

Extreme cross-polarization extinction method for quantification of residual crystal anisotropy

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(Received 10 April 2026; revised 2 July 2026; accepted 17 August 2026; published 26 August 2026)

The high-extinction cross-polarization measurement of a paraxial laser beam results in a Hermite-Gaussian (HG_{11})-like four lobes with a dark-centered cross pattern. The azimuthally spin-separated mode pattern is understood to result from the spin-orbit interaction of light. The perfect dark region at the beam center, due to extreme cross-polarization extinction (EXE), is used here to accurately determine the on-axis residual polarization of the laser beam and, using it, the residual elliptical birefringence of optical crystals. By propagating the laser beam through uniaxial optical crystals with their optic axis cut either parallel or perpendicular to the beam propagation direction and performing the EXE measurement, we demonstrate a simple approach to precisely quantify the residual elliptical birefringence of crystals, which are otherwise assumed to be absent, to a high degree of extinction of 1 part in 10^{-8} . The experimental results are interpreted using Jones-matrix-based calculations for a paraxial laser beam and upon its propagation through the optical crystals. The high sensitivity and high accuracy of the EXE technique make it well suited for measuring and characterizing anisotropy and its dispersion of monolayers, ultrathin films, and two-dimensional materials.

DOI: [10.1103/hd3s-n7gd](https://doi.org/10.1103/hd3s-n7gd)

I. INTRODUCTION

Understanding the effects of light propagation through an anisotropic inhomogeneous medium, such as a transparent, birefringent, and optically active crystal, is of immense interest to physicists, chemists, biologists, crystallographers, and engineers. The anisotropy characteristics of such media have been theoretically modeled, experimentally measured, and used in various applications, assuming plane-wave beams and ideal crystals. However, the simultaneous presence of linear birefringence (LB) and circular birefringence (CB) (or optical activity) in such crystals makes the crystal elliptically birefringent (EB) and the natural polarization modes elliptical. Propagating a linearly polarized light through such a crystal results in an elliptically polarized output beam with the ellipse axis rotated away from the input polarization plane. Understanding the propagation of light through such a generic, inhomogeneous, anisotropic medium is quite complicated due to the helical trajectory of the elliptically polarized modes [1]. Thus, propagating a plane-wave beam ($\vec{k}_0$) through the crystal whose optical axis (OA) is cut perpendicular to or parallel to it minimizes the effect of EB, simplifying the crystals to be understood as linear birefringent or circular birefringent,

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ignoring the effects due to CB and LB, respectively. Such crystals with ignorable weak elliptic birefringence are relatively easy to model and understand under the assumption that the LB and CB effects are independent of each other [2,3]. Several experimental techniques have been developed and demonstrated to measure the elliptical birefringence of such media under the assumption that the effects due to CB $\ll$ LB [4–8]. However, not ignoring the overall effects due to the weak EB results in an output beam whose state of polarization is elliptical and its plane of polarization is rotated with respect to the input linear polarized plane-wave beam. The effects are characterized by the azimuth angle ϕ and the ellipticity angle ϵ of the eigen-polarization state and are experimentally measured to characterize crystal birefringence. Though weak, these effects are present in almost all commercial zero- and multiorder wave plates and rotators, but are typically ignored due to the inability of existing measurement techniques to quantify.

Instead of an ideal plane-wave beam, we consider here a paraxial (weakly diverging) beam of light from a laser and its propagation through an anisotropic crystal with its OA oriented either parallel or perpendicular to the surface. A paraxial beam of light from a laser with a Gaussian intensity distribution and a finite angle of divergence has a spherical wave front [9]. To satisfy the Maxwell equations and the transversality of the electromagnetic field, such a beam should have a nonzero longitudinal component (E_z) and a transverse cross-polarization component (E_y) in addition to the input field component (E_x), and the component field strengths are individually invariant under free-space propagation [10]. Passing such a beam through an ideal polarizer, crossed with respect to the principal polarization component of the beam, transmits only the Hermite-Gaussian (HG₁₁) like cross-polarization component [11,12]. The vector superposition of the simultaneously present principal (HG₀₀) and cross-polarization (HG₁₁) components in any cross section of the propagating beam shows that the polarization state of the laser radiation is spatially inhomogeneous [13]. Under the assumption that the polarization inhomogeneity is low, the effect has been neglected, and a laser beam is approximated as a scalar wave beam. However, a careful polarization analysis of a partially polarized laser beam transmitted through a quarter-wave plate (QWP) and a polarizer (P) shows that the scalar beam assumption of a typical laser system is incorrect. In addition, the phase difference between the propagating orthogonal polarized HG modes results in an elliptically polarized output beam with a transverse position-dependent ellipticity angle [13]. Although these free-space propagation effects are very small, the transmission of the paraxial beam through a crystal amplifies them via the accumulation of the Pancharatnam-Berry-type geometric phase.

Propagating through uniaxial crystals, the accumulation of the phase difference between orthogonally polarized ordinary (o) and extraordinary (e) ray-beams results in distance-dependent exchange of spin and orbital angular momentum (AM) [14] in the beam. This has led to the generation of optical vortex [15] and polarization-singular beams [16] of light. Thus, a modern approach to understanding EB in optical crystals and its spatial variation is based on the spin-orbit interaction (SOI) of light [17]. Although the spin and the (extrinsic) orbital AM components of light, respectively, the circular polarization ($\sigma = \pm 1$) and trajectory ($\vec{k}$) are separable in a plane-wave beam [17] and upon its propagation through homogeneous—isotropic medium [18], they become coupled in inhomogeneous—anisotropic media. Optical fibers [19], space variant birefringent [20], and form-birefringent [21] optical elements are some of the early examples of spin-orbit converters for a paraxial beam.

While the cross-polarization component of the laser is a HG₁₁-type four-lobe pattern in the linear basis, it is a Laguerre-Gaussian (LG)-like vortex beam of charge $l = \pm 2$ in the circular basis, with a perfect null at the beam center. Understood as due to SOI, the appearance of the HG₁₁ or the LG₂₀ beam at the focus or due to propagation through a crystal has been used to demonstrate the spin-Hall effect [22] and spin-to-orbit conversion [23]. The appearance of a perfect intensity null at the center of the beam provides the best contrast (relative to the rest of the beam intensity) for high-resolution measurements. The dark cross region has been used for the detection of nanoparticles [24], to measure the photonic spin-Hall effect induced by surface reflection [22], and to measure surface chirality [25]. These measurements demonstrate the importance of the perfect null intensity along the beam axis, enabled by the cross-polarization technique and its significant advantages.

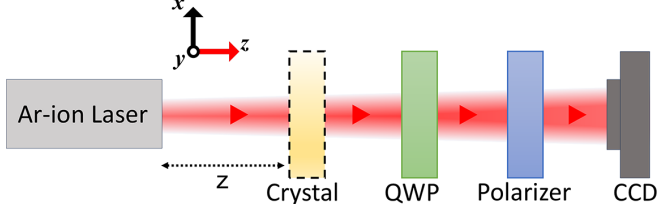


FIG. 1. Schematic of the optical setup used in the simulation and experiment. Ar^+ laser of wavelength $\lambda = 514.5$ nm, sample EB crystal, QWP—quarter-wave plate, P—polarizer, and CCD—camera.

II. WORKING PRINCIPLE

The principle behind the proposed technique and the experimental demonstration is to measure the cross-polarization characteristics of a partially polarized paraxial Gaussian laser beam and use it as a reference to quantify the residual EB of quartz and calcite crystals whose OA are either along or perpendicular to the beam propagation direction. The high spatial resolution (of the size of the charge coupled device (CCD) pixel $\sim 5 \mu\text{m}$) and the sensitivity (up to 10^{-8} , relative to parallel polarization and down to the electronic noise floor of the CCD camera) [26,27] provide unprecedented precision for the method, termed the extreme cross-polarization extinction (EXE) method. The EXE method is two dimensional, which allows us to measure and quantify the polarization ellipticity (ϵ) and the polarization ellipse orientation (azimuth angle) (ϕ) with very high precision in the dark cross region of the beam. Using the laser beam's cross-polarization characteristics, the residual EB of crystals with different orientations is measured. The polarization characteristics of the paraxial beam of light after passing through the crystal are simulated using the Jones matrix method, and the results match well with the experimental measurements. The measured and calculated crystal birefringence values are compared with those in the literature.

