

Multivariate Rényi divergences characterize betting games with multiple lotteries

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We provide an operational interpretation of the multivariate Rényi divergence in terms of economic-theoretic tasks based on betting, risk aversion, and multiple lotteries. Specifically, we show that the multivariate Rényi divergence $D_{\underline{\alpha}}(\vec{P}_X)$ of probability distributions $\vec{P}_X = (p_X^{(0)}, \dots, p_X^{(d)})$ and real-valued orders $\underline{\alpha} = (\alpha_0, \dots, \alpha_d)$ quantifies the economic-theoretic value that a rational agent assigns to d lotteries with odds $o_X^{(k)} \propto (p_X^{(k)})^{-1}$ ($k = 1, \dots, d$) on a random event described by $p_X^{(0)}$. In particular, when the odds are fair and the rational agent is allowed to maximize over all betting strategies, the economic-theoretic value (precisely, the isoelastic certainty equivalent) that the agent assigns to the lotteries is exactly given by $w_R^{\text{ICE}} = \exp[D_{\underline{\alpha}}(\vec{P}_X)]$, where $\underline{R} = (R_1, \dots, R_d)$ is a risk-aversion vector with $R_k = 1 + \alpha_k/\alpha_0$ being the risk-aversion parameter associated with lottery k . Furthermore, we introduce a conditional multivariate Rényi divergence that characterizes a generalized scenario where the rational agent is allowed to have access to side information. We prove that this quantity satisfies a data processing inequality that can be interpreted as the increment in the economic-theoretic value provided by side information; crucially, we show that such a data processing inequality is a consequence of the agent's economic-theoretically consistent risk-averse attitude toward every lottery and vice versa. Moreover, we demonstrate the applicability of these results to the resource theory of informative measurements within the operational framework of general probabilistic theories encompassing quantum theory. By establishing quantitative connections between information

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theory, physical theories, and theories from economics, our framework provides an operational foundation for quantum state betting games with multiple lotteries in the realm of quantum resource theories.

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I. INTRODUCTION

Expected utility theory (EUT) is a theoretical framework first formalized during the 1940s in the work of von Neumann and Morgenstern within the theory of games and economic behavior [1]. EUT primarily deals with the development of mathematical models describing the behavior of rational agents when facing decision problems involving uncertainty [2]. A central object of study within EUT is the *utility function*, a real-valued function representing the rational agent’s level of satisfaction when acquiring “desirable assets” such as wealth, goods, or services. A highly nontrivial insight first put forward by economists during the 1950s establishes that the economic-theoretic concept of *risk aversion* is mathematically encoded in the curvature of the utility function [3–5]. This realization in turn opened the door in particular for the development of risk-aversion measures, thus establishing a concise mathematical framework for the quantification of risk aversion. The exploration of subjects related to risk aversion has remained a topic of active research in the economic sciences, with several of these efforts having been recognized with the Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel [6].

In addition to encoding the level of risk aversion of a rational agent, the utility function also resolves decision problems that involve *fixed* amounts of wealth, goods, or services. This follows from the fundamental guiding principle in the economic sciences that establishes that rational agents are utility maximizers and, consequently, they will naturally choose the alternative that provides the largest utility [1]. A more involved situation arises when the gains in question involve *uncertainty* or, in other words, when rational agents face decision problems with uncertain gains such as lotteries. In this more general regime, the concept known as the certainty equivalent (CE) is required. The certainty equivalent of a lottery represents the minimum certain amount of wealth that the rational agent is willing to accept in order to be persuaded not to participate in the given lottery. In other words, it represents the minimum *certain* amount of wealth that is *equivalent* (from the viewpoint of the rational agent) to the lottery. The certainty equivalent thus represents the economic-theoretic “value” that the rational agent assigns to a given lottery and consequently completely resolves decision problems involving uncertainty; the rational agent would choose the lottery that provides the largest certainty equivalent. The economic-theoretic concepts of decision-making rational agents, risk aversion, and the certainty equivalent are ubiquitous ideas that have emerged and found usefulness in various scientific disciplines including biology and behavioral ecology [7,8], neuroscience [9–11], thermodynamics [12–19], and information theory [20–27], with the latter being the main subject of study in this work.

Information theory, first formalized during the 1940s in the works of Shannon [28], can, pragmatically speaking, be described as the subarea of applied probability theory that deals with the operational and physical interpretation of information-theoretic quantities such as unconditional and conditional entropies, mutual information measures, capacities, unconditional and conditional divergences, among many others [29,30]. Among this zoo of quantities, the quantity known as the Rényi divergence stands out as a cornerstone quantity that acts as a “parent” to many of the abovementioned quantities, in addition to having a multitude of applications within the information sciences [31,32]. The Rényi divergence $D_\alpha(p_X^{(0)} \| p_X^{(1)})$ of order $\alpha \in [0, 1) \cup (1, \infty)$ of probability distributions $p_X^{(0)}$ and $p_X^{(1)}$ (on an alphabet $\mathcal{X}$) reads

$$D_\alpha(p_X^{(0)} \| p_X^{(1)}) = \frac{1}{\alpha - 1} \log \left[\sum_{x \in \mathcal{X}} (p_X^{(0)}(x))^\alpha (p_X^{(1)}(x))^{1-\alpha} \right]$$

and can roughly be interpreted as the level of dissimilarity between $p_X^{(0)}$ and $p_X^{(1)}$. In particular, the case $\alpha = 1$ recovers the Kullback-Leibler divergence (also known as the relative entropy) [29,33]. The Rényi divergence has found extensive use in information science [31,34,35]. Similarly, the extension of the Rényi divergence to the realm of quantum mechanics [36–41] has been an active and fruitful area of research [30,42–44]. Recently, a *multivariate* extension of the Rényi divergence taking multiple input probability distributions $\vec{P}_X = (p_X^{(0)}, \dots, p_X^{(d)})$ as well as multiple orders $\underline{\alpha} = (\alpha_0, \dots, \alpha_d) \in \mathbb{R}^{d+1}$ has received significant attention [45–51]. This multivariate Rényi divergence is given by

$$D_{\underline{\alpha}}(\vec{P}_X) = \frac{1}{\max_{0 \leq k \leq d} \alpha_k - 1} \log \left[\sum_{x \in \mathcal{X}} \prod_{k=0}^d (p_X^{(k)}(x))^{\alpha_k} \right].$$

The orders $\underline{\alpha}$ are required to satisfy $\sum_{k=0}^d \alpha_k = 1$ and either (1) $\alpha_k \geq 0$ for all $k = 0, \dots, d$ or (2) $\alpha_l > 1$ for some l and $\alpha_k \leq 0$ for the other $k \neq l$. These requirements are essential for $D_{\underline{\alpha}}$ to be an information-theoretically reasonable divergence. In particular, with these conditions for $\underline{\alpha}$, the quantity $D_{\underline{\alpha}}(\vec{P}_X)$ satisfies the data processing inequality with respect to postprocessings of the probability distributions $\vec{P}_X$, which is one of the most primitive properties that a divergence should exhibit. Interestingly, it was observed that under the same restrictions, the converse of the data processing inequality also holds in a certain sense, that is, a set of probability distributions $\vec{P}_X$ can “simulate” another set $\vec{Q}_X$ if $D_{\underline{\alpha}}(\vec{P}_X) \geq D_{\underline{\alpha}}(\vec{Q}_X)$ holds (for the precise statement, see Ref. [48]). The exploration of the multivariate Rényi divergence and adjacent topics is an active area of research that is arguably still in its infancy, with various avenues for research to potentially be uncovered and developed.

In this work, we show that the multivariate Rényi divergence characterizes operational tasks involving decision-making rational agents, risk aversion, and betting on multiple lotteries. The operational task of betting on horse races, or *horse betting games* for short, was first introduced and characterized by Kelly in 1956 within the realm of classical information theory [20] (the reader is also referred to Refs. [29,52] for a modern treatment). Using the language of economics and finance, more specifically the theory of expected utility, the original version of these games can be viewed as being played by a *risk-averse* rational agent. Recently, Bleuler *et al.* [21] extended the treatment of horse betting games to allow rational agents to display more general attitudes toward risk such as different degrees of *risk aversion*, as well as *risk-neutral* and *risk-seeking* tendencies. Later in Ref. [23], the central role of the “certainty equivalent” was put forward, and horse betting was further extended to the realm of quantum information theory, taking the form of quantum state betting (QSB) games. This development draws inspiration from horse betting as well as standard quantum state discrimination (SD), the latter being an operational task introduced during the 1970s in the works of Holevo and Helstrom in the context of quantum information theory [53,54]. Building on this interdisciplinary line of research, in this work we take one further step and introduce operational tasks involving risk aversion, betting on multiple lotteries, and show that these are characterized by the multivariate Rényi divergence, as well as by a *conditional* multivariate Rényi divergence that we propose. We furthermore apply these insights to the resource theory (RT) of measurement informativeness within general probabilistic theories (GPTs), namely, the broadest framework for describing physical phenomena that occur probabilistically. We schematically summarize these contributions in Fig. 1, and we succinctly describe them in what follows, together with the organization of this article.

Organization and summary of results

In Sec. II, we start by addressing the unconditional multivariate Rényi divergence, followed by the introduction of a new conditional multivariate Rényi divergence inspired by the bivariate Bleuler-Lapidoth-Pfister (BLP) conditional Rényi divergence [21]. As mentioned above, for the

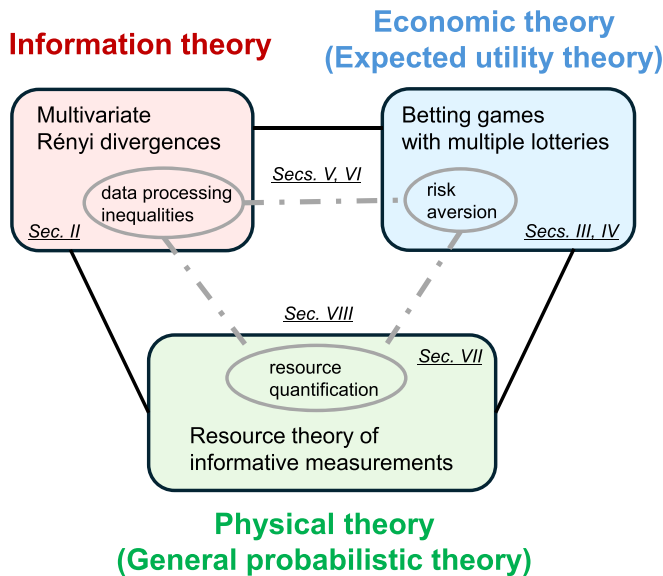


FIG. 1. Summary of the current work. We study three major themes in colored boxes and discuss their quantitative connections represented by gray lines. Technical details will be found in the underlined section attached with each topic.

(unconditional) multivariate Rényi divergence D_{α} to be a proper information-theoretic divergence, the orders α must satisfy certain conditions. This conditional quantity is defined for the same admissible orders and, with these orders, we show that it satisfies various data processing inequalities. In Sec. III, we review EUT and introduce the concepts of decision problems, risk aversion, the certainty equivalent, and the standard scenario for betting games with a single lottery.

In Sec. IV, we start with our main results by introducing operational tasks in the form of horse betting games with *multiple* lotteries, where a gambler receives multiple commodities as rewards from bets being placed on the winning horse in a horse race. We address these tasks as betting games with multiple lotteries, or *multilottery betting games* for short. In Sec. V, we show that the economic-theoretic value that a rational agent assigns to a multilottery betting game can be written in terms of the unconditional multivariate Rényi divergence. In particular, the admissible order constraints of the multivariate Rényi divergence can be reinterpreted as constraints on the gambler's risk-averse attitude toward the lotteries in question. This elucidates a direct link between the information-theoretic consistency of the multivariate divergence and the economic-theoretic consistency of the gambler's attitude toward risk. In Sec. VI, we take one step forward and consider a generalized scenario where the gambler is allowed to place bets after having access to side information. We reveal that the economic-theoretic value that the gambler assigns to the lotteries in this extended scenario can be written in terms of the conditional multivariate Rényi divergence introduced in Sec. II. In particular, we show that the data processing inequality captures the increment in economic-theoretic value due to the availability of side information, thus establishing an explicit quantitative connection between information-theoretic quantities on the one hand, and betting games within the theory of expected utility from the economic sciences on the other.

In Sec. VII, we further extend these ideas by introducing the operational tasks of state betting (SB) games within the framework of GPTs [55–61]. GPTs provide the most general description for probabilistic events involving states and measurements, including the theory of quantum mechanics as a particular case. A state betting game is constituted by replacing the classical horse race with an ensemble of states of a GPT, and considering the gambler being allowed to perform a measurement to extract side information from the received state. We show how the connection between the

conditional multivariate Rényi divergence and the economic value of a multilottery state betting game can be formulated in terms of states and measurements. In Sec. VIII, we show the applicability of these insights to the resource theory of informative measurements [62–65]. In particular, we propose a new resource measure for measurement informativeness based on multivariate Rényi divergences and interpret this measure from an economic-theoretic perspective. These latter results, in particular, allow for the extension of quantum state betting games to scenarios involving multiple lotteries within the realm of quantum resource theories [66,67]. Overall, we establish explicit connections between information theory, physical theories, and theories of economics by establishing quantitative relations between data processing inequalities, the concept of risk aversion, and resource quantifiers. In Sec. IX, we conclude with a summary and future perspectives.

II. MULTIVARIATE RÉNYI DIVERGENCES

In this section, we introduce unconditional and conditional multivariate Rényi divergences and derive various data processing inequalities.

A. Some remarks on notation

The notation to be used is as follows. Consider random variables $(X, G, \dots)$ with finite alphabets (or outcome sets) $(\mathcal{X}, \mathcal{G}, \dots)$. In this paper, the underlying probability spaces of those random variables are not explicitly shown because they do not affect the argument. The probability mass function (PMF) $p_X : \mathcal{X} \rightarrow [0, 1]$ of a random variable X is defined such that $p_X(x) = \Pr[X = x]$ ($\forall x \in \mathcal{X}$). We also call a function $p_X : \mathcal{X} \rightarrow [0, 1]$ a PMF if it satisfies $p_X(x) > 0$ ($\forall x \in \mathcal{X}$) and $\sum_{x \in \mathcal{X}} p_X(x) = 1$. In this case, the subscript X is simply an index highlighting that p_X is a function on $\mathcal{X}$, and we often do not strictly distinguish these two usages. Similarly, a function $f : \mathcal{X} \rightarrow \mathbb{R}$ on $\mathcal{X}$ is often expressed as f_X . Its expectation value with respect to a PMF p_X is denoted by $\mathbb{E}_{p_X}[f_X] := \sum_{x \in \mathcal{X}} f_X(x) p_X(x)$. Our analysis in this paper also focuses on $(d + 1)$ PMFs on the same alphabet $\mathcal{X}$. In this case, the k th PMF is written as $p_X^{(k)}$ ($k = 0, \dots, d$). We note that any PMF p_X is assumed to be with full support [$p_X(x) > 0, \forall x \in \mathcal{X}$]. In Appendix A, we loosen the assumptions that outcome sets are finite and PMFs are with full support, and study more general cases. Joint and conditional PMFs are denoted by p_{XG} and $p_{G|X}$, respectively. When considering multiple conditional PMFs, we use the expression $p_{X|G}^{(k)}$ ($k = 0, \dots, d$) similarly to the unconditional case.

