover physics depends primarily on d/a . The connection to Wigner crystals in moiré superlattices [35,36] further motivates understanding how interaction range controls 2D crystalline stability. Our results are also relevant to electrons-on-helium quantum computing platforms [18,19], where crystalline order underpins qubit isolation and coherent manipulation; the demonstrated tunability of the melting transition via film thickness provides a practical control parameter for optimizing the stability of such ordered electron arrays.

In summary, we have shown that the melting mechanism of 2D diplon crystals is controlled by d/a , evolving from strongly first order near $d/a = 0.01$ to behavior consistent with continuous melting near $d/a = 0.6$. The crossover is quantified through thermal hysteresis, the hexatic order parameter, orientational correlations, defect statistics, and a defect-core-energy analysis. The placement of T_m and the d/a ordering are independent of the choice between four alternative definitions of T_m (see Sec. S6 of the Supplemental Material [30]), and the small- d/a first-order classification is reproducible across three independent random seeds at $d = 100 \text{ \AA}$, $N = 10\,000$ (see Sec. S7 of the Supplemental Material [30]). In the large- d/a limit, the transition is approaching the KTHNY hexatic boundary ($\eta_6 \approx 1/4$) but with concurrent disclination unbinding rather than a clean two-step

transition; any hexatic phase is confined to $\Delta T/T_m < 0.5\%$, and the corresponding renormalized Frank constant $K_A \approx 18.5 k_B T$ is already below the universal-jump value $72/\pi$ at the last algebraic temperature, consistent with the absence of a stable hexatic intermediate. Within the investigated system sizes and numerical resolution, the large- d/a data are therefore consistent with continuous melting; conclusively excluding a very weakly first-order transition would require size scaling of the hexatic susceptibility, of the orientational Binder cumulant, and of the orientational correlation length, which we leave to future work. These results establish the dipylon system as a tunable platform for studying the full range of 2D melting phenomenology and provide a practical handle for tuning crystalline stability in electrons-on-helium platforms proposed for quantum computing [18,19]; a fully quantum extension applicable to bare-electron qubits remains a natural next step.

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Data availability. The code and data that support the findings of this article are openly available at Zenodo [38].

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