The accurate measurement of the polarization characteristics of the laser beam is the basis for the high-sensitivity quantification of the weak residual anisotropy of EB crystals. Our first task is thus to measure the polarization characteristics of the laser beam in detail and to establish the baseline settings for the EXE technique. The schematic of the optical system used is shown in Fig. 1. The paraxial Gaussian intensity beam from a Ar^+ laser ($\lambda = 514.5$ nm) with a divergence angle of 0.036° serves as the light source for the development of the EXE method. The Brewster cavity with a dispersion prism (for wavelength selection) produces a spatially partially polarized laser beam. A CCD camera is used to observe and measure the output beam intensity and its characteristics. To implement the EXE technique and measure the inhomogeneity in the state of elliptical polarization (SoEP) of the light beam, we use spatially resolved Stokes polarimetry and weak polarimetry with a QWP and a polarizer placed before the CCD. Subsequently, sample crystals are kept in the laser beam as shown in Fig. 1 to measure the modifications imparted on the beam's polarization characteristics to accurately quantify the residual polarization ellipticity (ϵ) and the ellipse orientation (ϕ) along the mean beam axis ($\langle \vec{k}_0 \rangle$). Knowing both polarization parameters under EXE conditions allows us to extract high-sensitivity information on the residual EB properties of the crystal, as discussed.

III. THEORY

The polarization characteristics of the paraxial laser beam are modeled using the vectorial angular spectrum method of Richards and Wolf [28]. The spatial distribution of the electric field components in the focal region $\mathbf{E}_f(\rho, \psi, Z)$ is given by

$$\mathbf{E}_f = C_1 \begin{bmatrix} I_0 + I_2 e^{-2i\psi} \\ -iI_0 + iI_2 e^{-2i\psi} \end{bmatrix} + C_2 \begin{bmatrix} I_0 + I_2 e^{2i\psi} \\ iI_0 - iI_2 e^{2i\psi} \end{bmatrix}, \quad (1)$$

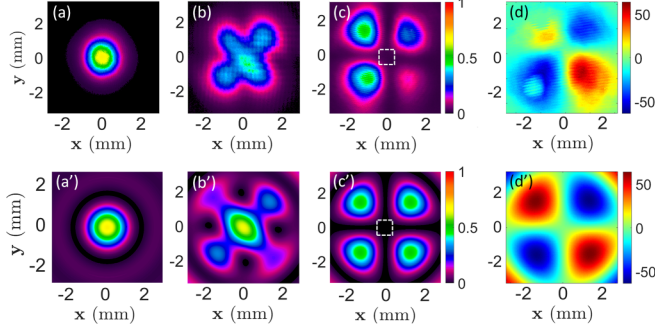


FIG. 2. Experimentally measured [panels (a) and (b)] and simulated [panels (a') and (b')] transverse intensity distribution of the paraxial laser beam for different orientation of the polarization elements. After passing through the optimized analyzer and QWP angles, the total intensity of the beam becomes a minimum, and the nonuniform cross-polarization intensity patterns of panels (b) and (b') become a four-lobe structure [panels (c) and (c')], due to the compensation of the nonuniform elliptical polarization of the laser beam. Panels (d) and (d') are the experimental and simulated S_3 Stokes parameter showing azimuthal spin variation.

where

$$\begin{aligned}
 I_0 &= \int_0^{\theta_{\max}} A(\theta, Z)(1 + \cos \theta) J_0(k \rho \sin \theta) d\theta, \\
 I_1 &= \int_0^{\theta_{\max}} A(\theta, Z) \cos \theta J_1(k \rho \sin \theta) d\theta, \\
 I_2 &= \int_0^{\theta_{\max}} A(\theta, Z)(1 - \cos \theta) J_2(k \rho \sin \theta) d\theta, \\
 A(\theta, Z) &= \frac{i \pi}{\lambda} f E_o \sqrt{\cos \theta} \sin \theta e^{-ikZ \cos \theta}, \\
 C_1 &= e^{i\beta} \cos \alpha \frac{1}{\sqrt{2}} \quad \text{and} \quad C_2 = e^{-i\beta} \sin \alpha \frac{1}{\sqrt{2}}.
 \end{aligned}$$

Here, $\alpha \in [0, \frac{\pi}{2}]$ is the polarization ellipticity and $\beta \in [0, \pi)$ is the azimuthal orientation of the polarization ellipse. Together, α and β are used to characterize the SoEP of the laser beam. To model a TM-polarized input beam (linear polarization parallel to the x axis), we set $\beta = 0^\circ$ and $\alpha = 45^\circ$. The polarization state of the input beam is then varied using the ellipticity angle α and the orientation angle β , to simulate experimental beam conditions with greater accuracy. Here, Z is the axial distance from the laser output, with $Z = 0$ corresponding to the laser output plane and $Z > 0$ indicating positions beyond, along the direction of propagation of the beam.

IV. EXE TECHNIQUE TO CHARACTERIZE LASER BEAM

We first investigate the polarization characteristics of the Ar^+ laser beam using only the polarizer and the CCD camera (Fig. 1). With the Glan-Thomson polarizer (GTH10M, Thorlabs, USA; with $1:10^6$ extinction ratio) mounted on a NanoRotator Stage (NR360S/M, Thorlabs, USA) set to 0° , the TM-polarized beam intensity recorded by the CCD shows a Gaussian intensity profile [Fig. 2(a)], consistent with the expected fundamental mode of the laser. Rotating the polarizer to 90° reveals a distorted beam structure that indicates nonzero polarization ellipticity in the beam cross section with residual central intensity [Fig. 2(b)]. A QWP (AQWP05M-630, Thorlabs, USA) mounted on a motorized rotation stage (K10CR1/M, Thorlabs, USA) is inserted before the polarizer and both elements are rotated simultaneously to minimize the total beam intensity, resulting in a clear

four-lobe pattern and a dark cross pattern [Fig. 2(c)]. The analyzer and QWP angles to achieve a perfect dark center in the cross pattern were 90.374° and -1.303° , respectively (Table I). Although the unequal intensity of the four lobes is attributed to the nonuniform gain in the beam cross section, the stability of the lobes confirms that the effect originates from the intrinsic polarization variation of the laser beam and its divergence [11,12].

The experimentally measured beam profiles are simulated considering the laser beam with a divergence angle of 0.036° and a distance of $Z = 2$ m. Using the polarization parameters (α, β) , simulations were performed using the vectorial Debye-Richards-Wolf model described by Eq. (1). The azimuthal orientation of the polarization ellipse β is taken to be 0° , and with $C1 = \frac{1}{\sqrt{2}} \cos \alpha = a_1$ and $C2 = \frac{1}{\sqrt{2}} \sin \alpha = a_2$, Eq. (1) becomes

$$\mathbf{E}_f = \begin{bmatrix} (a_2 + a_1)(I_0 + I_2 \cos 2\psi) + i(a_2 - a_1)I_2 \sin 2\psi \\ i(a_2 - a_1)(I_0 + I_2 \cos 2\psi) + (a_2 + a_1)I_2 \sin 2\psi \end{bmatrix}. \quad (2)$$

The Jones matrix for the polarizer and the QWP is

$$J_{\text{QWP}} = \frac{1}{\sqrt{2}} \begin{bmatrix} (1 - i) \cos 2\theta_Q & -i \sin 2\theta_Q \\ -i \sin 2\theta_Q & (1 + i) \cos 2\theta_Q \end{bmatrix}, \quad (3)$$

$$J_P = \begin{bmatrix} \cos^2 \theta_P & \cos \theta_P \sin \theta_P \\ \cos \theta_P \sin \theta_P & \sin^2 \theta_P \end{bmatrix}. \quad (4)$$

To obtain the simulated TM component, the beam [Eq. (2)] is passed through the polarizer oriented at $\theta_P = 0^\circ$. Using $\alpha = 44.626^\circ$ gives $(a_2 - a_1) = 0.0007$ and $(a_2 + a_1) = 0.999$, and Eq. (2) simplifies to

$$\mathbf{E}_{f\text{TM}} = \begin{bmatrix} (0.999)(I_0 + I_2 \cos 2\psi) + i(0.0007)I_2 \sin 2\psi \\ 0 \end{bmatrix}.$$

The simulated TM component [Fig. 2(a')] reproduces the Gaussian beam intensity distribution, since the dominant term of $|E_{f\text{TM}}|^2$ is I_0 .