For finite sets $\mathcal{X}$ and $\mathcal{Y}$, a stochastic operator (or Markov kernel) $T_{Y|X}$ can be understood as a set of conditional PMFs $\{t_{X|Y}(\cdot|x)\}_{x \in \mathcal{X}}$ on $\mathcal{Y}$ or simply a stochastic matrix. It maps a PMF p_X on $\mathcal{X}$ to a PMF q_Y on $\mathcal{Y}$ by

$$q_Y(y) = \sum_{x \in \mathcal{X}} t_{Y|X}(y|x) p_X(x), \quad (1)$$

In this case, q_Y is written as $q_Y = T_{Y|X}(p_X)$ and said to be obtained through postprocessing p_X . We also use the pseudoinverse $T_{X|Y}^\dagger$ of $T_{Y|X}$, which is a set of conditional PMFs $\{t_{X|Y}^\dagger(\cdot|y)\}_{y \in \mathcal{Y}}$ on $\mathcal{X}$ defined in terms of Eq. (1) by

$$t_{X|Y}^\dagger(x|y) = \frac{p_X(x)}{q_Y(y)} t_{Y|X}(y|x).$$

This pseudoinverse is used when postprocessing the conditioning systems of conditional PMFs (see Proposition 2).

B. Multivariate Rényi divergences

We now introduce an extension of the Rényi divergence allowing multiple input PMFs.

Definition 1 (Unconditional multivariate Rényi divergence [48,49]). Let $\vec{P}_X = (p_X^{(0)}, \dots, p_X^{(d)})$ be $(d + 1)$ PMFs with full support. The unconditional multivariate Rényi divergence of $\vec{P}_X$ and of

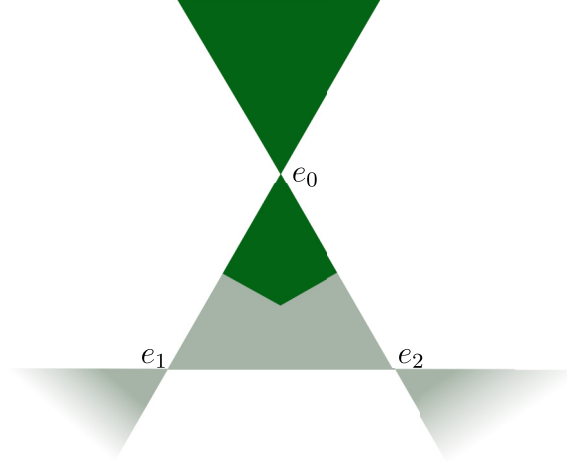


FIG. 2. The area of allowed parameter vectors $\underline{\alpha} = (\alpha_0, \alpha_1, \alpha_2) \in \mathbb{R}^3$ for the multivariate Rényi divergence $D_{\underline{\alpha}}$ in the case $d = 2$. The area is confined onto the affine plane $\sum_{k=0}^2 \alpha_k = 1$ in $\mathbb{R}^3$. The special points fixing this plane are $e_0 = (1, 0, 0)$, $e_1 = (0, 1, 0)$, and $e_2 = (0, 0, 1)$.

orders $\underline{\alpha} = (\alpha_0, \dots, \alpha_d) \in \mathbb{R}^{d+1}$ is denoted as $D_{\underline{\alpha}}(\vec{P}_X)$ and given by

$$D_{\underline{\alpha}}(\vec{P}_X) := \frac{1}{\alpha_* - 1} \log \left[\sum_{x \in \mathcal{X}} \prod_{k=0}^d (p_X^{(k)}(x))^{\alpha_k} \right],$$

where $\alpha_* := \max_{0 \leq k \leq d} \alpha_k$. The orders $\underline{\alpha} = (\alpha_0, \dots, \alpha_d)$ are real parameters satisfying $\sum_{k=0}^d \alpha_k = 1$ and either of the following two conditions:

- (1) $\alpha_k \geq 0$ for all $k = 0, \dots, d$,
- (2) $\alpha_l > 1$ for some l and $\alpha_k \leq 0$ for the other $k \neq l$.

In Fig. 2, we illustrate the parameter region for the multivariate Rényi divergence for the particular case of $d = 2$. The parameter conditions [Eq. (2)] are required so that the quantity $D_{\underline{\alpha}}(\vec{P}_X)$ appropriately behaves as a “divergence.” In fact, for any $\underline{\alpha}$ within the allowed parameter region, the quantity $D_{\underline{\alpha}}(\vec{P}_X) = D_{\underline{\alpha}}(p_X^{(0)}, \dots, p_X^{(d)})$ satisfies the data processing inequality, i.e.,

$$D_{\underline{\alpha}}(p_X^{(0)}, \dots, p_X^{(d)}) \geq D_{\underline{\alpha}}(T(p_X^{(0)}), \dots, T(p_X^{(d)})) \quad (3)$$

holds for any stochastic operator T [48]. This can be understood in the sense that operating with T makes the PMFs $p_X^{(k)}$ more similar to each other and the set $\vec{P}_X$ of PMFs less informative. Moreover, it satisfies the positive definiteness, i.e., $D_{\underline{\alpha}}(\vec{P}_X) \geq 0$ for any $\vec{P}_X = (p_X^{(0)}, \dots, p_X^{(d)})$ and $D_{\underline{\alpha}}(\vec{P}_X) = 0$ if and only if $p_X^{(0)} = \dots = p_X^{(d)}$. In Appendix B, we discuss some monotonicity properties of the divergence $D_{\underline{\alpha}}(\vec{P}_X)$ with respect to the order parameter $\underline{\alpha}$. We finally note that the case $d = 1$ recovers the (bivariate) Rényi divergence of two PMFs $p_X^{(0)}$ and $p_X^{(1)}$ and of order $\alpha \in [0, 1) \cup (1, \infty)$ as [31, 32]

$$D_{\alpha}(p_X^{(0)} \| p_X^{(1)}) = \frac{1}{\alpha - 1} \log \left[\sum_{x \in \mathcal{X}} (p_X^{(0)}(x))^{\alpha} (p_X^{(1)}(x))^{1-\alpha} \right] \quad (4)$$

up to the coefficient. Furthermore, the case $\alpha = 1$ reproduces the Kullback-Leibler divergence (also known as the relative entropy) [29, 33].

We now move on to introduce a conditional extension of the multivariate Rényi divergence. We assume that, in addition to random variables $X^{(0)}, \dots, X^{(d)}$ with the same alphabet $\mathcal{X}$, there is a random variable G on another alphabet $\mathcal{G}$ describing side information. Taking inspiration from the (bivariate) BLP conditional Rényi divergence [21], we introduce a conditional multivariate Rényi divergence as follows.

Definition 2 (Conditional multivariate Rényi divergence). Let $\vec{P}_{X|G} = (p_{X|G}^{(0)}, \dots, p_{X|G}^{(d)})$ be $(d+1)$ conditional PMFs with respect to a PMF p_G with full support. The conditional multivariate Rényi divergence of $\vec{P}_{X|G}$ and of orders $(\underline{\alpha}, \beta)$, with $\underline{\alpha} = (\alpha_0, \dots, \alpha_d) \in \mathbb{R}^{d+1}$ satisfying $\sum_{k=0}^d \alpha_k = 1$ and either (1) or (2) of Eq. (2) and $\beta \in \mathbb{R}_{\geq 0}$, is denoted as $D_{\underline{\alpha}, \beta}(\vec{P}_{X|G}|p_G)$ and given by

$$D_{\underline{\alpha}, \beta}(\vec{P}_{X|G}|p_G) := \frac{\beta}{\alpha_* - 1} \log \left[\sum_{g \in \mathcal{G}} p_G(g) \left(\sum_{x \in \mathcal{X}} \prod_{k=0}^d (p_{X|G}^{(k)}(x|g))^{\alpha_k} \right)^{1/\beta} \right], \quad (5)$$

where $\alpha_* := \max_{0 \leq k \leq d} \alpha_k$ and each $p_{X|G}^{(k)}(\cdot|g)$ is a PMF on X ($\forall g \in \mathcal{G}, k = 0, \dots, d$).

We note that the case $d = 1$ with $\beta = \alpha_*$ recovers the (bivariate) conditional Rényi divergence of order $\alpha \in (-\infty, 0) \cup (0, 1) \cup (1, \infty)$ introduced by BLP [21] up to the coefficient

$$D_{\alpha}^{\text{BLP}}(p_{X|G}^{(0)} || p_{X|G}^{(1)} | p_G) = \frac{\alpha}{\alpha - 1} \log \left[\sum_{g \in \mathcal{G}} p_G(g) \left(\sum_{x \in \mathcal{X}} (p_{X|G}^{(0)}(x|g))^{\alpha} (p_{X|G}^{(1)}(x|g))^{1-\alpha} \right)^{1/\alpha} \right].$$

To develop another information-theoretic understanding of the conditional multivariate Rényi divergence, we now address data processing inequalities that the quantity satisfies. To our knowledge, there is not yet a commonly accepted definition for data processing for conditional PMFs, so establishing general data processing inequalities for the conditional divergence is problematic. However, in Ref. [21] some steps toward this were taken in the bivariate case, and let us generalize them to our multivariate case [Eq. (5)]. The first inequality is about postprocessing the PMFs $(p_{X|G}^{(0)}, \dots, p_{X|G}^{(d)})$.

Proposition 1 (Data processing inequality with respect to the main system). Consider conditional PMFs $\vec{P}_X = (p_{X|G}^{(0)}, \dots, p_{X|G}^{(d)})$ and the conditional multivariate Rényi divergence $D_{\underline{\alpha}, \beta}(\vec{P}_X | p_G)$ in Eq. (5). Let $T_{Y|XG}$ be a stochastic operator taking each conditional PMF $p_{X|G}^{(k)}$ on $\mathcal{X} \times \mathcal{G}$ into a conditional PMF $q_{Y|G}^{(k)} := T_{Y|XG}(p_{X|G}^{(k)})$ on $\mathcal{Y} \times \mathcal{G}$ ($k = 0, \dots, d$) by

$$q_{Y|G}^{(k)}(y|g) = \sum_{x \in \mathcal{X}} t_{Y|XG}(y|x, g) p_{X|G}^{(k)}(x|g),$$

where $t_{Y|XG}(\cdot|x, g)$ is the conditional PMF on $\mathcal{Y}$ defined by $T_{Y|XG}$ ($\forall x \in \mathcal{X}, g \in \mathcal{G}$). Then,

$$D_{\underline{\alpha}, \beta}(p_{X|G}^{(0)}, \dots, p_{X|G}^{(d)} | p_G) \geq D_{\underline{\alpha}, \beta}(q_{Y|G}^{(0)}, \dots, q_{Y|G}^{(d)} | p_G)$$

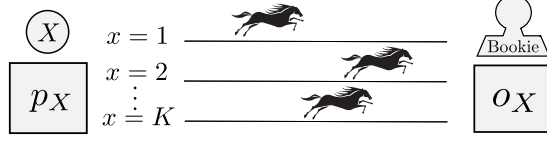
holds.

The proof follows immediately from the data processing inequality for the unconditional $D_{\underline{\alpha}}$ in Eq. (3) and the form of Eq. (5) for the multivariate conditional divergence. We now consider postprocessing of p_G .

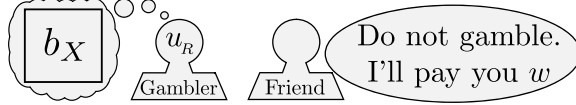
Proposition 2 (Data processing inequality with respect to the conditioning system). Consider conditional PMFs $\vec{P}_{X|G} = (p_{X|G}^{(0)}, \dots, p_{X|G}^{(d)})$ and assume $p_{X|G}^{(k)} = r_X^{(k)}$ ($k = 1, \dots, d$) to be independent of G . Through a stochastic operator $T_{H|G}$ and its pseudoinverse $T_{G|H}^{\dagger}$, we define a PMF q_H on $\mathcal{H}$ and a conditional PMF $q_{X|H}^{(0)}$ on $\mathcal{X} \times \mathcal{H}$, respectively, by

$$q_H(h) = \sum_{g \in \mathcal{G}} t_{H|G}(h|g) p_G(g), \quad q_{X|H}^{(0)}(x|h) = \sum_{g \in \mathcal{G}} t_{G|H}^{\dagger}(g|h) p_{X|G}^{(0)}(x|g) = \sum_{g \in \mathcal{G}} \frac{p_G(g)}{q_H(h)} t_{H|G}(h|g) p_{X|G}^{(0)}(x|g),$$

1. A horse race takes place (random event)
2. A bookmaker proposes an odds function



3. A gambler considers a betting strategy
4. A friend comes and offers an alternative



5. Will the gambler take the friend's offer?

FIG. 3. Illustration of a single-lottery betting game. A random event (horse race) is described by a random variable X and distributed according to a probability mass function p_X . A bookmaker proposes an odds function o_X , and a gambler considers placing a betting strategy b_X . The gambler is assumed to have a risk-averse character represented by a utility function u_R , with R the risk-averse parameter. Before the gambler fully committing to playing the game, a friend proposes to pay the gambler a fixed amount, say w , to persuade him to walk away from the betting game. Will the gambler take the friend's offer over gambling on the lottery? The solution to this decision problem is given by the isoelastic certainty equivalent $w_R^{\text{ICE}} \equiv w^{\text{ICE}}(p_X, o_X, b_X, u_R)$ of the betting game as it represents the economic-theoretic value that the gambler assigns to the lottery in question. Explicitly, the gambler will choose to gamble whenever $w_R^{\text{ICE}} > w$, and will take the friend's offer otherwise.

where $t_{H|G}$ and $t_{H|G}^\dagger$ are the conditional PMFs associated, respectively, with $T_{H|G}$ and $T_{G|H}^\dagger$. Then, with $\alpha_0 = \max_k \alpha_k$ and $\beta = \alpha_0$, the multivariate conditional Rényi divergence $D_{\underline{\alpha}, \alpha_0}(\tilde{P}_{X|G}|p_G)$ satisfies

$$D_{\underline{\alpha}, \alpha_0}(p_{X|G}^{(0)}, r_X^{(1)}, \dots, r_X^{(d)}|p_G) \geq D_{\underline{\alpha}, \alpha_0}(q_{X|H}^{(0)}, r_X^{(1)}, \dots, r_X^{(d)}|q_H). \quad (6)$$

The proof of this statement is given in Appendix C. Suppose above that the stochastic operator $T_{H|G}$ does not depend on $g \in \mathcal{G}$, i.e., $t_{H|G}(h|g) = q_H(h)$ ($\forall g \in \mathcal{G}, h \in \mathcal{H}$) holds. It implies $q_{X|H}^{(0)} = p_X^{(0)}$ and thus an important corollary follows.

Corollary 1 (Data processing inequality).

$$D_{\underline{\alpha}, \alpha_0}(p_{X|G}^{(0)}, r_X^{(1)}, \dots, r_X^{(d)}|p_G) \geq D_{\underline{\alpha}}(p_X^{(0)}, r_X^{(1)}, \dots, r_X^{(d)}).$$

This last data processing inequality will play an important role in the subsequent sections, as it explicitly relates the conditional multivariate Rényi divergence to its unconditional counterpart. This will prove useful when providing an operational interpretation of multivariate Rényi divergences in terms of betting games with multiple lotteries. In order to do so, let us now address the theory of expected utility from the economic sciences.