The orthogonal cross-polarization component [Fig. 2(b')] obtained with $\alpha = 44.626^\circ$, and the polarizer angle $\theta_P = 90^\circ$ gives

$$\mathbf{E}_{f\text{TE}} = \begin{bmatrix} 0 \\ i(0.0007)(I_0 + I_2 \cos 2\psi) + (0.999)I_2 \sin 2\psi \end{bmatrix}.$$

It matches the distorted four-lobe pattern, as the dominant term here is I_2 , and $\sin 2\psi$ gives alternate intensity maxima with $\psi \in [0, 2\pi]$, along with nonzero I_0 .

To obtain the proper four-lobe pattern, a QWP is introduced before the polarizer at $\theta_Q = 0^\circ$. After the polarizer at $\theta_P = 90^\circ$, Eq. (2) becomes

$$\mathbf{E}_{f\text{QTE}} = \begin{bmatrix} 0 \\ \frac{(i-1)}{\sqrt{2}}(0.0007)(I_0 + I_2 \cos 2\psi) + \frac{(1+i)}{\sqrt{2}}(0.999)I_2 \sin 2\psi \end{bmatrix}.$$

The QWP compensated output, $|E_{f\text{QTE}}|^2$ —[Fig. 2(c')], shows a symmetric four-lobe structure.

From the experimental measurements, we calculated the Stokes parameter S_3 [Fig. 2(d)] using weak polarimetry to quantify the spatial distribution of polarization ellipticity or spin components in the four-lobe pattern [22]. The two images of the output beam intensity corresponding to the polarizer angles of $90.374 \pm 0.099^\circ$ and the fixed QWP angle of -1.303° are used to obtain the S_3 image shown in [Fig. 2(d)], which matches well with the simulated result [Fig. 2(d')]. Measurement of S_3 clearly shows that the two sets of diagonal lobes are orthogonal, elliptically polarized with a phase difference of π . In Figs. 2(c) and 2(c'), a white dotted line square of size 20×20 pixels ($93 \mu\text{m} \times 93 \mu\text{m}$) is marked at the center of the dark cross pattern, where all subsequent intensity measurements and simulations are carried out. This region corresponds to an angular divergence of 0.0013° around the beam axis, k_0 . The fields are taken to be transverse (E_x, E_y) and the role of

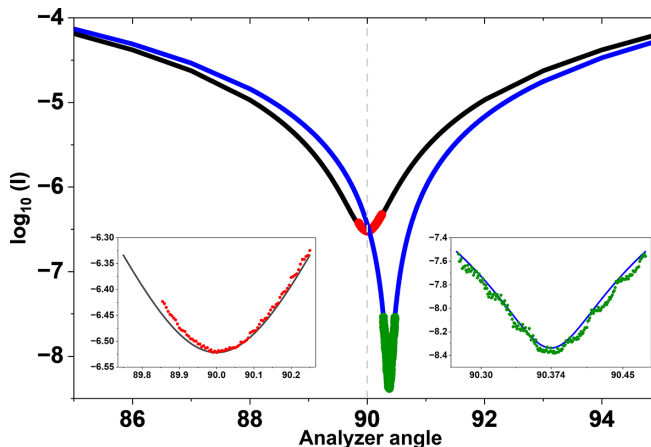


FIG. 3. Experimentally measured and simulated logarithmic intensity as a function of analyzer angle. The conventional polarization analysis, by rotating the polarizer around the cross position of 90° is shown by red circles (experiment) and a black continuous curve (simulation), with the left inset providing a magnified view. The EXE method, using optimized orientations of the QWP and polarizer, is shown by green circles (experiment) and a blue continuous curve (simulation), with the inset on the right providing a magnified view. The average intensity values are calculated from the 20×20 pixel region ($93 \mu\text{m} \times 93 \mu\text{m}$) centered on the beam axis. The EXE curve exhibits a sharper minimum, enabling more precise determination of the angles and, hence, the polarization ellipticity and ellipse orientation.

the axial (E_z) component of the field is neglected in all our calculations and measurements in the marked region.

The dynamic range of the CCD-measured intensity in the marked region varies by almost 8 orders of magnitude between the parallel and orthogonal polarizer orientations. To ensure the effectiveness of the EXE method, we calibrate the CCD using neutral density filters with an optical density (OD) of 5.56 in parallel orientation, remove all of them, and operate the CCD in the highest sensitivity regime for orthogonal polarization orientation. The logarithmic plot of the measured intensity as a function of the polarizer angle is shown in Fig. 3 without and with the oriented QWP. The intensity of the beam considered here corresponds to the white dotted-line square region shown in Figs. 2(c) and 2(c'). For the parallel-polarizer orientation, the intensity at the beam center for cross polarization decreases to the detector's electronic noise level. From the plots in Fig. 3, it is seen that the difference in the intensity versus the polarizer angle corresponding to the minimum intensity position without and with the QWP clearly shows that the minimum measurable intensity in the beam center is 3 orders of magnitude lower, and the polarizer angle where the minimum intensity is measured is slightly different. Replacing the CCD with a point detector with a lower noise floor will allow one to measure the QWP and polarizer angles with much higher accuracy and sensitivity. The angle of the polarization elements corresponding to the minimum intensity in the marked region is then used to extract the residual polarization ellipticity and the ellipse orientation of the light beam. This demonstrates the EXE method's ability to measure very small SoEP values of the beam from the CCD's lowest measurable intensity level.

The EXE technique involves extracting the intensity value from the dark region of a 20×20 pixel area centered on the beam, corresponding to a beam divergence angle of 0.0013° . For further calculations, we consider this region of the beam to be the beam axis $\langle k_0 \rangle$. From the QWP and polarizer angles corresponding to the null intensity, we reconstruct the SoEP of the beam at the beam center. The Jones vector of the beam is then determined from the QWP-polarizer angles in the null-intensity reconstruction method. For a given QWP angle θ_Q and analyzer angle θ_P , a null in

transmitted intensity implies that the input state J_{in} satisfies

$$P^\dagger(\theta_P) J_{\text{QWP}}(\theta_Q) J_{\text{in}} = 0, \quad (5)$$

where P is the transmission axis of the polarizer and $J_{\text{QWP}}(\theta_Q)$ is the Jones matrix of the rotated quarter-wave plate. The input polarization, therefore, lies in the null-space of this linear mapping. Solving the homogeneous equation yields the normalized Jones vector (unique up to global phase). A consistency check is performed for both possible retardance signs of the QWP, and the physically correct one is selected as the solution that reproduces the experimentally observed null. The amplitude (a) and phase (Δ) are calculated using the MATLAB program, and the Jones vector of the laser beam is determined using

$$\begin{bmatrix} E_x \\ E_y \end{bmatrix} = \begin{bmatrix} \cos a \\ \sin a e^{i\Delta} \end{bmatrix}. \quad (6)$$

From the field components and the phase difference between them, the Stokes parameters are calculated using

$$\begin{aligned} S_0 &= |E_x|^2 + |E_y|^2, & S_1 &= |E_x|^2 - |E_y|^2, \\ S_2 &= 2 \operatorname{Re}(E_x E_y^*), & S_3 &= 2 \operatorname{Im}(E_x E_y^*). \end{aligned} \quad (7)$$

Using Stokes parameters S_1, S_2, S_3 , the ellipticity and ellipse orientation of the polarization ellipse in the center of the beam are calculated using Eqs. (8) and (9) and the values are given in Table II:

$$\text{ellipticity, } \epsilon = \frac{1}{2} \sin^{-1} S_3, \quad (8)$$

$$\text{ellipse orientation, } \phi = \frac{1}{2} \tan^{-1} \frac{S_2}{S_1}. \quad (9)$$

V. RESIDUAL CRYSTAL ANISOTROPY

We now extend the application of the EXE method to extract the residual elliptical birefringence of different anisotropic crystals. Our samples are quartz and calcite, which are, respectively, positive ($n_e > n_o$) and negative ($n_e < n_o$) uniaxial crystals. The orientation of the OA with respect to the crystal surface or $\langle k_0 \rangle$ critically determines the birefringence-induced polarization transformation of the laser beam. We consider two representative cases: (1) OA parallel to the surface and (2) OA perpendicular to the surface. In general, it is assumed that a quartz crystal cut with its OA perpendicular to the surface (and parallel to the beam axis) behaves as an optical rotator, rotating the plane of linearly polarized light propagating through it without introducing ellipticity. Although such an optical element is predominantly circular birefringent, high-sensitivity EXE measurements reveal that the output beam shows weak ellipticity with a rotated plane of polarization due to the simultaneous presence of weak linear birefringence, making it an EB crystal, even along the OA. Independently, the quartz and calcite crystals, cut with their OA parallel to the surface (and perpendicular to the beam axis), are shown to behave as elliptical polarization rotators due to weak elliptical birefringence rather than just linear birefringent systems. The presence of orders of magnitude weak residual linear or circular birefringence, respectively, in the above two cases, makes the crystals EB, in general. However, the measurement of orders of magnitude weak birefringence effects, in the presence of the dominant birefringence of the crystal, is quite challenging and is rarely considered to be measurable.

To enable measurement and calculation, we first outline the theoretical framework for a paraxial Gaussian beam transmitted through the anisotropic medium and use the EXE protocol to characterize weak crystal anisotropy. To model the propagation of light through these anisotropic crystals, we use the Jones matrix for the elliptical retarder, which accounts for the vectorial nature of the electric field and the anisotropic dielectric response, including residual effects. Considering the crystals

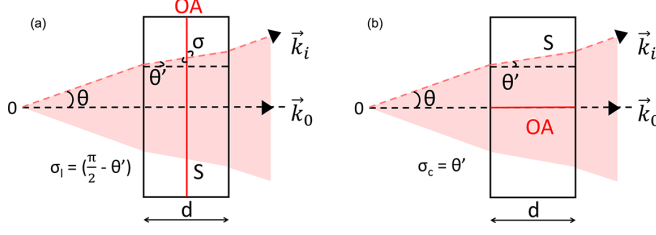


FIG. 4. Schematic diagram of a paraxial beam of light from a laser propagating through a uniaxial crystal. (a) The OA of the crystal is parallel to the surface and perpendicular to $\mathbf{k}_0$, “ d ” is the physical thickness of the crystal. The off-axis wave vector $\mathbf{k}_i$ makes an angle $\sigma_i = (\frac{\pi}{2} - \theta')$ with the OA. (b) The OA of the crystal is perpendicular to the crystal surface and parallel to $\mathbf{k}_0$. The off-axis wave vector $\mathbf{k}_i$ makes an angle $\sigma_c = \theta'$ with the OA.

as an elliptical retarder, with orthogonal eigenpolarizations χ_{ef} , and χ_{es} (with $\chi_{ef} \chi_{es}^* = -1$) and eigenvalues $e^{i\delta/2}$, and $e^{-i\delta/2}$, the transmission matrix is written as [29]

$$T_{ER} = T_0 \begin{bmatrix} e^{i\frac{\delta}{2}} + \chi_{ef} \chi_{ef}^* e^{-i\frac{\delta}{2}} & 2i\chi_{ef} \sin \frac{\delta}{2} \\ 2i\chi_{ef} \sin \frac{\delta}{2} & e^{-i\frac{\delta}{2}} + \chi_{ef} \chi_{ef}^* e^{i\frac{\delta}{2}} \end{bmatrix}, \quad (10)$$

where $T_0 = (1 + \chi_{ef} \chi_{ef}^*)^{-1}$ and $\chi_{ef} = (\frac{\tan \phi + i \tan \epsilon}{1 - i \tan \phi \tan \epsilon})$, with ϕ, ϵ the azimuth and ellipticity angles of χ_{ef} , respectively.

A. OA parallel to the surface: Linear retarder

Figure 4(a) shows the schematic for the linear retarder case where the orientation of the OA is parallel to the crystal surface. The coordinate system is defined such that the z axis is the direction of beam propagation, and the OA lies in the x - y plane. The incident paraxial beam propagates through the crystal with a divergence angle of θ . The angle the wave vector makes after entering the crystal is θ' . The angle between the axial component of the wave vector $\mathbf{k}_0$ and the OA of the crystal is $\frac{\pi}{2}$; propagating through the crystal, the off-axis wave vector $\mathbf{k}_i$ makes an angle ($\sigma = \frac{\pi}{2} - \theta'$) with the OA. With d as the thickness of the crystal and S the light path through the crystal, using Snell's law, we get

$$\theta' = \arcsin \left(\frac{\sin \theta}{n_o} \right), \quad S = \frac{d}{\cos \theta'}.$$

Except for $\mathbf{k}_0$, the conical divergent beam propagating through the crystal accumulates radial phase differences due to n_o and n_e (dependent on θ'), as the beam propagates through the crystal. This makes the output beam, after propagating through the crystal, elliptically polarized with spatially varying ellipticity and ellipse orientation. For a linear retarder, ellipticity, $\epsilon = 0$, $\chi_{ef} = \tan \phi$. Using this in Eq. (10), the transmission matrix for a linear retarder becomes

$$T_{LR} = \begin{bmatrix} e^{i\frac{\delta}{2}} \cos^2 \phi + e^{-i\frac{\delta}{2}} \sin^2 \phi & 2i \sin \phi \cos \phi \sin \frac{\delta}{2} \\ 2i \sin \phi \cos \phi \sin \frac{\delta}{2} & e^{i\frac{\delta}{2}} \sin^2 \phi + e^{-i\frac{\delta}{2}} \cos^2 \phi \end{bmatrix}. \quad (11)$$

The linear birefringence LB [30] is given by

$$\delta(\theta')_{LB} = \frac{2\pi}{\lambda} [n_e(\theta') - n_o] \frac{d}{\cos \theta'} \quad (12)$$

$$\text{and } \frac{1}{n_e^2(\theta')} = \frac{\sin^2 \theta'}{n_e^2} + \frac{\cos^2 \theta'}{n_o^2}. \quad (13)$$

For the central wave vector $\mathbf{k}_0$,

$$\delta_{LB} = \frac{2\pi}{\lambda} [n_e - n_o] d. \quad (14)$$

B. OA perpendicular to the surface: Circular retarder

Figure 4(b) shows the schematic of the case where the OA is perpendicular to the crystal surface and along the propagation direction, $\mathbf{k}_0$. The central wave vector $\mathbf{k}_0$, is normally incident, forming an angle 0° , with the OA, while an off-axis wave vector $\mathbf{k}_i$, makes an angle θ' with the OA. In this configuration, the light beam propagating along the OA, and the polarization eigenstates are circular. The phase difference between the orthogonal circular eigenstates makes the component a polarization rotator, causing the linearly polarized light propagating along the OA to rotate the plane of polarization. However, the paraxial beam divergence introduces small angular deviation, leading to accumulation of spatially dependent elliptical birefringence as the right- and left-circularly polarized components accumulate unequal phases depending on the local incidence angle. For the circular retarder (or optical rotator), the ellipticity $\epsilon = 1$, $\phi = 0^\circ$, and $\chi_{ef} = \pm i$, and the corresponding transmission matrix is

$$T_{CR} = \begin{bmatrix} \cos \frac{\delta}{2} & \pm \sin \frac{\delta}{2} \\ \mp \sin \frac{\delta}{2} & \cos \frac{\delta}{2} \end{bmatrix}. \quad (15)$$

The CB is given by [6]

$$\delta(\theta')_{CB} = \frac{2\pi}{\lambda} (n_L - n_R) \frac{d}{\cos \theta'}, \quad (16)$$

$$\delta(\theta')_{CB} = \frac{2\pi}{\lambda} \left(\frac{g_{11} \sin^2 \theta' + g_{33} \cos^2 \theta'}{\sqrt{n_o n_e}} \right) \frac{d}{\cos \theta'}. \quad (17)$$

Here, g_{11} and g_{33} are the components of the crystal gyration tensor. For quartz, $|g_{11}| = 6.1 \times 10^{-5}$ and $|g_{33}| = 12.81 \times 10^{-5}$ for $\lambda = 510$ nm [6]. For the central wave vector $\mathbf{k}_0$,

$$\delta_{CB} = \frac{2\pi}{\lambda} \frac{g_{33}}{\sqrt{n_o n_e}} d. \quad (18)$$

VI. CALCULATION OF BIREFRINGENCE FROM THE TRANSMITTED FIELD

We now describe the numerical procedure used to determine the elliptical birefringence due to the propagation of a paraxial beam of light through a uniaxial crystal. This is done from the measured transmitted optical fields. Consider an elliptically polarized input beam given by

$$\begin{bmatrix} E_{in}^x \\ E_{in}^y \end{bmatrix} = \begin{bmatrix} \cos a \\ \sin a e^{i\Delta} \end{bmatrix}. \quad (19)$$

For linear retarder, described by Eq. (11), the transmitted output light field with the OA azimuth $\phi = 0$ is

$$\begin{bmatrix} E_{out}^x \\ E_{out}^y \end{bmatrix} = \begin{bmatrix} e^{i\frac{\delta}{2}} & 0 \\ 0 & e^{-i\frac{\delta}{2}} \end{bmatrix} \begin{bmatrix} E_{in}^x \\ E_{in}^y \end{bmatrix}. \quad (20)$$

Solving this (Appendix A) gives the linear birefringence as

$$\delta_{LB} = i \ln(E_L) + 2\pi m, \quad (21)$$

where $E_L = \frac{E_{in}^x E_{out}^y}{E_{in}^y E_{out}^x}$ is a complex number.