III. EXPECTED UTILITY THEORY

We begin by describing the operational task of horse betting games (or simply betting games) [20, 21, 29, 52]. Consider three types of agents: a referee, a gambler, and a bookmaker (see Fig. 3). The referee is in charge of a probabilistic event governed by a PMF p_X on an outcome set $\mathcal{X}$. We interpret the event as a horse race; the value $p_X(x)$ exhibits the probability for each horse $x \in \mathcal{X}$ to win the race. The bookmaker is in charge of proposing a non-negative function $o_X : \mathcal{X} \rightarrow \mathbb{R}_{\geq 0}$,

the “odds” (or “payoff”) function, on $\mathcal{X}$ and announces it to the gambler. Under this setting, the gambler is going to “bet” on every possible horse $x \in \mathcal{X}$. In other words, the gambler can say, for instance: “I believe horse 1 will win with probability 20%” and “horse 2 with probability 30%,” and similarly with the rest of $\mathcal{X}$. This can be modeled by asking the gambler to provide a *betting strategy* in the form of a PMF b_X on $\mathcal{X}$ ($b_X(1) = 0.2$, $b_X(2) = 0.3$, ... in the previous example). In a standard scenario of economic theory, the amount of *wealth* (money, goods, etc.) is represented by positive real numbers. Naturally, we interpret the amount $w_1 \in \mathbb{R}_{\geq 0}$ as “more valuable” than $w_2 \in \mathbb{R}_{\geq 0}$ if $w_1 > w_2$. Now, let the gambler start with an initial amount of wealth $w^i > 0$. The gambler is assumed to distribute all the initial wealth as bets according to b_X , i.e., the gambler bets $b_X(x)w^i$ on each horse x . After the gambler defines the betting strategy b_X , the referee holds the race and reveals the winning horse, say, x' . The bookmaker then rewards the gambler with an amount of wealth (final wealth) proportional to the betting done on the correct index given as $w^f = b_X(x')o_X(x')w^i$ (and nothing otherwise), with o_X the pre-established odds function. From this point forward, and for simplicity, we assume that the gambler starts with initial wealth given by a unit of wealth $w^i = 1$, and so final wealth is simply $w^f = w_X(x') = b_X(x')o_X(x')$. One important property for later discussion is the *fairness* of the odds, which is given by $F := \sum_x 1/o_X(x)$. The odds are considered to be fair when $F = 1$, subfair when $F > 1$, and superfair when $F < 1$ [20,21,29,52].

Let us take these elements a step further from the perspective of EUT, arguably the main theoretical framework used in economics and finance since the 1940s [1]. A key ingredient of EUT is the notion of utility functions. A utility function $u : \mathbb{R} \rightarrow \mathbb{R}$ expresses the level of “satisfaction” of a rational agent when acquiring an amount of wealth; the value $u(w)$ quantifies how satisfied the agent is when receiving the amount of wealth w . For simplicity, in this paper we mainly study the case when wealth w and utility u are non-negative, while the converse situations express “loss” and “dissatisfaction,” respectively. A natural requirement is that the utility function u is monotonically increasing with wealth, i.e., $u(w_1) > u(w_2)$ if $w_1 > w_2$, simply meaning that a larger amount of wealth leads to a larger satisfaction for the agent. The principle of EUT is that under uncertain situations, rational agents make decisions based not on expected wealth but on expected utility. A specific context in which this principle plays a crucial role is the decision problem involving uncertain gains such as *lotteries*.

The main subject of this paper is the notion of lotteries in EUT. A lottery is a pair (p_X, w_X) of a PMF p_X on an outcome set $\mathcal{X}$ and a wealth (or reward) function w_X , describing the situation where wealth $\{w_X(x)\}_{x \in \mathcal{X}}$ is randomly distributed according to $\{p_X(x)\}_{x \in \mathcal{X}}$. It is clear that the setup of a betting game described before corresponds to a lottery with a wealth function given by $w_X = b_X o_X$ and the PMF p_X . Important to our discussion later on, here we note that this setup involves a single-commodity lottery. Economists have derived ways of quantifying how “attractive” lotteries are to rational agents. In short, the economic-theoretic “value” of a lottery (p_X, w_X) for an agent (a gambler) modeled by a utility function u is given by the CE $w^{\text{CE}} \equiv w^{\text{CE}}(p_X, w_X, u)$, which is defined as the amount of wealth that generates the same level of satisfaction as the expected utility of the lottery, and so is given by

$$u(w^{\text{CE}}) := \mathbb{E}_{p_X}[u(w_X)] = \sum_{x \in \mathcal{X}} p_X(x)u(w_X(x)). \quad (7)$$

This definition indicates that the expected (uncertain) profit of the lottery is “equivalent” to the certain gain of w^{CE} for the agent. That is, according to the principle of EUT, the agent chooses the lottery instead of a fixed amount of wealth, say w , whenever $w^{\text{CE}} > w$. In this sense, the quantity w^{CE} can be considered as a reasonable measure of the economic-theoretic value that the agent assigns to the lottery. From the perspective of economics, the certainty equivalent also expresses the agent’s attitude toward *risk*. To see this, suppose that $w^{\text{CE}} < \mathbb{E}_{p_X}[w_X]$. In this case, although the agent is expected to receive a larger amount of wealth through the gamble, the agent is satisfied enough by certainly gaining w^{CE} , and so this describes that the agent is risk averse. With similar arguments, when $w^{\text{CE}} > \mathbb{E}_{p_X}[w_X]$, the agent is seen as risk seeking, and when $w^{\text{CE}} = \mathbb{E}_{p_X}[w_X]$, the agent is considered risk neutral [3–5].

The certainty equivalent w^{CE} depends on the lottery in question as well as the utility function u . There are several utility functions of interest to economists, one of which is the *isoelastic utility function*

$$u_R(w) := \frac{w^{1-R} - 1}{1 - R} \quad (R \in \mathbb{R}). \quad (8)$$

The parameter R describes the agent's attitude of risk aversion through the degree of convexity (or concavity) of the function [23]; $R > 0$ describes a risk-averse attitude, $R < 0$ risk-seeking, and $R = 0$ risk neutral. Below, we mainly consider risk-averse agents and use a simpler form [68,69]

$$u_R(w) := \frac{w^{1-R}}{1 - R} \quad (R \in \mathbb{R}_{\geq 0}). \quad (9)$$

Later, we will see that the assumption of risk aversion is related to the data processing inequalities for multivariate Rényi divergences. Note that the function u_R is set as $u_R(w) = \log w$ for $R = 1$ by taking the limit in Eq. (8). In the following, the diverging factor $(1 - R)^{-1}$ is usually canceled, and thus the case $R = 1$ is treated by setting $R \rightarrow 1$ at the final stage of the calculation. The certainty equivalent of the isoelastic function, or simply the isoelastic certainty equivalent (ICE), for a betting game is then given by

$$w_R^{\text{ICE}} = \left(\sum_{x \in \mathcal{X}} p_X(x) (b_X(x) o_X(x))^{1-R} \right)^{\frac{1}{1-R}}.$$

The notion of the ICE will play a crucial role in the following analysis. This finishes the description of betting games and EUT with single-commodity lotteries. Let us move on to the new scenario of lotteries with multiple commodities that we introduce in this work.

IV. BETTING GAMES WITH MULTIPLE LOTTERIES

Expected utility theory can accommodate the analysis of multicommodity wealth $\vec{w} = (w^{(1)}, \dots, w^{(d)}) \in \mathbb{R}^d$ by considering utility functions $u : \mathbb{R}^d \rightarrow \mathbb{R}$ [70–73]. Similarly to the single-commodity case, here $u(\vec{w})$ represents the level of satisfaction that an agent experiences when acquiring the vector of commodities $\vec{w} \in \mathbb{R}^d$.

This situation can be combined with betting games to be rephrased as betting games with multiple lotteries or simply multilottery betting games. Specifically, in a d -lottery betting game, played by a referee, a gambler, and d bookmakers, the referee is in charge of a random event (a horse race) governed by a PMF $p_X^{(0)}$ on $\mathcal{X}$. Unlike the single-commodity case, here the gambler is rewarded with a set of multiple commodities described by wealth vectors $\vec{w} = (w^{(1)}, \dots, w^{(d)})$ from multiple bookmakers. That is, there are d bookmakers offering d odds functions $\vec{O}_X = (o_X^{(1)}, \dots, o_X^{(d)})$ in advance and the gambler placing d bets $\vec{B}_X = (b_X^{(1)}, \dots, b_X^{(d)})$ of PMFs, where $o_X^{(k)} : \mathcal{X} \rightarrow \mathbb{R}_{\geq 0}$ and $b_X^{(k)} : \mathcal{X} \rightarrow [0, 1]$ are, respectively, the odds and betting strategy for the k th component ($k = 1, \dots, d$). The referee then reveals the winning label (horse), say x' , and asks the bookmakers to reward the gambler with a *multicommodity wealth* vector $\vec{W}(x') = (w_X^{(1)}(x'), \dots, w_X^{(d)}(x'))$, with each $w_X^{(k)}(x') = b_X^{(k)}(x') o_X^{(k)}(x')$ ($k = 1, \dots, d$) a wealth function on $\mathcal{X}$. This scenario can be effectively understood as the gambler playing d lotteries $\{(p_X^{(0)}, w_X^{(k)})\}_{k=1}^d \equiv (p_X^{(0)}, \vec{W}) \equiv (p_X^{(0)}, \vec{O}_X, \vec{B}_X)$.

To clarify the economic-theoretic meaning of these betting fractions, one can intuitively regard the gambler as managing d separate budgets to bet on the lotteries. Unlike the standard single-lottery scenario where a single budget is distributed across all outcomes, here the gambler allocates a fraction $b_X^{(k)}(x)$ of the k th independent budget to horse x . This idea of assigning a separate budget to each lottery aligns with multiattribute utility theory, where an agent's total wealth is divided into independent components [74–76]. Consequently, each PMF $b_X^{(k)}$ operates as a standard single-commodity betting strategy, but is strictly confined to its respective budget.

As in the previous construction, we are interested in the economic-theoretic “value” that a rational agent with a utility function $u : \mathbb{R}^d \rightarrow \mathbb{R}$ assigns to the d lotteries $(p_X^{(0)}, \vec{W})$. One possible approach is to consider the CE vector $\vec{w}^{\text{CE}} := (w^{\text{CE}}, w^{\text{CE}}, \dots, w^{\text{CE}}) \in \mathbb{R}^d$ defined via

$$u(\vec{w}^{\text{CE}}) := \mathbb{E}_{p_X}[u(\vec{W})] = \sum_{x \in \mathcal{X}} p_X(x) u(\vec{W}(x)). \quad (10)$$

Similarly to the single-commodity scenario, here the CE $w^{\text{CE}}(p_X^{(0)}, \vec{W}, u) \equiv w^{\text{CE}}$ represents the amount of certain per-commodity wealth that generates the same level of satisfaction for the gambler as playing the d lotteries $(p_X^{(0)}, \vec{W})$. In other words, the gambler is willing to accept a certain amount of wealth w^{CE} per lottery (so, $d w^{\text{CE}}$ in total), instead of playing the d lotteries $(p_X^{(0)}, \vec{W}) = \{(p_X^{(0)}, w_X^{(k)})\}_k$ to obtain uncertain rewards $\vec{W}(x') = (w_X^{(1)}(x'), \dots, w_X^{(d)}(x'))$. There are several potential utility functions of interest, one of which is the multicommodity isoelastic utility function $u_R : \mathbb{R}^d \rightarrow \mathbb{R}$ generalizing Eq. (9) as [74,75]

$$u_R(\vec{w}) := \prod_{k=1}^d \frac{(w^{(k)})^{1-R_k}}{1-R_k}, \quad (11)$$

where $\underline{R} = (R_1, \dots, R_d) \in \mathbb{R}_{\geq 0}^d$ is called the *risk-aversion vector* implying that the gambler behaves in a risk-averse manner for every lottery $k = 1, \dots, d$. This multicommodity utility function satisfies various desirable properties expected in the multicommodity setting [76], as we will discuss in detail shortly. When applied to our betting game, it particularly gives the ICE $w_R^{\text{ICE}}(p_X^{(0)}, \vec{O}_X, \vec{B}_X, u_R)$ by

$$w_R^{\text{ICE}}(p_X^{(0)}, \vec{O}_X, \vec{B}_X, u_R) = \left(\sum_{x \in \mathcal{X}} p_X^{(0)}(x) \prod_{k=1}^d (b_X^{(k)}(x) o_X^{(k)}(x))^{1-R_k} \right)^{\frac{1}{\sum_{k=1}^d (1-R_k)}} \quad (12)$$

through the formula [Eq. (10)]. In Fig. 4, we further illustrate the task of multilottery betting games explained so far.

The economic-theoretic motivation for adopting this multiplicative structure rather than simply an additive formulation $u_R(\vec{w}) = \sum_k u_{R_k}(w^{(k)})$ is twofold. First, the additive form implies that the marginal utility of one commodity is completely independent of the others. In contrast, the multiplicative form treats the rewards holistically. This allows the utility function to reflect the interactions among different assets where, for instance, a payoff from one lottery can compensate for the risks taken in another thus capturing the agent’s comprehensive satisfaction. It moreover encompasses the situation where the agent does not treat all risk equally; the agent often assigns different levels of satisfaction to different types of wealth depending on the nature of the lottery. For example, a gambler may exhibit lower risk aversion toward a lottery with a familiar source, while applying a higher risk-aversion level to a more uncertain one. The multiplicative structure captures this economic behavior by merging the agent’s preference for specific commodities with their risk profile. Second, drawing from multiattribute utility theory, the product form provides the necessary flexibility to attach an independent risk-aversion parameter R_k to each lottery k while preserving a type of overall scale invariance, which we describe below. Mathematically, this product form stands out as an ideal candidate because it is continuous, twice differentiable, and crucially, its relative risk aversion (RRA) measure $\text{RRA}_{w^{(k)}}$ with respect to each variable $w^{(k)}$ is constant. Specifically, this measure is given by

$$\text{RRA}_{w^{(k)}}(\vec{w}, u_R) := -w^{(k)} \frac{\frac{\partial^2}{\partial (w^{(k)})^2} u_R(\vec{w})}{\frac{\partial}{\partial w^{(k)}} u_R(\vec{w})} = R_k.$$

That is, the product form $u_R(\vec{w})$ displays constant relative risk aversion (CRRA) with respect to the variable $w^{(k)}$ for all $k = 1, \dots, d$. This is an appealing property for economists, as it models situations where the risk-aversion measure is scale invariant, i.e., independent of the wealth vector

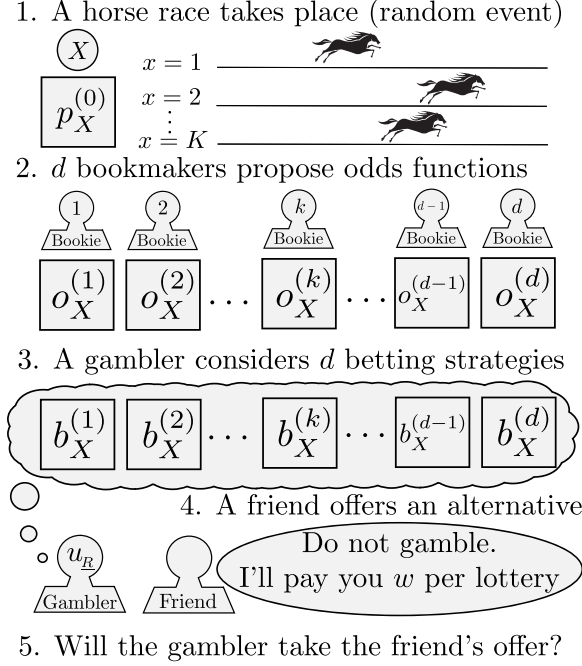


FIG. 4. Schematic representation of a betting game extended to a multilottery framework. A random event (horse race) is described by a random variable X distributed according to a probability mass function $p_X^{(0)}$. In this setting, a group of d bookmakers proposes d odds functions $\bar{O}_X = (o_X^{(1)}, \dots, o_X^{(d)})$, and a gambler places d betting strategies $\bar{B}_X = (b_X^{(1)}, \dots, b_X^{(d)})$. Accordingly, the gambler's risk profile is characterized by a risk-aversion vector $\bar{R} = (R_1, \dots, R_d) \in \mathbb{R}_{\geq 0}^d$ within the utility function $u_{\bar{R}}$, where each R_k is the risk-aversion parameter associated with the k th lottery. To persuade the gambler to withdraw from the betting, a fixed offer of w per lottery is presented. The decision to accept this offer or play depends on the isoelastic certainty equivalent $w_{\bar{R}}^{\text{ICE}} \equiv w^{\text{ICE}}(p_X^{(0)}, \bar{O}_X, \bar{B}_X, u_{\bar{R}})$, which quantifies the gambler's valuation of the lotteries. Explicitly, the gambler chooses to play (rejecting the persuasion) whenever $w_{\bar{R}}^{\text{ICE}} > w$.