For circular retarder, described by Eq. (15), the transmitted output light field is

$$\begin{bmatrix} E_{\text{out}}^x \\ E_{\text{out}}^y \end{bmatrix} = \begin{bmatrix} \cos \frac{\delta}{2} & -\sin \frac{\delta}{2} \\ +\sin \frac{\delta}{2} & \cos \frac{\delta}{2} \end{bmatrix} \begin{bmatrix} E_{\text{in}}^x \\ E_{\text{in}}^y \end{bmatrix}. \quad (22)$$

Solving this expression (Appendix B) gives the circular birefringence of the crystal as

$$\delta_{\text{CB}} = \pm \cos^{-1} \left(\frac{E_{C1}}{\sqrt{1 + 4E_{C2}^2}} \right) - \tan^{-1}(2E_{C2}), \quad (23)$$

$$\text{with } E_{C1} = \left(\frac{E_{\text{out}}^x{}^2 - E_{\text{out}}^y{}^2}{E_{\text{in}}^x{}^2 - E_{\text{in}}^y{}^2} \right), \quad E_{C2} = \left(\frac{E_{\text{in}}^x E_{\text{in}}^y}{E_{\text{in}}^x{}^2 - E_{\text{in}}^y{}^2} \right).$$

VII. EXE TECHNIQUE IN CRYSTAL TRANSMISSION

Next, we propagate the laser beam through different uniaxial crystals, as an application of the EXE method to extract weak residual crystal anisotropy. We consider uniaxial calcite and quartz crystals in which the OA is oriented perpendicular to the surface and right- (R) and left- (L)-handed quartz crystals and a calcite crystal. From Eqs. (11) and (15), it is seen that the sample thickness d directly governs the effective birefringence and, hence, the polarization evolution within the crystal, resulting in markedly different polarization characteristics of the output beam.

As shown in Fig. 1, a QWP and an analyzer (polarizer) are placed after the crystal and in front of the CCD. With the crystal kept normal to the beam, the QWP-P are rotated sequentially to produce a four-lobe pattern with a dark-centered cross pattern in the output beam. From the QWP-P angle, we calculate the residual elliptical birefringence of the crystal along the axial direction, corresponding to the dark-beam center. We first used the quartz crystal, $Q_{\parallel}$ (SIO-1a, OPTOGAMA, Lithuania), and then the calcite crystal, $C_{\parallel}$ (CACO-1a, OPTOGAMA, Lithuania), both with their OA parallel to the crystal surface. The crystals were placed in the laser beam with the azimuthal angle of OA $\phi = 0^\circ$. Ideally, neither crystal is expected to introduce any additional polarization ellipticity to the transmitted beam, as they are fabricated to work as multiorder half-wave plates. However, for the paraxial beam from the laser propagating through the crystals, the angle of incidence, the accumulated phase difference, and, hence, the EB in the output beam exhibit spatial variations. Due to varying angles of incidence of the weakly divergent laser beam, different regions of the wave front experience different amounts of EB, including along the propagation axis $\mathbf{k}_0$. It is important to note here that for a known input laser beam parameters, the measured elliptical birefringence is attributed predominantly to the residual crystal characteristics, beyond what is expected of an ideal behavior. This results in a complex output beam with spatially nonuniform elliptical polarization. The QWP and analyzer angles are optimized to obtain a dark-centered cross pattern and four-lobe pattern, as shown in Figs. 5(a) and 5(b). The QWP and analyzer angles (θ_Q, θ_A) to achieve the output beam pattern are listed in Table I. We use these angles to quantify the polarization ellipticity, ellipse orientation, and hence the residual elliptical birefringence of the crystals (Table II). The values obtained are compared with the literature and manufacturer-specified values. Since we are interested here only in the extraction of the axial elliptical birefringence of the crystals, we focus only on the dark-centered intersection of the cross pattern and ignore its spatial variation, the slight nonuniformity, and asymmetry in the four-lobe intensity pattern.

Next, we study two right-handed quartz samples $Q_{\perp 1}$ (SIO-1b, OPTOGAMA, Lithuania), $Q_{\perp 2}$ (300EXR, Rudolph Instruments, USA) and one left-handed quartz sample $Q_{\perp 3}$ (300EXL, Rudolph Instruments, USA) of different thickness, with their OA perpendicular to the surface of the crystal, along the beam propagation direction. Following the EXE procedure, we rotate the QWP and the analyzer sequentially, nullify the residual polarization ellipticity at the beam center, and obtain the angles at which the four-lobe structure with a dark-centered cross pattern is observed. The experimentally measured patterns are shown in Figs. 5(c)–5(e). Compared with the previous case of

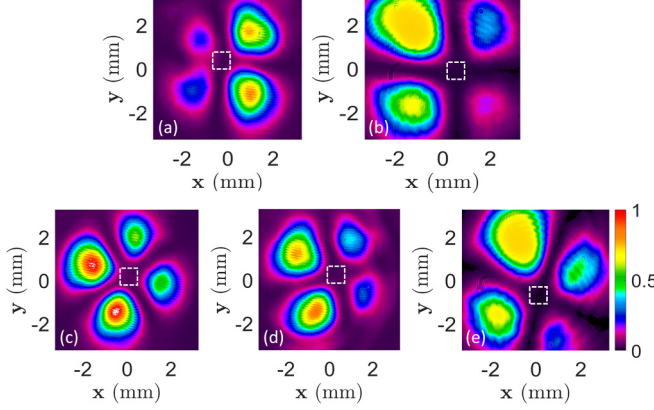


FIG. 5. Experimentally measured images of the four lobes with dark-centered cross pattern under EXE conditions for (a) quartz, $Q_{\parallel}$, and (b) calcite, $C_{\parallel}$, with the OA parallel to the crystal surface; and for different thickness quartz crystals with the OA perpendicular to the surface: (c) $Q_{\perp 1}$, (d) $Q_{\perp 2}$, and (e) $Q_{\perp 3}$.

crystals with their OA parallel to the surface, the four lobes are rotated by different angles as a result of the crystal's optical activity. As these quartz crystals have different thicknesses and therefore rotate the polarization of the beam by different angles, the experimentally obtained angles (θ_Q , θ_A) to obtain the patterns shown in Figs. 5(c)–5(e) are also different and given in Table I. Like in the previous case, we ignore the slight nonuniformity and asymmetry in the four-lobe pattern and focus only on the dark-centered intersection of the cross pattern. The circular birefringence of a quartz rotator, cut with its OA perpendicular to the surface and kept along the beam axis ($\mathbf{k}_0$), rotates the linear plane-polarized beam of light by a fixed amount that depends on its thickness. In our experiments, however, we use a paraxial beam of light rather than a plane-wave propagating through the crystal. As a result, different parts of the paraxial laser beam with different amounts of SoEP (as was demonstrated in Sec. IV) pick up different amounts of polarization ellipticity while propagating through the crystals. This can be seen in Figs. 5(c)–5(e), where we notice that the off-axis variations in the four-lobe pattern are quite different due to the spatially varying coupling between the laser polarization and the crystal's elliptical birefringence. We do not address these complex effects here and reserve them for future investigations.

After obtaining the optimized four-lobe structures with a dark-centered cross pattern for both OA orientations (Fig. 5), we select the 20×20 pixel region centered on the beam axis and extract the average intensity as a function of the analyzer angle. Since the intensity in this region, as a function of (θ_A) varies by several orders of magnitude (as shown in Fig. 3), we plot the logarithm of the intensity as a function of (θ_A) [for a fixed (θ_Q)] only in a small angle range for all crystals as shown in Fig. 6. For reference, the log-intensity curve of the input laser beam (without any crystal sample) is also plotted using black dots connected by a thin black line in the figure. The minimum

TABLE I. The table lists the orientation angles of the QWP and polarizer (θ_Q , θ_P) that satisfy the EXE condition for the input laser beam, for different crystal thicknesses and OA orientations, as indicated.

Sample	Thickness (mm)	QWP, θ_Q°	Analyzer, θ_P°
Laser	—	−1.303	90.374
$Q_{\parallel}$	1	−0.895	90.219
$C_{\parallel}$	1	−1.739	91.774
$Q_{\perp 1}$	1	−32.762	121.904
$Q_{\perp 2}$	0.5223	−16.488	105.536
$Q_{\perp 3}$	0.632	17.104	71.968

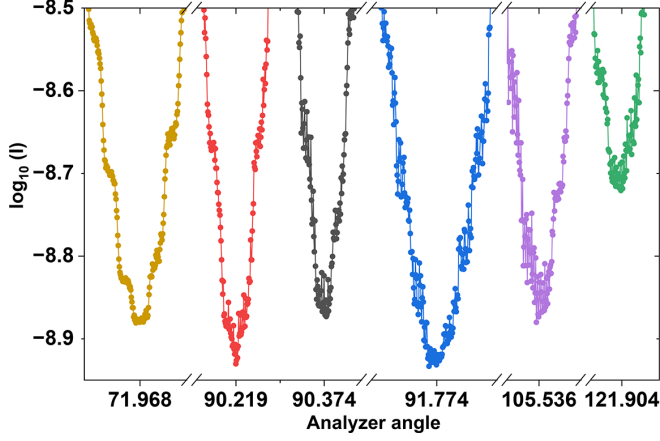


FIG. 6. Experimentally measured average beam intensity (I) in the 20×20 pixel region centered on the beam axis as a function of analyzer angle (θ_p), obtained using the EXE method. The black dots connected by a thin black line represent the laser's reference behavior. The plots for $Q_{\parallel}$, $C_{\parallel}$, $Q_{\perp 1}$, $Q_{\perp 2}$, and $Q_{\perp 3}$ are shown as red, blue, green, purple, and dark yellow dots, respectively, each connected by thin lines of the same color. The different fixed QWP angles (θ_Q) used to obtain the behavior are given in Table I.

recorded intensity occurs at an analyzer angle of 90.374° , and the corresponding QWP angle is -1.303° , serving as a reference for comparison with crystal samples. The polarization ellipticity of the laser beam was subtracted from subsequent measurements propagating through the crystals to isolate and quantify only the residual EB introduced by the different crystals. The QWP and analyzer angles obtained from the plots (Fig. 6) corresponding to the minimum intensity measured in the dark-central region of (Fig. 5) are given in Table I along with the crystal thickness provided by the manufacturer.

VIII. QUANTIFICATION OF RESIDUAL ELLIPTICAL BIREFRINGENCE

The birefringence calculations described in Sec. VI require the input and transmitted electric fields $E_{\text{in}}^x, E_{\text{in}}^y$ and $E_{\text{out}}^x, E_{\text{out}}^y$, respectively. The input electric field ($E_{\text{in}}^x, E_{\text{in}}^y$) is first reconstructed from the experimentally determined QWP (θ_Q) and analyzer (θ_p) angles corresponding to the EXE null condition of the incident laser beam and expressed in the amplitude-phase form given by Eq. (6). The same EXE procedure is then repeated after introducing the crystal sample into the beam path to reconstruct the transmitted electric field ($E_{\text{out}}^x, E_{\text{out}}^y$). These reconstructed electric fields are subsequently substituted into Eqs. (21) and (23) to calculate the linear (δ_{LB}) and circular (δ_{CB}) birefringence of the crystal, respectively.

It is important to highlight the significance of the proposed and demonstrated EXE method here. Although the laser beam is assumed to be linearly polarized, due to the Brewster-cut window cavity, it is actually elliptically polarized. The calculated ellipticity and ellipse orientation of the laser are given in Table II. The EXE method's ability to measure such small values is important to extend it to subsequently measure the EB of the crystals. The calculated values of crystal birefringence δ_{LR} for $Q_{\parallel}, C_{\parallel}$ and δ_{CB} for $Q_{\perp 1}, Q_{\perp 2}, Q_{\perp 3}$ are listed in Table II. The crystals $Q_{\parallel}, C_{\parallel}$ with their OA cut parallel to the surface and with the input field oriented parallel to the OA are expected to only be linearly birefringent. However, from the values given in Table II, based on the EXE measurements, we find that the output beam is elliptically polarized due to residual EB introduced by the crystal. Quartz crystals $Q_{\perp 1}, Q_{\perp 2}, Q_{\perp 3}$ with their OA cut perpendicular to the surface and kept along the beam propagation direction are expected to behave as polarization rotators due to circular birefringence. In these crystals, we also observe that the output beam is elliptically polarized, with

TABLE II. The polarization ellipticity and ellipse orientation of the transmitted beam through different crystal samples, calculated using Eqs. (8) and (9), are listed. The residual birefringence, δ_{LB} , for $Q_{\parallel}$ and $C_{\parallel}$, and δ_{CB} , for $Q_{\perp 1}$, $Q_{\perp 2}$, and $Q_{\perp 3}$, of the crystals are calculated from the experimentally obtained θ_Q , θ_P angles, using Eqs. (21) and (23). The refractive index difference Δn values calculated from the residual EB of the crystals are compared with literature values. Using Eqs. (C1) and (C2), the Δn calculation errors are, respectively, $Q_{\parallel} = \pm 3.242 \times 10^{-6}$, $C_{\parallel} = \pm 7.194 \times 10^{-6}$, and $Q_{\perp 1,2,3} = \pm 5.74 \times 10^{-7}$.

Sample	Ellipticity (deg)	Ellipse orientation (deg)	$(\delta_{LB}/\delta_{CB})$	$ \Delta n $ $(n_e - n_o \text{ or } n_L - n_R)$	$ \Delta n $
Laser	-0.9293	1.3033	—	—	—
$Q_{\parallel}$	-0.6760	0.8950	$(113.0700 - 0.3558i)$	$(9.258 - 0.0291i) \times 10^{-3}$	9.228×10^{-3} [31]
$C_{\parallel}$	0.0354	1.7386	$(2136.9231 + 0.0830i)$	$(0.17498 + i 6.79 \times 10^{-6})$	0.175 [32]
$Q_{\perp 1}$	-0.8584	32.7624	$(1.0975 - 0.0109i)$	$(8.987 - 0.089i) \times 10^{-5}$	8.15×10^{-5} [33]
$Q_{\perp 2}$	-0.9518	16.4878	$(0.5293 - 0.0132i)$	$(8.297 - 0.207i) \times 10^{-5}$	8.15×10^{-5} [33]
$Q_{\perp 3}$	-0.9283	-17.1037	$(0.5507 + 0.0491i)$	$(8.314 + 0.204i) \times 10^{-5}$	8.15×10^{-5} [33]

the plane of polarization rotated away from the input plane. These additional rotations are adjusted by the QWP and the analyzer to different angles to null the beam intensity at the beam center. From these angles, we calculated the residual EB of the crystals. The ellipticity and ellipse orientation of the transmitted beam through the sample crystals calculated using Eq. (8) [Eq. (9)], along with the calculated values of crystal birefringences using Eqs. (14) and (16), are listed in Table II. The values calculated using the EXE method are compared with the Δn values reported in the literature. Although the real part of the birefringence calculated using the EXE method matches the values reported in the literature, we also observed effects from the imaginary part of the refractive indices, leading to polarization ellipticity in these crystals, which was previously assumed to be nonexistent due to measurement limitations. The high sensitivity of the EXE method to measure small changes in the SoEP in the output beam confirms the unexpected EB in these predominantly LB and CB uniaxial crystals.

IX. SUMMARY

We presented a generalized framework to characterize the residual elliptical birefringence due to weak optical anisotropy in uniaxial crystals by exploiting the intrinsic SOI of a paraxial laser beam. Using the vectorial Debye-Richards-Wolf formalism, we simulate the nonuniform spatial polarization structure of an Ar^+ laser and confirm its SOI signature experimentally via spatially spin-separated four-lobe intensity pattern under EXE conditions. This establishes the sensitivity of the proposed method to measure very small SoEP values in the laser beam and use it to quantify the EB of different uniaxial crystals. We model the transmission of the laser beam through uniaxial crystals in two configurations: with their optic axis parallel or perpendicular to the surface. We derive and use generic transmission matrices that account for the weak EB in otherwise linear or circular birefringent crystals. To quantify the changes in the weak SoEP in the laser beam propagating through the crystals, we extract the average intensity in the 20×20 pixel region of the dark-centered cross pattern, corresponding to the beam axis as a function of analyzer angle, for fixed θ_Q , achieving a high extinction ratio down to 10^{-8} W. This allows us to extract the residual SoEP due to propagation through different crystals and calculate the corresponding birefringence and Δn values. Our results demonstrate the effectiveness of using SOI-induced polarization variations in a laser beam to quantify the residual EB of crystals with high accuracy and sensitivity. Our demonstrated ability to measure beam intensity down to the detector's noise floor and use it to calculate the residual EB of the crystals is unique and enables a sensitive, practical tool for anisotropy measurements, extending its applicability to characterize monolayers, two-dimensional materials, ultrathin films, and metasurfaces. The potential of the EXE method as an alternative to the popular high-accuracy universal polarimeter and ellipsometry methods deserves further systematic study.