$\bar{w}$. Additional desirable properties involving the mixed partial derivatives $\partial^2 u_{\bar{R}} / \partial w^{(i)} \partial w^{(j)}$ were discussed in Ref. [76]. For these reasons, CRRA-type utility functions of product form are widely and recurrently used in the literature of economics and finance [74–76].

From the information-theoretic perspective, our objective is to give the multivariate Rényi divergences of Definition 1 and their conditional counterparts of Definition 2 an operational interpretation through multilottery betting games. Not only does the utility function of Eq. (11) satisfy natural monotonicity properties and describe risk aversion in the multicommodity setting [76], but we will see that this utility function has a natural connection to the (conditional) multivariate Rényi divergences, thus providing the desired utility-theoretic interpretation for these entropic quantities. Since the multivariate Rényi divergences are central among all multivariate entropic quantities in the sense that they span all other general multivariate divergences [50,77,78] (in the set of which they represent extreme rays), we argue that the connection subsequently established between the multicommodity isoelastic utility function and the multivariate Rényi divergences gives further support for the use of the former to describe the behavior of rational agents in multicommodity environments.

To conclude this section, we note that the “rational agents” here considered are not restricted to human gamblers, but more abstractly can represent any entity capable of decision-making under uncertainty, including living beings subject to evolutionary pressures [7,8]. In this biological context,

multiple commodities can naturally be considered as analogues of wealth. Two examples are the animal's physical integrity (safety) as well as the resources it needs to survive (like food or energy intake). Higher levels of safety and energy intake naturally generate a higher level of satisfaction, which directly translates to greater evolutionary fitness. Building on this biophysical lens, we can also find a natural interpretation of the certainty equivalent, or the “buyout” mechanism. Consider an animal facing an uncertain environment (a risky lottery) where foraging in an exposed area may yield abundant food, but could also lead to a predator encounter that compromises its safety. In this scenario, accepting the certainty equivalent corresponds to choosing a deterministic alternative without uncertainty such as entering into a state akin to “hibernation.” With this latter choice, both the animal's physical safety and a minimal energy intake could be completely guaranteed, thus effectively trading the uncertain high reward for a risk-free buyout. Importantly, while foundational studies on evolutionary risk aversion have focused on a single lottery [7,8], the multilottery scenario here considered could more realistically capture these biological settings. Living organisms usually need to evaluate multiple commodities simultaneously, and so this framework could elegantly model this by assigning decoupled, independent risk-aversion parameters R_k to each commodity. This flexibility would allow us to capture realistic scenarios where, for instance, an animal exhibits extreme risk aversion regarding its physical safety, but lower risk aversion regarding its food intake. This highlights that the multivariate extension here considered is not merely of mathematical curiosity, but a necessary step to consistently model complex decision-making in real systems.

V. UNCONDITIONAL MULTIVARIATE RÉNYI DIVERGENCE AND MULTILOTTERY BETTING GAMES

We are now in position to derive a connection between the certainty equivalent of multilottery betting games and the multivariate Rényi divergence.

Let us consider a d -lottery betting game involving a random event (horse race), d bookmakers ($d \geq 1$), and a gambler. The random event is described by a PMF $p_X^{(0)}$, and the d bookmakers offer d odds $\vec{O}_X = (o_X^{(1)}, \dots, o_X^{(d)})$. The gambler then proceeds to place d betting strategies described by PMFs $\vec{B}_X = (b_X^{(1)}, \dots, b_X^{(d)})$, thus effectively establishing d lotteries as $\{(p_X^{(0)}, w_X^{(k)})\}_{k=1}^d \equiv (p_X^{(0)}, \vec{O}_X, \vec{B}_X)$, with $w_X^{(k)} = b_X^{(k)} o_X^{(k)}$ ($k = 1, \dots, d$). The gambler is also assumed to behave according to a multicommodity isoelastic utility function $u_{\underline{R}}$, where $\underline{R} = (R_1, \dots, R_d)$ is a risk-aversion vector with each risk-aversion value $R_k \in \mathbb{R}_{\geq 0}$ specific to lottery k and satisfying either of the following conditions:

- (1) $R_k \geq 1$ ($k = 1, \dots, d$),
 - (2) $0 < R_k < 1$ ($k = 1, \dots, d$) and $\sum_{k=1}^d R_k > d - 1$.
- (13)

Under this setup, the economic-theoretic value that the gambler assigns to the d lotteries in question is given, according to expected utility theory, by the ICE $w_{\underline{R}}^{\text{ICE}} \equiv w^{\text{ICE}}(p_X^{(0)}, \vec{O}_X, \vec{B}_X, u_{\underline{R}})$ via Eq. (12). This value can be characterized by the multivariate Rényi divergence as follows.

Theorem 1 (Economic-theoretic interpretation of the unconditional multivariate Rényi divergence). Given the setup above, the log isoelastic certainty equivalent assigned by the gambler can be decomposed as

$$\log w^{\text{ICE}}(p_X^{(0)}, \vec{O}_X, \vec{B}_X, u_{\underline{R}}) = D_{\underline{\alpha}}(p_X^{(0)}, p_X^{(1)}, \dots, p_X^{(d)}) + \sum_{k=1}^d \frac{\alpha_k}{\alpha_0 - 1} (D_{S_k}(q_X^{(k)} \| b_X^{(k)}) + \log F^{(k)}),$$
(14)

where $D_{\underline{\alpha}}(p_X^{(0)}, p_X^{(1)}, \dots, p_X^{(d)})$ is the multivariate Rényi divergence of PMFs $p_X^{(0)}$ and

$$p_X^{(k)} := \frac{1}{F^{(k)} o_X^{(k)}}, \quad F^{(k)} := \sum_{x \in \mathcal{X}} \frac{1}{o_X^{(k)}(x)},$$
(15)

and $D_{S_k}(q_X^{(k)} \| b_X^{(k)})$ is the bivariate Rényi divergence of $b_X^{(k)}$ and a PMF $q_X^{(k)}$ independent of $b_X^{(k)}$ ($k = 1, \dots, d$). The orders $\underline{\alpha} = (\alpha_0, \dots, \alpha_d)$ and S_k for these divergences are determined by the risk-aversion vector and given, respectively, as

$$\alpha_0 := \left(1 + \sum_{k=1}^d (R_k - 1)\right)^{-1}, \quad \alpha_k := (R_k - 1)\alpha_0, \quad (16)$$

and

$$S_k := \frac{\sum_{l=0}^k \alpha_l}{\sum_{l=0}^{k-1} \alpha_l} \quad (\geq 0).$$

The full proof of this statement is in Appendix D 1, where we gave the explicit expression of $q_X^{(k)}$ and also developed a continuous extension of the result to general outcome sets. There are several aspects of this result worth analyzing. First, this result is effectively a mathematical identity and thus holds very generally for all PMFs $p_X^{(0)}$, all types of odds $\vec{O}_X$ proposed by the bookmakers, all types of betting strategies $\vec{B}_X$ placed by the gambler, as well as for arbitrary agents with general attitudes toward risk described by the multicommodity utility function u_R as in Eq. (11). Second, it is worth highlighting that whilst the left-hand side in Eq. (14) corresponds to a purely economic-theoretically operational task, specifically the certainty equivalent of d lotteries, the right-hand side contains information-theoretic quantities in the form of a multivariate Rényi divergence as well as d bivariate Rényi divergences. This result thus establishes a direct link between notions in the realm of the theory of games and economic behavior on the one hand, and information-theoretic quantities on the other. It is also of particular interest that the quantitative characterization of the gambler's risk-averse attitude for each lottery completely reproduces the order restrictions for the multivariate Rényi divergence. To see this, let us remark that the conditions [Eqs. (13) and (16)] for the risk-aversion vector are equivalent to Eq. (2) and $R_k = 1 + \alpha_k/\alpha_0$. The former constraints display an information-theoretically clear interpretation; the gambler treats every lottery k with a similar degree of risk aversion: $R_k \geq 1$ (sensitive to risk) or $0 < R_k < 1$ (insensitive to risk) for every k . In other words, the seemingly peculiar conditions for $\underline{\alpha}$ can now be understood as not just mathematical requirements that make $D_{\underline{\alpha}}$ information-theoretically consistent, but also as expressing the gambler's economic-theoretic consistency toward risk in a multilottery betting game, where the gambler never prioritizes a specific lottery and acts analogously for all lotteries. Finally, the decomposition in the right-hand side of Eq. (14) can be divided into three terms: a first term concerning the multivariate Rényi divergence of PMFs $p_X^{(k)}$, a second term concerning the bivariate Rényi divergences, and a third term containing the coefficients $\{F^{(k)}\}_k$. These terms are rich in explanatory power, which we exploit next.

Let us start by analyzing the term in Eq. (14) containing the d bivariate Rényi divergences. The coefficient $\alpha_k/(\alpha_0 - 1)$ is nonpositive, and taking into account that the bivariate Rényi divergence $D_{S_k}(q_X^{(k)} \| b_X^{(k)})$ is non-negative, this makes these terms nonpositive. We can use this decomposition to extract the optimal betting strategies for the maximization of the certainty equivalent and, consequently, the term can be interpreted as the penalization to the gambler's satisfaction for not implementing the optimal bets. Explicitly, the term $D_{S_k}(q_X^{(k)} \| b_X^{(k)})$ is the gap between the gambler's bet $b_X^{(k)}$ and the optimal bet $q_X^{(k)}$ for lottery k . We summarize this in the following corollary.

Corollary 2 (Optimal betting strategies). The log isoelastic certainty equivalent of a d -lottery betting game with a gambler implementing optimal betting strategies reads

$$\max_{\vec{B}_X} \log w^{\text{ICE}}(p_X^{(0)}, \vec{O}_X, \vec{B}_X, u_R) = D_{\underline{\alpha}}(\vec{P}_X) + \sum_{k=1}^d \frac{\alpha_k}{\alpha_0 - 1} \log F^{(k)},$$

where the optimization is taken over all betting strategies represented by PMFs $\vec{B}_X$.

We continue the analysis of the decomposition in Eq. (14) and consider now the term containing the constants $F^{(k)} = \sum_x 1/o_X^{(k)}(x)$. This quantity depends exclusively on the odds proposed by the bookmakers and, more specifically, describes the fairness of the odds. When all the odds are subfair ($F^{(k)} > 1, \forall k$), the value of the lottery decreases [taking into account that the coefficient $\alpha_k/(\alpha_0 - 1)$ is nonpositive], as one can intuitively expect. Similarly, when all the odds are superfair ($F^{(k)} < 1, \forall k$), the value of the lotteries on the other hand increases, which rarely happens as bookmakers usually offer odds that are subfair. Finally, when all the odds are fair ($F^{(k)} = 1, \forall k$), these terms disappear. Additionally, with this restriction, we obtain $p_X^{(k)} = (o_X^{(k)})^{-1}$ ($k = 1, \dots, d$) as per Eq. (15). Writing d fair odds functions as $\vec{O}_X^{\text{Fair}}$, we can summarize the latter case in the following corollary.

Corollary 3 (The unconditional multivariate Rényi divergence characterizes multilottery betting, optimal betting strategies, and fair odds). The log isoelastic certainty equivalent of a d -lottery betting game with fair odds and a gambler implementing optimal betting strategies reads

$$\max_{\vec{B}_X} \log w^{\text{ICE}}(p_X^{(0)}, \vec{O}_X^{\text{Fair}}, \vec{B}_X, u_{\underline{R}}) = D_{\underline{R}}(\vec{P}_X),$$

where the optimization is taken over all betting strategies.

This latter statement provides a conceptually clean interpretation of the multivariate Rényi divergence as the economic-theoretic value that rational agents assign to multilottery betting games, when allowed to maximize over betting strategies, and when all the odds are fair. In particular, one appealing feature of this statement is that the Rényi orders α_k can now also be interpreted, as they explicitly represent the degree of risk aversion manifested by the agent toward each lottery in question as $R_k = 1 + \alpha_k/\alpha_0$ ($k = 1, \dots, d$). In addition to being of purely mathematical interest, the multivariate Rényi divergence is now endowed with an economic-theoretic interpretation. In order to reinforce these conceptual aspects, we will further elaborate on this idea in subsequent sections by introducing tasks in the framework of resource theories within general probabilistic theories, where these results will find further applicability. We now move on to consider a generalized scenario where the rational agent has access to side information.

VI. CONDITIONAL MULTIVARIATE RÉNYI DIVERGENCE AND MULTILOTTERY BETTING GAMES

We now consider multilottery betting games where the gambler can place betting strategies after having access to side information. In Fig. 5, we illustrate this scenario and fully describe it as follows.

We consider a d -lottery game constituted almost in the same way as before; a random event (a horse race) takes place according to a PMF $p_X^{(0)}$ on an outcome set $\mathcal{X}$, d bookmakers offer d odds functions $\vec{O}_X = (o_X^{(1)}, \dots, o_X^{(d)})$, and a gambler makes d bets. However, this time we assume that the random variable X is correlated to another random variable G whose outcome set is $\mathcal{G}$ and thus there is a joint PMF $p_{XG}^{(0)}$ on $\mathcal{X} \times \mathcal{G}$. Also, we allow the gambler to access G and propose d betting strategies according to its outcome. The gambler's bets are now d conditional PMFs $\vec{B}_{X|G} = (b_{X|G}^{(1)}, \dots, b_{X|G}^{(d)})$ on $\mathcal{X} \times \mathcal{G}$ and yield multicommodity wealth $(b_{X|G}^{(1)}(x|g)o_X^{(1)}(x), \dots, b_{X|G}^{(d)}(x|g)o_X^{(d)}(x))$ when the winning label is $X = x$ and the gambler receives $G = g$. The scenario is effectively equivalent to the gambler playing d lotteries $\{p_{XG}^{(0)}, b_{X|G}^{(k)}o_X^{(k)}\}_{k=1}^d \equiv (p_{XG}^{(0)}, \vec{O}_X, \vec{B}_{X|G})$, where $b_{X|G}^{(k)}o_X^{(k)} : (x, g) \mapsto b_{X|G}^{(k)}(x|g)o_X^{(k)}(x)$ is the k th wealth function. Then, by means of the formula [Eq. (12)], the value $w_R^{\text{ICE}}(p_{XG}^{(0)}, \vec{O}_X, \vec{B}_{X|G}, u_{\underline{R}})$ of this d -lottery betting game for the gambler modeled by the multicommodity isoelastic utility function $u_{\underline{R}}$ in Eq. (11) is

$$w_{\underline{R}}^{\text{ICE}}(p_{XG}^{(0)}, \vec{O}_X, \vec{B}_{X|G}, u_{\underline{R}}) = \left(\sum_{x \in \mathcal{X}} p_{XG}^{(0)}(x, g) \prod_{k=1}^d (b_{X|G}^{(k)}(x|g)o_X^{(k)}(x))^{1-R_k} \right)^{\frac{1}{\sum_{k=1}^d (1-R_k)}}, \quad (17)$$

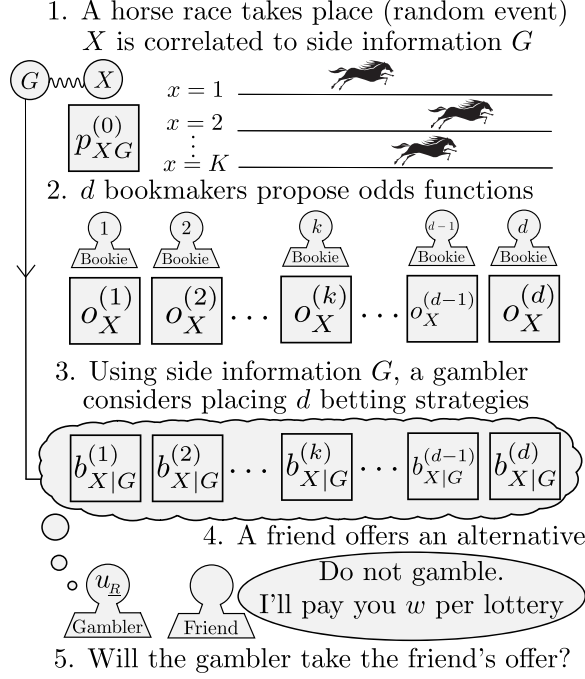


FIG. 5. Multilottery betting games with side information. Unlike standard multilottery betting games described in Fig. 3, the random variable X is now correlated to another random variable G and jointly distributed according to a joint probability mass function $p_{XG}^{(0)}$. The gambler places d betting strategies $\vec{b}_{X|G} = (b_{X|G}^{(1)}, \dots, b_{X|G}^{(d)})$ having access to the side information G . The economic-theoretic value of the lotteries is given by $w_{\underline{R}}^{\text{ICE}} \equiv w^{\text{ICE}}(p_{XG}^{(0)}, \vec{O}_X, \vec{B}_{X|G}, u_{\underline{R}})$.

where $\underline{R} = (R_1, \dots, R_d)$ is a risk-aversion vector with each risk-aversion value $R_k \in \mathbb{R}_{\geq 0}$ specific to lottery k and assumed to satisfy either of the following conditions:

- (1) $R_k \geq 1 \quad (k = 1, \dots, d)$,
- (2) $0 < R_k < 1 \quad (k = 1, \dots, d)$ and $\sum_{k=1}^d R_k > d - 1$.

We can now present the conditional counterpart of Theorem 1.

Theorem 2 (Economic-theoretic interpretation of the conditional multivariate Rényi divergence). Given the setup above, the log isoelastic certainty equivalent assigned by the gambler can be upper bounded as

$$\log w^{\text{ICE}}(p_{XG}^{(0)}, \vec{O}_X, \vec{B}_{X|G}, u_{\underline{R}}) \leq D_{\underline{\alpha}, \alpha_0}(p_{X|G}^{(0)}, p_X^{(1)}, \dots, p_X^{(d)} | p_G^{(0)}) + \sum_{k=1}^d \frac{\alpha_k}{\alpha_0 - 1} (D_{S_k}(q_{XG}^{(k)} \| r_{XG}^{(k)}) + \log F^{(k)}), \quad (19)$$

where $D_{\underline{\alpha}, \alpha_0}(p_{X|G}^{(0)}, p_X^{(1)}, \dots, p_X^{(d)} | p_G^{(0)})$ is the conditional multivariate Rényi divergence of PMFs $p_{X|G}^{(0)}$ and

$$p_X^{(k)} := \frac{1}{F^{(k)} o_X^{(k)}}, \quad F^{(k)} := \sum_{x \in \mathcal{X}} \frac{1}{o_X^{(k)}(x)} \quad (k = 1, \dots, d)$$

and $D_{S_k}(q_{XG}^{(k)} \| r_{XG}^{(k)})$ is the bivariate Rényi divergence of joint PMFs $q_{XG}^{(k)} = q_G^{(k)} q_{X|G}^{(k)}$ and $r_{XG}^{(k)} = q_G^{(k)} b_{X|G}^{(k)}$, with $q_{X|G}^{(k)}$ independent of $b_{X|G}^{(k)}$. The orders $\underline{\alpha} = (\alpha_0, \dots, \alpha_d)$ and S_k for these divergences are determined by the risk-aversion vector and given, respectively, as

$$\alpha_0 := \left(1 + \sum_{k=1}^d (R_k - 1)\right)^{-1}, \quad \alpha_k := (R_k - 1)\alpha_0, \quad (20)$$

and

$$S_k := \frac{\sum_{l=0}^k \alpha_l}{\sum_{l=0}^{k-1} \alpha_l} \quad (\geq 0).$$

This result can be proven using similar methods to Theorem 1 and the full proof is presented in Appendix D.2. Again, this statement holds very generally for all joint PMFs $p_{XG}^{(0)}$, all types of odds $\vec{O}_X$ proposed by the bookmakers, all types of betting strategies placed by the gambler and described by conditional PMFs $\vec{B}_{X|G}$, as well as for arbitrary agents with risk-averse attitudes described by the multicommodity utility function u_R in Eq. (11). Also, the order restriction is rephrased by the gambler's risk-averse attitude through Eqs. (18) and (20). The major difference being that it is now the conditional multivariate Rényi divergence, the quantity that characterizes the certainty equivalent. Similarly to Theorem 1, we can now extract the optimal betting strategies in this generalized scenario.

Corollary 4 (Optimal betting strategies). The log isoelastic certainty equivalent of a d -lottery betting game with side information and a gambler implementing optimal betting strategies reads

$$\max_{\vec{B}_{X|G}} \log w^{\text{ICE}}(p_{XG}^{(0)}, \vec{O}_X, \vec{B}_{X|G}, u_R) = D_{\underline{\alpha}, \alpha_0}(\vec{P}_{X|G} | p_G^{(0)}) + \sum_{k=1}^d \frac{\alpha_k}{\alpha_0 - 1} \log F^{(k)},$$

where the optimization is taken over all betting strategies represented by conditional PMFs $\vec{B}_{X|G}$.

The analysis concerning the fairness of the odds is similar to that for Theorem 1 and we thus obtain the following corollary.

Corollary 5 (Conditional multivariate Rényi divergence characterizes multilottery betting games, access to side information, optimal betting strategies, and fair odds). The log isoelastic certainty equivalent of a d -lottery betting game with side information, fair odds, and a gambler implementing optimal betting strategies reads

$$\max_{\vec{B}_{X|G}} \log w^{\text{ICE}}(p_{XG}^{(0)}, \vec{O}_X^{\text{Fair}}, \vec{B}_{X|G}, u_R) = D_{\underline{\alpha}, \alpha_0}(\vec{P}_{X|G} | p_G^{(0)}),$$

where the optimization is taken over all betting strategies.

This provides an interpretation of the conditional multivariate Rényi divergence as the economic-theoretic value that a rational agent assigns to a multilottery betting game, when all the bookmakers propose fair odds, when the agent has access to side information, and when the agent is allowed to maximize over betting strategies. Taking into account that there can be multiple ways of defining conditional multivariate Rényi divergences, one technical aspect worth highlighting is that this approach singles out a specific conditional multivariate Rényi divergence (the BLP type) and, among these, it furthermore specifies the second order to be $\alpha_0 = \max_k \alpha_k$. Finally, we can now also provide an economic-theoretic interpretation of the data processing inequality (Corollary 1) as follows.

Corollary 6 (Economic-theoretic interpretation of the data processing inequality). We compare a scenario of a d -lottery betting game where the gambler has access to some side information G against a situation where the gambler does not have access to side information. In both situations, the gambler is allowed to maximize over all betting strategies (over conditional PMFs $\vec{B}_{X|G}$ when having access to G , and over PMFs B_X when there is no side information). Using the data processing inequality for the conditional multivariate Rényi divergence, we have for their log isoelastic

certainty equivalents

$$\max_{\bar{B}_{X|G}} \log w^{\text{ICE}}(p_{XG}^{(0)}, \bar{O}_X^{\text{Fair}}, \bar{B}_{X|G}, u_R) \geq \max_{\bar{B}_X} \log w^{\text{ICE}}(p_X^{(0)}, \bar{O}_X^{\text{Fair}}, \bar{B}_X, u_R). \quad (21)$$

In other words, having access to side information increases the economic-theoretic value that the gambler assigns to the d lotteries.

We therefore observe that having access to side information (Theorem 2) increases the certainty equivalent when compared to having no side information (Theorem 1). In words, side information increases the “value” of the d lotteries in question for a risk-averse gambler. We again emphasize that the gambler’s economic-theoretically consistent attitude [Eq. (18)] of risk aversion for every lottery now implies the data processing inequality as an information-theoretic consequence. Specifically, Corollary 5 and Corollary 6 reveal a quantitative connection between the economic-theoretic benefit and the data processing inequality through the conditional multivariate Rényi divergence. This finishes our presentation of the results concerning the characterization of multilottery betting games in terms of multivariate unconditional and conditional Rényi divergences. We are now ready to show the applicability of these results within the operational framework of general probabilistic theories and reveal the connection between the information, economic, and physical theories.

VII. STATE BETTING GAMES IN GENERAL PROBABILISTIC THEORIES AND MULTIVARIATE RÉNYI DIVERGENCES

A. General probabilistic theories

We have so far considered betting on classical labels. In this section, we introduce the task of *state betting*, where the gambler bets on states being allowed to perform measurements as side information for the bets. This scenario is well described by notions in GPTs. A GPT is a pair of subsets $(\Omega, \mathcal{E})$ of a finite-dimensional real Euclidean space V . The set Ω , called a *state space*, is a compact and convex subset of V whose linear and affine hulls satisfy $\text{lin}(\Omega) = V$ and $\text{aff}(\Omega) \not\ni O$ (the origin of V), respectively. Elements of Ω are called *states* and they mathematically represent physical systems. On the other hand, the set $\mathcal{E}$, called an *effect space*, is defined as $\mathcal{E} = \{m \in V \mid \langle m, \omega \rangle \in [0, 1], \forall \omega \in \Omega\}$ and its elements are called *effects*. Here, $\langle m, \omega \rangle$ denotes the Euclidean inner product of two vectors $\omega, m \in V$. The element u of $\mathcal{E}$ satisfying $\langle u, \omega \rangle = 1$ ($\forall \omega \in \Omega$) is called the *unit effect*. A *measurement* M with an outcome set $\mathcal{A}$ is defined as a family of effects (more precisely, an *effect-valued measure*) $M = \{m_a\}_{a \in \mathcal{A}}$ such that $\sum_{a \in \mathcal{A}} m_a = u$. The measurement M is a mathematical representation of a measurement apparatus. The output $\{\langle m_a, \omega \rangle\}_{a \in \mathcal{A}}$ represents the statistics observed when the measurement is performed using the apparatus on a physical system described by a state $\omega \in \Omega$. In particular, each effect m_a outputs the probability of observing an outcome $a \in \mathcal{A}$ in the experiment. Throughout this article, we assume the *no-restriction hypothesis* [58] for mathematical simplicity; we assume that all elements of $\mathcal{E}$ are physically valid, but this requirement is not imposed in certain settings [79–81].

The most important and physically relevant examples of GPTs are quantum and classical probability theories. With an n -dimensional complex Hilbert space $\mathcal{H}$ ($n < \infty$), a finite-dimensional quantum theory is described in terms of the n^2 -dimensional real vector space $\mathcal{L}_S(\mathcal{H})$ of self-adjoint operators on $\mathcal{H}$. The space $\mathcal{L}_S(\mathcal{H})$ is regarded as a Euclidean space equipped with the Hilbert-Schmidt inner product $\langle X, Y \rangle_{\text{HS}} = \text{Tr}[X^\dagger Y]$. A finite-dimensional quantum theory is then given by the pair $(\mathcal{S}(\mathcal{H}), \mathcal{E}(\mathcal{H}))$, where $\mathcal{S}(\mathcal{H})$ is the set of all positive semidefinite and unit-trace operators (density operators) on $\mathcal{H}$ and $\mathcal{E}(\mathcal{H}) = \{E \in \mathcal{L}_S(\mathcal{H}) \mid \langle \rho, E \rangle_{\text{HS}} \in [0, 1], \forall \rho \in \mathcal{S}(\mathcal{H})\}$. We note that $E \in \mathcal{E}(\mathcal{H})$ if and only if E is bounded by the zero operator O and the identity operator (the unit effect) I on $\mathcal{H}$, namely, $O \leq E \leq I$ [82,83]. Measurements in the theory $(\mathcal{S}(\mathcal{H}), \mathcal{E}(\mathcal{H}))$ are called positive operator-valued measures. On the other hand, a finite-dimensional classical probability theory is described by a probability simplex [84]. With $V = \mathbb{R}^n$, it is given as $(\mathcal{P}_n, \mathcal{E}(\mathcal{P}_n))$, where $\mathcal{P}_n = \{(p_i)_{i=1}^n \in V \mid p_i \geq 0, \sum_i p_i = 1\}$ is the set of all n -outcome PMFs and

$\mathcal{E}(\mathcal{P}_n) = \{(v_i)_{i=1}^n \in V \mid v_i \in [0, 1]\}$. This finishes a concise summary of GPTs. A more rigorous formulation and advanced topics can be found in Refs. [85–87].

B. Economic-theoretic values of multilottery state betting games

Let us move on to generalizing betting games discussed in previous sections and introduce state betting games. The task of state betting was first proposed in the realm of quantum theory [23]. In this article, we generalize the scenario in terms of multiple lotteries and states in GPTs.