ACKNOWLEDGMENTS

The authors acknowledge the Science and Engineering Research Board (SERB), Government of India, for financial support for this work and the Ministry of Social Justice and Empowerment, Government of India, for the research fellowship (U.B.).

The authors declare that they have no conflict of interest.

DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX A: CALCULATION OF LINEAR BIREFRINGENCE

For linear birefringence, from Eq. (11), the output transmitted light field with optic axis azimuth $\phi = 0$ is

$$\begin{bmatrix} E_{\text{out}}^x \\ E_{\text{out}}^y \end{bmatrix} = \begin{bmatrix} e^{i\frac{\delta}{2}} & 0 \\ 0 & e^{-i\frac{\delta}{2}} \end{bmatrix} \begin{bmatrix} E_{\text{in}}^x \\ E_{\text{in}}^y \end{bmatrix}. \quad (\text{A1})$$

So, $E_{\text{out}}^x = e^{i\frac{\delta}{2}} E_{\text{in}}^x$ and $E_{\text{out}}^y = e^{-i\frac{\delta}{2}} E_{\text{in}}^y$:

$$E_{\text{out}}^x E_{\text{out}}^y = e^{i\frac{\delta}{2}} E_{\text{in}}^x E_{\text{out}}^y, \quad (\text{A2})$$

$$E_{\text{out}}^x E_{\text{out}}^y = e^{-i\frac{\delta}{2}} E_{\text{in}}^y E_{\text{out}}^x. \quad (\text{A3})$$

From Eqs. (A2) and (A3), we get

$$\frac{E_{\text{in}}^x E_{\text{out}}^y}{E_{\text{in}}^y E_{\text{out}}^x} = e^{-i\delta}. \quad (\text{A4})$$

Considering $\frac{E_{\text{in}}^x E_{\text{out}}^y}{E_{\text{in}}^y E_{\text{out}}^x} = E_L$, here E_L is a complex number:

$$E_L = e^{-i\delta}, \quad \ln(E_L) = i(-\delta + 2\pi m), \quad \delta_{\text{LB}} = i \ln(E_L) + 2\pi m. \quad (\text{A5})$$

APPENDIX B: CALCULATION OF CIRCULAR BIREFRINGENCE

From Eq. (15), the output transmitted light field for a circular retarder is

$$\begin{bmatrix} E_{\text{out}}^x \\ E_{\text{out}}^y \end{bmatrix} = \begin{bmatrix} \cos \frac{\delta}{2} & -\sin \frac{\delta}{2} \\ +\sin \frac{\delta}{2} & \cos \frac{\delta}{2} \end{bmatrix} \begin{bmatrix} E_{\text{in}}^x \\ E_{\text{in}}^y \end{bmatrix},$$

$$E_{\text{out}}^x = \cos \frac{\delta}{2} E_{\text{in}}^x - \sin \frac{\delta}{2} E_{\text{in}}^y, \quad (\text{B1})$$

$$E_{\text{out}}^y = \sin \frac{\delta}{2} E_{\text{in}}^x + \cos \frac{\delta}{2} E_{\text{in}}^y, \quad (\text{B2})$$

$$(E_{\text{out}}^x)^2 - (E_{\text{out}}^y)^2 = \left(\cos^2 \frac{\delta}{2} - \sin^2 \frac{\delta}{2} \right) E_{\text{in}}^x{}^2 - \left(\cos^2 \frac{\delta}{2} - \sin^2 \frac{\delta}{2} \right) E_{\text{in}}^y{}^2 - 4 \cos \frac{\delta}{2} \sin \frac{\delta}{2} E_{\text{in}}^x E_{\text{in}}^y,$$

$$(E_{\text{out}}^x)^2 - (E_{\text{out}}^y)^2 = \cos \delta (E_{\text{in}}^x{}^2 - E_{\text{in}}^y{}^2) - 2 \sin \delta E_{\text{in}}^x E_{\text{in}}^y,$$

$$\left(\frac{E_{\text{out}}^x{}^2 - E_{\text{out}}^y{}^2}{E_{\text{in}}^x{}^2 - E_{\text{in}}^y{}^2} \right) = \cos \delta - 2 \sin \delta \left(\frac{E_{\text{in}}^x E_{\text{in}}^y}{E_{\text{in}}^x{}^2 - E_{\text{in}}^y{}^2} \right). \quad (\text{B3})$$

Considering

$$\begin{aligned}
 \left(\frac{E_{\text{out}}^x{}^2 - E_{\text{out}}^y{}^2}{E_{\text{in}}^x{}^2 - E_{\text{in}}^y{}^2} \right) &= E_{C1} \quad \text{and} \quad \left(\frac{E_{\text{in}}^x E_{\text{in}}^y}{E_{\text{in}}^x{}^2 - E_{\text{in}}^y{}^2} \right) = E_{C2} \\
 E_{C1} &= \cos \delta - 2 \sin \delta E_{C2} \\
 R \cos(\delta + \alpha) &= \cos \delta - 2 \sin \delta E_{C2} \\
 R \cos \delta \cos \alpha - R \sin \delta \sin \alpha &= \cos \delta - 2 \sin \delta E_{C2} \\
 R \cos \alpha &= 1; \quad R \sin \alpha = 2E_{C2} \\
 R &= \pm \sqrt{1 + 4E_{C2}^2}; \quad \alpha = \tan^{-1}(2E_{C2}) \\
 E_{C1} &= \pm \sqrt{1 + 4E_{C2}^2} \cos(\delta + \tan^{-1}(2E_{C2})) \\
 \delta_{CB} &= \pm \cos^{-1} \left(\frac{E_{C1}}{\sqrt{1 + 4E_{C2}^2}} \right) - \tan^{-1}(2E_{C2}) \quad (\text{B4})
 \end{aligned}$$

APPENDIX C: ERROR CALCULATION

We used a Monte Carlo error analysis to estimate the uncertainties in the reconstructed polarization parameters. The QWP was mounted on a motorized rotation stage (K10CR1/M, Thorlabs, USA) with a specified angular repeatability of $\sigma_Q = \pm 0.00344^\circ$, while the analyzer was mounted on a NanoRotator stage (NR360S/M, Thorlabs, USA) with a specified angular repeatability of $\sigma_P = \pm 0.00278^\circ$. These values were used as the angular uncertainties in the Monte Carlo simulations.

For sampling,

$$\begin{aligned}
 Q_k &= Q_0 + \sigma_Q N(0, 1), \quad P_k = P_0 + \sigma_P N(0, 1), \\
 \text{where } \sigma_Q &= 0.00344^\circ \quad \text{and} \quad \sigma_P = 0.00278^\circ, \\
 \text{and } k &= 1, \dots, N, \quad \text{with } N = 5000.
 \end{aligned}$$

Sampling parameters:

- (1) Q_0 : measured QWP angle and P_0 : measured analyzer angle.
- (2) Q_k : QWP and P_k : analyzer angle in the k th Monte Carlo realization.
- (3) $N(0, 1)$: standard normal random variable with mean 0 and standard deviation 1.
- (4) N : total number of Monte Carlo realizations ($N = 5000$).
- (5) k : realization index ($k = 1, 2, \dots, N$).

Using Eq. (5), the amplitude (a) and phase (Δ) are reconstructed from Eq. (6). In the following discussion, the phase will be denoted by ζ to distinguish it from the uncertainty notation used in the error analysis. For each realization,

$$(a_k, \zeta_k) = f(Q_k, P_k),$$

where f is the polarization-state reconstruction algorithm.

Now, using the Monte Carlo error analysis, the following are obtained:

Amplitude uncertainty, $\Delta A = \text{std}(a_k)$; here, $\text{std}(\cdot)$ is the standard deviation of all Monte Carlo realizations.