Consider again three types of agents: a referee, a gambler, and d bookmakers. This time, instead of classical labels, the referee has an ensemble Λ of states of a GPT $(\Omega, \mathcal{E})$, namely, $\Lambda = \{p_X^{(0)}(x), \omega_x\}_{x \in \mathcal{X}}$, with $\omega_x \in \Omega$ and $p_X^{(0)}$ a PMF on $\mathcal{X}$. The referee starts the game by sending the gambler one of the states, say $\omega_{x'}$, according to $p_X^{(0)}$. The goal of the game is for the gambler to correctly identify the state (or the index x') that was sent. In order to do this, the gambler is allowed to perform a measurement $\mathbf{M} = \{m_a\}_{a \in \mathcal{A}}$ and extract information from the state. The gambler is also allowed to postprocess the measurement outcome a to produce an index g as the guess of the state. Until this point, the game is essentially the same as standard SD, where the gambler “wins” whenever $g = x'$ and “loses” otherwise. In SD, the quantity of interest is the probability of successfully identifying the correct state. It is calculated as

$$p_{\text{succ}}^{\text{SD}}(\mathbf{M}, \Lambda) = \max_{s_{G|A}} \sum_{g, a, x} \delta_{x, g} s_{G|A}(g|a) \langle m_a, \omega_x \rangle p_X^{(0)}(x),$$

where the optimization is over all postprocessing strategies $s_{G|A}$ represented by conditional PMFs. We now generalize the task of SD incorporating the idea of betting games. In an SB game, the gambler does not provide a unique index g , as a guess of the state that was sent, but instead to “bet” on all possible outcomes according to the measurement and its postprocessing. The scenario closely resembles the betting games discussed so far. Here, we further employ the multicommodity setting to constitute a multilottery state betting game. That is, d bookmakers propose d odds functions $\vec{O}_X = (o_X^{(1)}, \dots, o_X^{(d)})$ in addition to the referee’s ensemble $\Lambda = \{p_X^{(0)}(x), \omega_x\}_{x \in \mathcal{X}}$. On the other hand, the gambler performs the measurement $\mathbf{M} = \{m_a\}_{a \in \mathcal{A}}$ on the received state $\omega_{x'}$ and postprocesses the outcome through the conditional PMF $s_{G|A}$ to obtain a label $g \in \mathcal{G}$. The gambler then provides d betting strategies $\vec{B}_{X|G=g} = (b_{X|G=g}^{(1)}, \dots, b_{X|G=g}^{(d)})$ in the form of conditional PMFs $b_{X|G=g}^{(k)} = \{b_{X|G=g}^{(k)}(x|g)\}_{x \in \mathcal{X}}$ on $\mathcal{X}$ according to the side information g on the state. After placing the d bets $\vec{B}_{X|G=g}$, the referee reveals the state that was sent (or the index x'), and the bookmakers reward the gambler with multicommodity wealth $(w_{XG}^{(1)}(x', g), \dots, w_{XG}^{(d)}(x', g))$, where $w_{XG}^{(k)}(x', g) = b_{X|G=g}^{(k)}(x'|g) o_X^{(k)}(x')$. This SB scenario can be interpreted as a d -lottery betting game with side information, where the side information is conveyed by the state and extracted by the gambler by means of the measurement and postprocessing such that $p_{XG}^{(0)}[\mathbf{M}, \Lambda, s_{G|A}](x, g) = \sum_{a \in \mathcal{A}} s_{G|A}(g|a) \langle m_a, \omega_x \rangle p_X^{(0)}(x)$. A schematic representation of this operational setting is presented in Fig. 6.

Now, let us present another main result on the economic-theoretic “value” of the multilottery SB game. Our analysis above implies that the SB game can directly be related to the variant of a d -lottery betting game, which involves the agent having access to side information. In particular, the scenario is identified with the gambler playing d lotteries $\{(p_{XG}^{(0)}[\mathbf{M}, \Lambda, s_{G|A}], b_{X|G}^{(k)} o_X^{(k)})\}_{k=1}^d \equiv (\mathbf{M}, \Lambda, s_{G|A}, \vec{O}_X, \vec{B}_{X|G})$. The value of the lotteries is given by the isoelastic certainty equivalent $w^{\text{ICE}}(\mathbf{M}, \Lambda, s_{G|A}, \vec{B}_{X|G}, \vec{O}_X, u_{\vec{R}})$, which is defined via Eq. (17) with the multicommodity isoelastic utility function $u_{\vec{R}}$ in Eq. (11) of a risk-aversion vector $\vec{R} = (R_1, \dots, R_d)$ with each risk-aversion value $R_k \in \mathbb{R}_{\geq 0}$ specific to lottery k and assumed to satisfy either of the following conditions:

- (1) $R_k \geq 1$ ($k = 1, \dots, d$),
- (2) $0 < R_k < 1$ ($k = 1, \dots, d$) and $\sum_{k=1}^d R_k > d - 1$. (22)

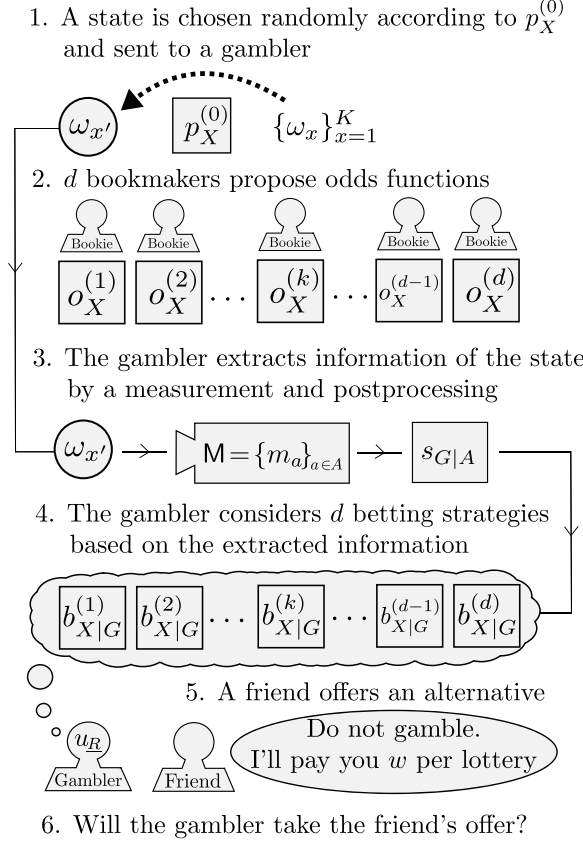


FIG. 6. Illustration of a multilottery state betting game. Instead of a classical horse race, a state is randomly chosen according to an ensemble of states $\Lambda = \{p_X^{(0)}(x), \omega_x\}_x$. A group of d bookmakers proposes d odds functions $\vec{O}_X = (o_X^{(1)}, \dots, o_X^{(d)})$, and the gambler extracts information of the state from a measurement $\mathbf{M} = \{m_a\}_a$ and a postprocessing strategy $s_{G|A}$ to place d betting strategies $\vec{B}_{X|G} = (b_{X|G}^{(1)}, \dots, b_{X|G}^{(d)})$. The economic-theoretic value that the gambler assigns to the gambling is given by the isoelastic certainty equivalent $w^{\text{ICE}}(\mathbf{M}, \Lambda, s_{G|A}, \vec{B}_{X|G}, \vec{O}_X, u_R)$.

This motivates us to rephrase Theorem 2 in terms of SB.

Theorem 3 (Economic-theoretic characterization of the multilottery state betting game through the multivariate Rényi divergence). The log isoelastic certainty equivalent assigned to the state betting is upper bounded by

$$\begin{aligned}
 & \log w^{\text{ICE}}(\mathbf{M}, \Lambda, s_{G|A}, \vec{O}_X, \vec{B}_{X|G}, u_R) \\
 & \leq D_{\underline{\alpha}, \alpha_0}(p_{X|G}^{(0)}[\mathbf{M}, \Lambda, s_{G|A}], p_X^{(1)}, \dots, p_X^{(d)} | p_G^{(0)}[\mathbf{M}, \Lambda, s_{G|A}]) \\
 & + \sum_{k=1}^d \frac{\alpha_k}{\alpha_0 - 1} (D_{S_k}(q_{XG}^{(k)} \| r_{XG}^{(k)}) + \log F^{(k)}),
 \end{aligned} \tag{23}$$

where the PMFs $p_{X|G}^{(0)}[\mathbf{M}, \Lambda, s_{G|A}]$ and $p_G^{(0)}[\mathbf{M}, \Lambda, s_{G|A}]$ are determined through the joint PMF $p_{XG}^{(0)}[\mathbf{M}, \Lambda, s_{G|A}]$ and

$$p_X^{(k)} := \frac{1}{F^{(k)} o_X^{(k)}}, \quad F^{(k)} := \sum_{x \in \mathcal{X}} \frac{1}{o_X^{(k)}(x)} \quad (k = 1, \dots, d),$$

and the joint PMFs $q_{XG}^{(k)}$ and $r_{XG}^{(k)}$ are of the form $q_{XG}^{(k)} = q_G^{(k)} q_{X|G}^{(k)}$ and $r_{XG}^{(k)} = q_G^{(k)} b_{X|G}^{(k)}$, with $q_{X|G}^{(k)}$ independent of $b_{X|G}^{(k)}$. The orders $\underline{\alpha} = (\alpha_0, \dots, \alpha_d)$ and S_k for the divergences $D_{\underline{\alpha}, \alpha_0}$ and D_{S_k} are determined by the risk aversion and given, respectively, as

$$\alpha_0 := \left(1 + \sum_{k=1}^d (R_k - 1) \right)^{-1}, \quad \alpha_k := (R_k - 1) \alpha_0,$$

and

$$S_k := \frac{\sum_{l=0}^k \alpha_l}{\sum_{l=0}^{k-1} \alpha_l} \quad (\geq 0). \quad (24)$$

We omit the proof because it proceeds in the same way as Theorem 2. We note that the PMF $p_{XG}^{(0)}[\mathbf{M}, \Lambda, s_{G|A}] : (x, g) \mapsto \sum_a s_{G|A}(g|a) \langle m_a, \omega_x \rangle p_X^{(0)}(x)$ on $\mathcal{X} \times \mathcal{G}$ is postprocessing of another PMF $p_{XA}^{(0)}[\mathbf{M}, \Lambda] : (x, a) \mapsto \langle m_a, \omega_x \rangle p_X^{(0)}(x)$ on $\mathcal{X} \times \mathcal{A}$ via $s_{G|A}$. It enables us to apply the data processing inequality in Proposition 2 and optimize the ICE on all postprocessing strategies.

Corollary 7 (Optimal betting and postprocessing strategies for the state betting game). When optimizing over the betting $\vec{B}_{X|G}$ and postprocessing strategies $s_{G|A}$, the upper bound of Eq. (23) can be reached:

$$\begin{aligned} & \max_{\vec{B}_{X|G}, s_{G|A}} \log w^{\text{ICE}}(\mathbf{M}, \Lambda, s_{G|A}, \vec{O}_X, \vec{B}_{X|G}, u_R) \\ &= D_{\underline{\alpha}, \alpha_0}(p_{X|A}^{(0)}[\mathbf{M}, \Lambda], p_X^{(1)}, \dots, p_X^{(d)} | p_A^{(0)}[\mathbf{M}, \Lambda]) + \sum_{k=1}^d \frac{\alpha_k}{\alpha_0 - 1} \log F^{(k)}, \end{aligned}$$

where $p_{X|A}^{(0)}[\mathbf{M}, \Lambda]$ and $p_A^{(0)}[\mathbf{M}, \Lambda]$ are determined through the joint PMF $p_{XA}^{(0)}[\mathbf{M}, \Lambda](x, a) = p_X^{(0)}(x) \langle m_a, \omega_x \rangle$. In particular, when the odds are fair ($F^{(k)} = 1, \forall k$), it reduces to

$$\max_{\vec{B}_{X|G}, s_{G|A}} \log w^{\text{ICE}}(\mathbf{M}, \Lambda, s_{G|A}, \vec{O}_X^{\text{Fair}}, \vec{B}_{X|G}, u_R) = D_{\underline{\alpha}, \alpha_0}(p_{X|A}^{(0)}[\mathbf{M}, \Lambda], p_X^{(1)}, \dots, p_X^{(d)} | p_A^{(0)}[\mathbf{M}, \Lambda]),$$

with $p_X^{(k)} = (o_X^{(k)})^{-1}$ ($k = 1, \dots, d$).

The results again provide a practical interpretation of the multivariate Rényi divergence through the economic-theoretic value of a task involving betting with multiple lotteries. What is specific to the current situation is that the influence of states (precisely, an ensemble) Λ and a measurement $\mathbf{M}$ on the task is incorporated through the PMFs $p_{XG}^{(0)}[\mathbf{M}, \Lambda, s_{G|A}]$ and $p_{XA}^{(0)}[\mathbf{M}, \Lambda]$. In other words, here we have revealed a quantitative connection between information-theoretic, economic-theoretic, and physical notions. In the subsequent section, we will develop a resource-theoretic aspect of the task of SB.

Let us conclude the section with several remarks. First, we can check that SB games generalize SD. In fact, if we restrict the single-lottery scenario ($d = 1$), set the odds as constant $o_X^{(1)}(x) := C > 0$ ($\forall x$), and the gambler to have a neutral attitude toward risk ($R = 0$), it is straightforward to check that the optimal CE becomes

$$\max_{b_{X|G}^{(1)}, s_{G|A}} w_{R=0}^{\text{ICE}}(\mathbf{M}, \Lambda, s_{G|A}, C, b_{X|G}^{(1)}) = C p_{\text{succ}}^{\text{SD}}(\mathbf{M}, \Lambda), \quad (25)$$

where $b_{X|G}^{(1)}(x|g) = \delta_{x,g}$ realizes the maximum over all $b_{X|G}^{(1)}$. Second, the SB discussed here is a natural extension of QSB introduced in Ref. [23] to GPTs. When the GPT is quantum, namely, $(\Omega, \mathcal{E}) = (\mathcal{S}(\mathcal{H}), \mathcal{E}(\mathcal{H}))$, the task of SB reduces to QSB. Also, (multilottery) betting games are reproduced simply by applying a classical probability theory $(\Omega, \mathcal{E}) = (\mathcal{P}_n, \mathcal{E}(\mathcal{P}_n))$. These observations reinforce the operational aspect of the multivariate Rényi divergence.

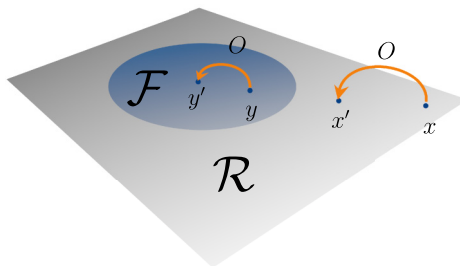


FIG. 7. A depiction of a resource theory with resource set $\mathcal{R}$ and free set $\mathcal{F}$. A free operation $O \in \mathcal{O}_{\mathcal{F}}$ is required to map a free resource $y \in \mathcal{F}$ into a free resource. A genuine resource $x \in \mathcal{R} \setminus \mathcal{F}$ is mapped by O often in some sense “closer” to the free set, and this kind of monotonicity is quantitatively expressed by the notion of resource monotones.