Phase uncertainty (after phase unwrapping)

$$\Delta \zeta = \text{std}(\zeta_k).$$

Ellipticity,

$$\epsilon_k = \frac{1}{2} \sin^{-1}[\sin(2a_k) \sin(\zeta_k)], \quad \Delta\epsilon = \text{std}(\epsilon_k).$$

Ellipse orientation,

$$\phi_k = \frac{1}{2} \tan^{-1}[\tan(2a_k) \cos(\zeta_k)], \quad \Delta\phi = \text{std}(\phi_k).$$

Linear retardance,

$$\delta_{\text{LB},k} = \zeta_{0,k} - \zeta_{i,k}, \quad \Delta\delta_{\text{LB}} = \text{std}(\delta_{\text{LB},k}).$$

Linear birefringence,

$$\Delta n_{\text{LB},k} = \frac{\lambda}{2\pi d} \delta_{\text{LB},k}, \quad (\sigma_{\Delta n})_{\text{LB}} = \text{std}(\Delta n_{\text{LB},k}). \quad (\text{C1})$$

Circular retardance,

$$U_k = \tan(2a_{i,k}) \cos(\zeta_{i,k}), \quad D_k = \sqrt{1 + U_k^2},$$

$$\delta_{\text{CB},k} = \cos^{-1} \left[\frac{\cos(2a_{o,k})}{\cos(2a_{i,k}) D_k} \right] - \tan^{-1}(U_k),$$

$$\Delta\delta_{\text{CB}} = \text{std}(\delta_{\text{CB},k}).$$

Circular birefringence,

$$\Delta n_{\text{CB},k} = \frac{\lambda}{2\pi d} \delta_{\text{CB},k}, \quad (\sigma_{\Delta n})_{\text{CB}} = \text{std}(\Delta n_{\text{CB},k}). \quad (\text{C2})$$

Using the abovementioned method, errors in Δn are calculated for $Q_{\parallel}$, $C_{\parallel}$, and $Q_{\perp 1,2,3}$.

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- [1] G. B. Malykin, V.L. Ginzburg's helical elliptically polarized modes and their application, [Phys. Usp. **59**, 1245 \(2016\)](#).
 - [2] G. Ramachandran and S. Ramaseshan, Crystal optics, in *Kristallografik Beugung/Crystal Optics Diffraction* (Springer, Berlin, 1961), pp. 1–217.
 - [3] J. Nye, *Physical Properties of Crystals* (Clarendon Press, Oxford, 1985), Vol. 19762, p. 3.
 - [4] J. Kobayashi and Y. Uesu, A new optical method and apparatus ‘HAUP’ for measuring simultaneously optical activity and birefringence of crystals. I. Principles and construction, [J. Appl. Crystallogr. **16**, 204 \(1983\)](#).
 - [5] C. Chou, Y.-C. Huang, and M. Chang, Effect of elliptical birefringence on the measurement of the phase retardation of a quartz wave plate by an optical heterodyne polarimeter, [J. Opt. Soc. Am. A **14**, 1367 \(1997\)](#).
 - [6] O. Arteaga, A. Canillas, and G. E. Jellison Jr., Determination of the components of the gyration tensor of quartz by oblique incidence transmission two-modulator generalized ellipsometry, [Appl. Opt. **48**, 5307 \(2009\)](#).
 - [7] A. T. Martin, S. M. Nichols, V. L. Murphy, and B. Kahr, Chiroptical anisotropy of crystals and molecules, [Chem. Commun. **57**, 8107 \(2021\)](#).
 - [8] P. Yang, B. Xie, and S. Feng, Simultaneous measurement of polarization rotation angle and ellipticity at the quantum noise limit, [J. Opt. Soc. Am. B **40**, 2900 \(2023\)](#).
 - [9] U. Levy, Y. Silberberg, and N. Davidson, Mathematics of vectorial Gaussian beams, [Adv. Opt. Photon. **11**, 828 \(2019\)](#).
 - [10] R. Simon, E. Sudarshan, and N. Mukunda, Cross polarization in laser beams, [Appl. Opt. **26**, 1589 \(1987\)](#).
 - [11] Y. Fainman and J. Shamir, Polarization of nonplanar wave fronts, [Appl. Opt. **23**, 3188 \(1984\)](#).

- [12] W. L. Erikson and S. Singh, Polarization properties of Maxwell-Gaussian laser beams, *Phys. Rev. E* **49**, 5778 (1994).
- [13] A. Sokolov, Polarization of spherical waves, *Opt. Spectrosc.* **92**, 936 (2002).
- [14] A. Ciattoni, G. Cincotti, D. Provenziani, and C. Palma, Paraxial propagation along the optical axis of a uniaxial medium, *Phys. Rev. E* **66**, 036614 (2002).
- [15] A. Ciattoni, G. Cincotti, and C. Palma, Circularly polarized beams and vortex generation in uniaxial media, *J. Opt. Soc. Am. A* **20**, 163 (2003).
- [16] A. Volyar, T. Fadeeva, and Y. A. Egorov, Vector singularities of Gaussian beams in uniaxial crystals: Optical vortex generation, *Tech. Phys. Lett.* **28**, 958 (2002).
- [17] L. Allen, S. Barnett, and M. Padgett, *Optical Angular Momentum* (IOP Publishing Ltd. and individual contributors, Bristol, UK, 2003).
- [18] N. Simpson, K. Dholakia, L. Allen, and M. Padgett, Mechanical equivalence of spin and orbital angular momentum of light: An optical spanner, *Opt. Lett.* **22**, 52 (1997).
- [19] V. S. Liberman and B. Y. Zel'dovich, Spin-orbit interaction of a photon in an inhomogeneous medium, *Phys. Rev. A* **46**, 5199 (1992).
- [20] L. Marrucci, C. Manzo, and D. Paparo, Optical spin-to-orbital angular momentum conversion in inhomogeneous anisotropic media, *Phys. Rev. Lett.* **96**, 163905 (2006).
- [21] G. M. Lerman and U. Levy, Generation of a radially polarized light beam using space-variant subwavelength gratings at 1064 nm, *Opt. Lett.* **33**, 2782 (2008).
- [22] U. Baishya, N. Kumar, and N. K. Viswanathan, Dark-field spin Hall effect of light, *Opt. Lett.* **47**, 4479 (2022).
- [23] N. Kumar and N. K. Viswanathan, Spin-orbit conversion-enabled generation of cylindrical vector beams, *Opt. Commun.* **591**, 132089 (2025).
- [24] X. Hong, E. M. van Dijk, S. R. Hall, J. B. Götte, N. F. van Hulst, and H. Gersen, Background-free detection of single 5 nm nanoparticles through interferometric cross-polarization microscopy, *Nano Lett.* **11**, 541 (2011).
- [25] U. Baishya and N. K. Viswanathan, Measurement of surface chirality at near-normal incidence, *Appl. Phys. Lett.* **122**, 261102 (2023).
- [26] M. Benelajla, E. Kammann, B. Urbaszek, and K. Karrai, Physical origins of extreme cross-polarization extinction in confocal microscopy, *Phys. Rev. X* **11**, 021007 (2021).
- [27] P. Steindl, J. A. Frey, J. Norman, J. E. Bowers, D. Bouwmeester, and W. Löffler, Cross-polarization-extinction enhancement and spin-orbit coupling of light for quantum-dot cavity quantum electrodynamics spectroscopy, *Phys. Rev. Appl.* **19**, 064082 (2023).
- [28] B. Richards and E. Wolf, Electromagnetic diffraction in optical systems, II. Structure of the image field in an aplanatic system, *Proc. R. Soc. London A* **253**, 358 (1959).
- [29] R. M. Azzam, N. M. Bashara, and S. S. Ballard, *Ellipsometry and Polarized Light* (North-Holland, Amsterdam, 1978).
- [30] M. Born and E. Wolf, *Principles of Optics: Electromagnetic Theory of Propagation, Interference and Diffraction of Light* (Elsevier, Amsterdam, 2013).
- [31] N. Nagib, S. Khodier, and H. Sidki, Retardation characteristics and birefringence of a multiple-order crystalline quartz plate, *Opt. Laser Technol.* **35**, 99 (2003).
- [32] G. Ghosh, Dispersion-equation coefficients for the refractive index and birefringence of calcite and quartz crystals, *Opt. Commun.* **163**, 95 (1999).
- [33] F. Song, L. Yu, Y. Yang, and H. Tan, Color separation of argon laser light with the effect of dispersion of optical activity, *Appl. Opt.* **33**, 5513 (1994).