VIII. RESOURCE-THEORETIC EXPRESSIONS OF STATE BETTING GAMES

Let us now analyze multilottery SB games from the perspective of RTs. In this paper, we mainly study RTs in the realm of GPTs generalizing quantum resource theories. The basis of any RT is the set $\mathcal{R}$ of all resources. This set may consist of various elements such as states and measurements. Mathematically, $\mathcal{R}$ is often modeled as a subset of an affine plane or a vector space; For states and measurements, $\mathcal{R}$ is a subset of the underlying vector space that embeds the state and effect spaces, respectively. Among the resources in $\mathcal{R}$, some are considered to possess no resource. The set of all such elements $\mathcal{F} \subset \mathcal{R}$ is called the *free set*, because we usually think of zero-resource elements as available to us without restriction. In contrast, elements of $\mathcal{R} \setminus \mathcal{F}$ are called *resourceful*. The free set $\mathcal{F}$ is often convex, in which case the RT is also called a *convex resource theory*. Another important concept in RTs is the set $\mathcal{O}_{\mathcal{F}}$ of *free operations*, suitable maps on $\mathcal{R}$, which are somehow freely available. Essentially always a minimal requirement for $O \in \mathcal{O}_{\mathcal{F}}$ is that $O(x) \in \mathcal{F}$ whenever $x \in \mathcal{F}$, i.e., free maps cannot create resources out of free elements. Figure 7 gives a very general picture of RTs. The main object of this section is the RT of informative measurements [62,65].

Definition 3 (Resource theory of informative measurements). Consider a GPT $(\Omega, \mathcal{E})$. The set $\mathcal{R}$ of all measurements $\mathbf{M} = \{m_a\}_{a \in \mathcal{A}}$ on a discrete outcome set constitutes the set of all resources. We say that a measurement $\mathbf{N} = \{n_a\}_{a \in \mathcal{A}} \in \mathcal{R}$ is *uninformative* (or *trivial*) if all effects n_a are multiples of the unit effect u , i.e., $n_a = q_A(a)u$ for all $a \in \mathcal{A}$ with a PMF $q_A : \mathcal{A} \rightarrow [0, 1]$. The resource theory with its free set $\mathbb{U} \subset \mathcal{R}$ of uninformative measurements is called the resource theory of informative measurements. As a set $\mathcal{O}_{\mathbb{U}}$ of free operations, we consider postprocessing maps introduced as follows: We pick another discrete set $\mathcal{B}$ and a conditional PMF $t_{B|A}$. We define a map $T_{B|A}$ that maps a measurement $\mathbf{M} = \{m_a\}_{a \in \mathcal{A}}$ with outcomes $\mathcal{A}$ to a measurement $T_{B|A}(\mathbf{M}) = \{m'_b\}_{b \in \mathcal{B}}$ with outcomes $\mathcal{B}$, where

$$m'_b = \sum_{a \in \mathcal{A}} t_{B|A}(b|a) m_a. \quad (26)$$

These postprocessing maps clearly satisfy $T_{B|A}(\mathbf{N}) \in \mathbb{U}$ whenever $\mathbf{N} \in \mathbb{U}$.

A central question in RTs is to find reasonable quantifications of resources. Such quantifiers are often required to be monotonic with respect to free operations. More precisely, for an RT $(\mathcal{R}, \mathcal{F}, \mathcal{O}_{\mathcal{F}})$, a function $f : \mathcal{R} \rightarrow \mathbb{R}$ on the resource set $\mathcal{R}$ is called a *resource measure* (or *resource monotone*) if $f(x) \geq 0$ for any $x \in \mathcal{R}$, $f(x) = 0$ whenever $x \in \mathcal{F}$, and $f(x) \geq f(O(x))$ for any $x \in \mathcal{R}$ and $O \in \mathcal{O}_{\mathcal{F}}$ [66,67]. Among resource measures, information-theoretically and physically (operationally) meaningful quantifiers are particularly important. Here, we propose a resource measure for informative measurements based on multivariate Rényi divergence and reveal its practical interpretation through the task of multilottery SB games.

Let $\underline{\alpha} = (\alpha_0, \dots, \alpha_d) \in \mathbb{R}^{d+1}$ be orders satisfying $\alpha_0 = \max_{0 \leq k \leq d} \alpha_k$, $\sum_{k=0}^d \alpha_k = 1$, and one of the conditions of Eq. (2). Consider a GPT $(\Omega, \mathcal{E})$. For an ensemble of states $\Lambda = \{p_X^{(0)}, \omega_x\}_{x \in \mathcal{X}}$ and a measurement $\mathbf{M} = \{m_a\}_{a \in \mathcal{A}}$, we define a joint PMF $p_{XA}^{(0)}[\mathbf{M}, \Lambda](x, a) = \sum_{a \in \mathcal{A}} \langle m_a, \omega_x \rangle p_X^{(0)}(x)$ on $\mathcal{X} \times \mathcal{A}$. As earlier, we also introduce a d -lottery SB game involving a referee, d bookmakers ($d \geq 1$), and a gambler. The referee possesses an ensemble of states $\Lambda = \{p_X^{(0)}(x), \omega_x\}_{x \in \mathcal{X}}$ to choose one of the states according to the PMF $p_X^{(0)}$ and send it to the gambler. The d bookmakers offer d odds functions $\vec{O}_X = (o_X^{(1)}, \dots, o_X^{(d)})$. The gambler is allowed to perform a measurement $\mathbf{M} = \{m_a\}_{a \in \mathcal{A}}$ on the received state and its postprocessing via $s_{G|A}$, and use them to place d betting strategies $\vec{B}_{X|G} = (b_{X|G}^{(k)})_{k=1}^d$. It effectively establishes d lotteries $\{(p_{XG}^{(0)}[\mathbf{M}, \Lambda, s_{G|A}], b_{X|G}^{(k)} o_X^{(k)})\}_{k=1}^d \equiv (\mathbf{M}, \Lambda, s_{G|A}, \vec{O}_X, \vec{B}_{X|G})$, with $p_{XG}^{(0)}[\mathbf{M}, \Lambda, s_{G|A}](x, g) = \sum_{a \in \mathcal{A}} s_{G|A}(g|a) \langle m_a, \omega_x \rangle p_X^{(0)}(x)$. We consider the ratio

$$\mathcal{W}_{\mathbb{UI}}(\mathbf{M}, \Lambda, \vec{O}_X, u_{\underline{R}}) := \frac{\max_{\vec{B}_{X|G}, s_{G|A}} w^{\text{ICE}}(\mathbf{M}, \Lambda, s_{G|A}, \vec{O}_X, \vec{B}_{X|G}, u_{\underline{R}})}{\max_{\mathbf{N} \in \mathbb{UI}} \max_{\vec{B}_{X|G}, s_{G|B}} w^{\text{ICE}}(\mathbf{N}, \Lambda, s_{G|B}, \vec{O}_X, \vec{B}_{X|G}, u_{\underline{R}})}$$

of the optimal isoelastic certainty equivalent representing the figure of merit in the SB game provided by a measurement $\mathbf{M}$ with outcomes $\mathcal{A}$ against all free measurements $\mathbf{N}$ with outcomes $\mathcal{B}$.

Theorem 4 (Resource measures for informative measurements based on multilottery state betting games). (1) For any uninformative measurement $\mathbf{N} = \{n_b\}_{b \in \mathcal{B}} \in \mathbb{UI}$, ensemble of states Λ , and d PMFs $\{r_X^{(k)}\}_{k=1}^d$ on $\mathcal{X}$, the multivariate Rényi divergences satisfy

$$D_{\underline{\alpha}, \alpha_0}(p_{X|B}^{(0)}[\mathbf{N}, \Lambda], r_X^{(1)}, \dots, r_X^{(d)} | p_B^{(0)}[\mathbf{N}, \Lambda]) = D_{\underline{\alpha}}(p_X^{(0)}, r_X^{(1)}, \dots, r_X^{(d)}),$$

and thus the quantity

$$\mathcal{D}_{\underline{\alpha}, \alpha_0}(\mathbf{M}, \Lambda, \{r_X^{(k)}\}_{k=1}^d) := D_{\underline{\alpha}, \alpha_0}(p_{X|B}^{(0)}[\mathbf{M}, \Lambda], r_X^{(1)}, \dots, r_X^{(d)} | p_B^{(0)}[\mathbf{M}, \Lambda]) - D_{\underline{\alpha}}(p_X^{(0)}, r_X^{(1)}, \dots, r_X^{(d)})$$

is a resource measure for the resource theory of informative measurements. That is, $\mathcal{D}_{\underline{\alpha}, \alpha_0}(\mathbf{M}, \Lambda, \{r_X^{(k)}\}) \geq 0$ and $\mathcal{D}_{\underline{\alpha}, \alpha_0}(\mathbf{M}, \Lambda, \{r_X^{(k)}\}) = 0$ whenever $\mathbf{M} \in \mathbb{UI}$, and for any free operation $T_{B|A}$, the monotonicity

$$\mathcal{D}_{\underline{\alpha}, \alpha_0}(\mathbf{M}, \Lambda, \{r_X^{(k)}\}) \geq \mathcal{D}_{\underline{\alpha}, \alpha_0}(T_{B|A}(\mathbf{M}), \Lambda, \{r_X^{(k)}\}) \quad (27)$$

holds.

(2) With a risk-aversion vector $\underline{R} = (R_1, \dots, R_d) \in \mathbb{R}_{\geq 0}^d$ defined via the parameters $\underline{\alpha}$ as

$$R_k = 1 + \frac{\alpha_k}{\alpha_0} \quad (k = 1, \dots, d),$$

and PMFs

$$p_X^{(k)} := \frac{1}{F^{(k)} o_X^{(k)}}, \quad F^{(k)} := \sum_{x \in \mathcal{X}} \frac{1}{o_X^{(k)}(x)} \quad (k = 1, \dots, d)$$

determined by the odds $\vec{O}_X$, the resource measure $\mathcal{D}_{\underline{\alpha}, \alpha_0}(\mathbf{M}, \Lambda, \{p_X^{(k)}\}_{k=1}^d) \equiv \mathcal{D}_{\underline{\alpha}, \alpha_0}(\mathbf{M}, \Lambda, \vec{O}_X)$ is expressed as

$$\mathcal{D}_{\underline{\alpha}, \alpha_0}(\mathbf{M}, \Lambda, \vec{O}_X) = \log \mathcal{W}_{\mathbb{UI}}(\mathbf{M}, \Lambda, \vec{O}_X, u_{\underline{R}}). \quad (28)$$

The proof is presented in Appendix D3. This result proposes a resource measure $\mathcal{D}_{\underline{\alpha}, \alpha_0}$ for measurement informativeness based on the difference between the unconditional and conditional multivariate Rényi divergences. The measure is also economic-theoretically meaningful; in terms of Eq. (28), it reflects the natural intuition that using informative measurements is more beneficial to the gambler in a multilottery SB game than using uninformative measurements. In other words, the result can be understood as providing an operational, physical interpretation of the

data processing inequality for multivariate Rényi divergences, which was initially seen as an information-theoretic relation, through multilottery SB games. It is again notable that the data processing inequality (Corollary 1) for multivariate Rényi divergences plays a crucial role when showing the monotonicity [Eq. (27)]. Now, the order conditions [Eq. (2)] together with an additional requirement $\alpha_0 = \max_k \alpha_k$ are not only rephrased as the economic-theoretically consistent behavior of a risk-averse gambler toward all lotteries but also lead to a new quantifier of how beneficial a measurement is in any GPT, the most general framework of physical theory. We also note that the measure $\mathcal{D}_{\underline{\alpha}, \alpha_0}$ reproduces a *complete set of monotones* for measurement informativeness in certain cases. This can be confirmed by recalling that the set of success probabilities $\{p_{\text{succ}}^{\text{SD}}(\mathbf{M}, \Lambda) \mid \Lambda\}$ of SDs in a GPT is a complete set of monotones; for two measurements $\mathbf{M}_1$ and $\mathbf{M}_2$, there exists a postprocessing map $T \in \mathcal{O}_{\mathbb{U}}$ such that $\mathbf{M}_2 = T(\mathbf{M}_1)$ if and only if $p_{\text{succ}}^{\text{SD}}(\mathbf{M}_1, \Lambda) \geq p_{\text{succ}}^{\text{SD}}(\mathbf{M}_2, \Lambda)$ for every state ensemble Λ [62, 65]. Then, by setting $\underline{\alpha} = (1, 0, 0, \dots, 0)$ and applying constant odds, we can reduce the ICE $w_{R=0}^{\text{ICE}}(\mathbf{M}, \Lambda, s_{G|A}, C, b_{X|G}^{(1)})$ of the corresponding SB to the success probability $p_{\text{succ}}^{\text{SD}}(\mathbf{M}, \Lambda)$ [see Eq. (25)] and thus verify the claim.

IX. SUMMARY AND OUTLOOK

In this work, we showed that multivariate Rényi divergences characterize multilottery betting games. Specifically, we revealed that the unconditional multivariate Rényi divergence represents the economic-theoretic value that a rational agent (gambler) assigns to an operational task based on risk aversion and betting on multiple lotteries when the odds are fair, and when the gambler is allowed to maximize over all betting strategies. Furthermore, we studied a generalized scenario where the gambler is allowed to implement betting strategies after having access to side information. The analysis of this extended scenario led us to introduce a new conditional multivariate Rényi divergence, as the appropriate quantity that describes this generalized scenario involving side information. In particular, we derived a data processing inequality for this new conditional multivariate Rényi divergence, and proved that it can be endowed with an economic-theoretic interpretation; having access to side information increases the economic-theoretic value that the gambler assigns to the lotteries in question. This latter statement is somewhat intuitive, and can now be grounded in a clean mathematical statement of the data processing inequality. It is also particularly notable that the order restrictions for the conditional multivariate Rényi divergence to satisfy the data processing inequality are derived from the gambler's economic-theoretic consistency of risk aversion for all lotteries. More precisely, we proved the quantitative equivalency between the gambler's treatment of every lottery with a similar sensitivity to risk aversion and the mathematical requirements for the order parameters of the information-theoretically proper divergence. Finally, we incorporated these results into the scenario of multilottery state betting games and studied their implications for the resource theory of informative measurements within the operational framework of GPTs. It was revealed that multivariate Rényi divergences define a resource monotone of measurement informativeness and that the monotone can be interpreted as expressing the economic-theoretic benefit of an informative measurement for a gambler when playing a multilottery state betting game. These observations were obtained in the realm of GPTs, which are physical theories with minimal operational requirements. Consequently, they establish a quantitative connection between information, economic, and physical theories with a high degree of generality.

An interesting future direction of this study is to develop physical or other practical aspects of the majorization property of the multivariate Rényi divergence. In Ref. [48], a curious characterization of the unconditional multivariate Rényi divergence was given stating that a set $\vec{P}_X$ of PMFs asymptotically majorizes another $\vec{Q}_X$ (in our language, $\vec{Q}_X$ is obtained by postprocessing of $\vec{P}_X$) with large samples if and only if $D_{\underline{\alpha}}(\vec{P}_X) \geq D_{\underline{\alpha}}(\vec{Q}_X)$ holds for all the admissible orders $\underline{\alpha}$. It is natural to expect this majorization property to be possibly incorporated into the framework of this study. In particular, to investigate whether the conditional multivariate Rényi divergence exhibits a similar property is not only an information-theoretically motivated problem but may also yield

several economic-theoretic insights in a quantitative way. From the resource-theoretic perspective, the problem may be related to convertibility of resource, which is a central problem in the field of resource theory. It is also an interesting future problem to investigate whether a set of monotones $\{\mathcal{D}_{\underline{\alpha}, \alpha_0}(\mathbf{M}, \Lambda, \vec{O}_X) \mid \Lambda, \vec{O}_X\}$ introduced in Theorem 4 is a complete set of monotones for a fixed $\underline{\alpha}$. This set generalizes the complete set of monotones given by state discriminations, and this problem will bring another operational and practical interpretation to the field of resource theory.

Our resource-theoretic results involve entropic quantities obtained from measurement outcome statistics. This is why these quantities are somewhat “classical” in nature. When studying the resource theory of (ensembles of) quantum states or quantum channels (completely positive and trace-preserving linear maps on operators), it seems natural that the relevant resource monotones are “more quantum” in nature. Such quantities could be related to quantum relative entropies [36–41] or their multivariate extensions [45,47,88] (see also Sec. 4 of Ref. [50] and references therein). However, the study of the latter is still in its infancy and investigation into conditional multivariate quantum divergences is, to the best of our knowledge, completely nonexistent. Some quantum divergences, like the “sandwiched” quantum relative entropies [39], can be obtained as optimized and regularized measured (thus “classical”) quantities (see, e.g., Proposition 4.12 of Ref. [30]). It remains to be seen if similar techniques might work in the multivariate (and conditional) settings, yielding meaningful resource monotones.

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DATA AVAILABILITY

No external datasets or software were generated or analyzed during this study. All mathematical derivations and theoretical frameworks supporting the results are contained entirely within the article.

APPENDIX A: MULTIVARIATE RÉNYI DIVERGENCES FOR CONTINUOUS SAMPLE SPACES

Let us make a couple of discussions regarding general sample spaces. We first study the unconditional multivariate Rényi divergence in Definition 1 on a measurable space $(\mathcal{X}, \Sigma)$, where Σ is a σ -algebra of subsets of an alphabet $\mathcal{X}$. We consider probability measures $p_X^{(k)} : \Sigma \rightarrow [0, 1]$ ($k = 0, \dots, d$) on $(\mathcal{X}, \Sigma)$. When treating probability densities on a general sample space, we need a reference measure M (e.g., the Lebesgue measure) on it. Here, we set $M = p_X^{(0)}$ for simplicity, but generalizing the following observations to a general M is straightforward by formally replacing $dp_X^{(0)}$ with dM . Now, we assume that all other probability measures p are dominated by $p_X^{(0)}$, i.e., whenever $p_X^{(0)}(A) = 0$ for some $A \in \Sigma$ then $p(A) = 0$. This means that p is absolutely continuous with respect to $p_X^{(0)}$ (expressed as $p \ll p_X^{(0)}$), so that we may define the Radon-Nikodým derivative (density) $dp/dp_X^{(0)}$ of p with respect to $p_X^{(0)}$. We still denote the expectation value of a measurable

function $f_X : \mathcal{X} \rightarrow \mathbb{R}$ with respect to $p_X^{(0)}$ by $\mathbb{E}_{p_X^{(0)}}[f_X] := \int_{\mathcal{X}} dp_X^{(0)}(x) f_X(x)$. In both cases of Eq. (2), we may define the multivariate Rényi divergence with orders $\underline{\alpha} = (\alpha_0, \dots, \alpha_d)$ for a family of probability measures (a *statistical experiment* [89]) $\vec{P}_X = (p_X^{(0)}, \dots, p_X^{(d)})$, with $p_X^{(k)} \ll p_X^{(0)}$ ($k = 1, \dots, d$) through

$$D_{\underline{\alpha}}(\vec{P}_X) = \frac{1}{\alpha_{\star} - 1} \log \mathbb{E}_{p_X^{(0)}} \left[\prod_{k=1}^d \left(\frac{dp_X^{(k)}}{dp_X^{(0)}} \right)^{\alpha_k} \right],$$

where $\alpha_{\star} := \max_{0 \leq k \leq d} \alpha_k$. Moreover, the (bivariate) Rényi divergence of degree $S \in [0, 1) \cup (1, \infty)$ for $p_X^{(1)}, p_X^{(2)} \ll p_X^{(0)}$ is defined through

$$D_S(p_X^{(1)} \| p_X^{(2)}) = \frac{1}{S - 1} \log \mathbb{E}_{p_X^{(0)}} \left[\left(\frac{dp_X^{(1)}}{dp_X^{(0)}} \right)^S \left(\frac{dp_X^{(2)}}{dp_X^{(0)}} \right)^{1-S} \right].$$

This divergence has to be set to infinity when the supports of $p_X^{(1)}$ and $p_X^{(2)}$ do not intersect and $S \in [0, 1)$, and when $p_X^{(1)}$ has support outside that of $p_X^{(2)}$ and $S > 1$. Regarding the support conditions, however, we essentially need to assume that $(\mathcal{X}, \Sigma)$ is standard Borel.

Similar arguments can be applied to the conditional case developed in Definition 2. We assume that the sample spaces are measurable spaces $(\mathcal{X}, \Sigma)$ and $(\mathcal{G}, \Gamma)$ where the latter describes side information. We will typically assume that these measurable spaces are standard Borel to make things simpler. The total sample space is $(\mathcal{X} \times \mathcal{G}, \Sigma \otimes \Gamma)$, where $\Sigma \otimes \Gamma$ is the coarsest σ -algebra on $\mathcal{X} \times \mathcal{G}$ that contains the product sets $A \times K$, where $A \in \Sigma$ and $K \in \Gamma$. We fix a probability measure $p^{(0)} : \Sigma \otimes \Gamma \rightarrow [0, 1]$ that dominates all other measures on G that we consider. We define the additional probability measures $p_X^{(0)} : \Sigma \rightarrow [0, 1]$ and $p_G^{(0)} : \Gamma \rightarrow [0, 1]$ through

$$p_X^{(0)}(A) = p^{(0)}(A \times \mathcal{G}), \quad p_G^{(0)}(K) = p^{(0)}(\mathcal{X} \times K)$$

for all $A \in \Sigma$ and $K \in \Gamma$. We may also define the positive measures $p_{A,G}^{(0)} : K \mapsto p^{(0)}(A \times K)$ for all $A \in \Sigma$. These measures are all absolutely continuous with respect to $p_G^{(0)}$ as one easily sees. It enables us to define the function $p_{X|G}^{(0)}(\cdot | \cdot) : \Sigma \times \mathcal{G} \rightarrow [0, \infty]$ through

$$p_{X|G}^{(0)}(A|g) = \frac{dp_{A,G}^{(0)}}{dp_G^{(0)}}(g), \quad A \in \Sigma, \quad g \in \mathcal{G}.$$

It is easy to see that $p_{X|G}^{(0)}(\cdot | g)$ is a probability measure for all $g \in \mathcal{G}$, i.e., $p_{X|G}^{(0)}(\cdot | \cdot)$ is a Markov kernel. It follows that

$$p^{(0)}(A \times K) = \int_K p_{X|G}^{(0)}(A|g) dp_G^{(0)}(g), \quad A \in \Sigma, \quad K \in \Gamma.$$

Let us pick $\alpha \geq 0$ in addition to the parameter vector $\underline{\alpha} = (\alpha_0, \dots, \alpha_d)$. For any probability measures $p_{X|G}^{(k)} : \Sigma \rightarrow [0, 1]$ ($k = 1, \dots, d$) such that $p_{X|G}^{(k)}(\cdot | g) \ll p_{X|G}^{(0)}(\cdot | g)$ for all $g \in \mathcal{G}$, we can define the continuous extension of the conditional multivariate Rényi divergence (with respect to $p_G^{(0)}$) through

$$\begin{aligned} & D_{\underline{\alpha}, \alpha}(p_{X|G}^{(0)}, \dots, p_{X|G}^{(d)} | p_G^{(0)}) \\ &= \frac{\alpha}{\alpha_{\star} - 1} \log \left\{ \int_{\mathcal{G}} dp_G^{(0)}(g) \left(\int_{\mathcal{X}} p_{X|G}^{(0)}(dx|g) \prod_{k=1}^d \left(\frac{dp_{X|G}^{(k)}(\cdot | g)}{dp_{X|G}^{(0)}(\cdot | g)} \right)^{\alpha_k} \right)^{1/\alpha} \right\}, \end{aligned}$$

where $\alpha_{\star} = \max_{0 \leq k \leq d} \alpha_k$. These quantities not only are straightforward generalizations of the conditional multivariate Rényi divergences of Definition 2 but their unconditional versions have many similar uses in the continuous setting as their finite-outcome counterparts [77, 78].

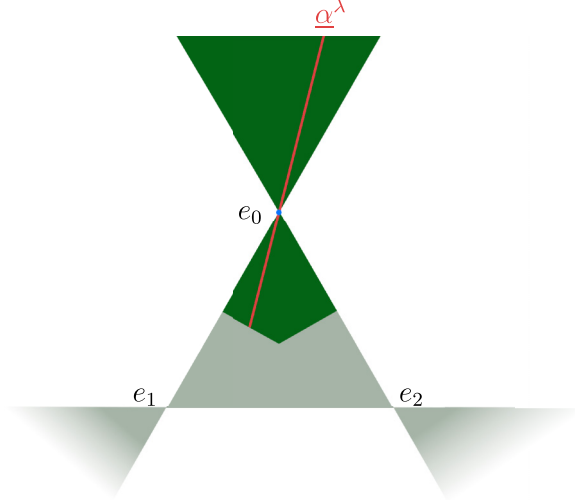


FIG. 8. The area of allowed parameter vectors $\underline{\alpha} = (\alpha_0, \alpha_1, \alpha_2) \in \mathbb{R}^3$ in the case $d = 2$. The area is confined onto the affine plane $\sum_{k=0}^2 \alpha_k = 1$. The special points fixing this plane are $e_0 = (1, 0, 0)$, $e_1 = (0, 1, 0)$, and $e_2 = (0, 0, 1)$. The total shaded area represents all Eq. (2) and the darker area is where α_0 is the largest among the α_k . The path of Eq. (B1) satisfying Eq. (B2) is depicted in red with the special point $\lambda = 1$ at e_0 in blue. As we move upward along this path, the value $D_{\underline{\alpha}^\lambda}(P)$ increases asymptotically approaching the limit quantity $D_{\underline{\beta}}^{\mathbb{T}}(P)$, where $\underline{\beta}$ depends on the angle of the path.

APPENDIX B: MONOTONICITY OF MULTIVARIATE RÉNYI DIVERGENCES FOR ORDER VECTORS

We may also identify certain monotonicity properties for the unconditional multivariate Rényi divergence $D_{\underline{\alpha}}(\vec{P}_X)$ as a function of $\underline{\alpha}$ in the allowed region for any fixed $\vec{P}_X = (p_X^{(0)}, \dots, p_X^{(d)})$ (Proposition 17 of Ref. [48]). Let us remain in the region of Eq. (2) with α_0 the largest among $k = 0, \dots, d$; the other sections of the parameter region can be dealt with in a similar fashion. Let us fix $\gamma_1, \dots, \gamma_d \geq 0$ such that $\gamma_1 + \dots + \gamma_d = 1$ and consider the path

$$[0, 1) \cup (1, \infty) \ni \lambda \mapsto \underline{\alpha}^\lambda = (\lambda, (1 - \lambda)\gamma_1, \dots, (1 - \lambda)\gamma_d) \in \mathbb{R}^{d+1} \quad (\text{B1})$$

within the parameter space. We illustrate the setting for the case $d = 3$ in Fig. 8. Note that we set

$$\lambda \geq \max_{1 \leq k \leq d} \frac{\gamma_k}{\gamma_k + 1}, \quad \lambda \neq 1, \quad (\text{B2})$$

so that $\alpha_0 = \lambda$ is the largest among the components of $\underline{\alpha}^\lambda$ and they are within the region characterized by Eq. (2). In the region described by Eq. (B2), the divergence $D_{\underline{\alpha}^\lambda}(\vec{P}_X)$ is nondecreasing in λ for any fixed $\vec{P}_X$. The pointwise limit at $\lambda \rightarrow 1$ exists and is given by

$$\lim_{\lambda \rightarrow 1} D_{\underline{\alpha}^\lambda}(\vec{P}_X) = \sum_{k=1}^d \gamma_k D_{\text{KL}}(p_X^{(0)} \| p_X^{(k)}),$$

where D_{KL} is the Kullback-Leibler divergence

$$D_{\text{KL}}(p_X \| q_X) = \mathbb{E}_{q_X} \left[\frac{p_X}{q_X} \log \frac{p_X}{q_X} \right].$$

Also the limit $\lambda \rightarrow \infty$ exists and is given by

$$\lim_{\lambda \rightarrow \infty} D_{\underline{\alpha}^\lambda}(\vec{P}_X) = \sup_{\lambda > 1} D_{\underline{\alpha}^\lambda}(\vec{P}_X) = D_{(1, -\gamma_1, \dots, -\gamma_d)}^{\mathbb{T}}(\vec{P}_X) := \max_{x \in \mathcal{X}} \log \frac{p_X^{(0)}(x)}{(p_X^{(1)}(x))^{\gamma_1} \dots (p_X^{(d)}(x))^{\gamma_d}}.$$

The limit quantities $D_{\underline{\beta}}^{\mathbb{T}}$ for any $\underline{\beta} = (\beta_0, \dots, \beta_d) \in \mathbb{R}^{d+1}$, with $\beta_0 > 0$, $\beta_1, \dots, \beta_d \leq 0$, and $\beta_0 + \dots + \beta_d = 0$ satisfy the data processing inequality as well as many other desirable properties [48].

APPENDIX C: PROOF OF PROPOSITION 2

Proof. Let us first assume that condition (1) of Eq. (2) is satisfied. Using Minkowski's inequality, we now have

$$\begin{aligned} & \sum_{h \in \mathcal{H}} q_H(h) \left(\sum_{x \in \mathcal{X}} \left(q_{X|H}^{(0)}(x|h) \prod_{k=1}^d (r_X^{(k)}(x))^{\alpha_k/\alpha_0} \right)^{\alpha_0} \right)^{1/\alpha_0} \\ &= \sum_{h \in \mathcal{H}} q_H(h) \left(\sum_{x \in \mathcal{X}} \left(\sum_{g \in \mathcal{G}} t_{G|H}^\dagger(g|h) p_{X|G}^{(0)}(x|g) \prod_{k=1}^d (r_X^{(k)}(x))^{\alpha_k/\alpha_0} \right)^{\alpha_0} \right)^{1/\alpha_0} \\ &\leq \sum_{g \in \mathcal{G}} \sum_{h \in \mathcal{H}} \underbrace{q_H(h) t_{G|H}^\dagger(g|h)}_{=p_G(g)} \left(\sum_{x \in \mathcal{X}} \left(p_{X|G}^{(0)}(x|g) \prod_{k=1}^d (r_X^{(k)}(x))^{\alpha_k/\alpha_0} \right)^{\alpha_0} \right)^{1/\alpha_0} \\ &= \sum_{g \in \mathcal{G}} p_G(g) \underbrace{\sum_{h \in \mathcal{H}} t_{H|G}(h|g)}_{=1} \left(\sum_{x \in \mathcal{X}} \left(p_{X|G}^{(0)}(x|g) \prod_{k=1}^d (r_X^{(k)}(x))^{\alpha_k/\alpha_0} \right)^{\alpha_0} \right)^{1/\alpha_0} \\ &= \sum_{g \in \mathcal{G}} p_G(g) \left(\sum_{x \in \mathcal{X}} \left(p_{X|G}^{(0)}(x|g) \prod_{k=1}^d (r_X^{(k)}(x))^{\alpha_k/\alpha_0} \right)^{\alpha_0} \right)^{1/\alpha_0}. \end{aligned}$$

In the case (2), the inequality on the third line above is reversed. Noting that $\alpha_0/(\alpha_0 - 1)$ is negative in case (1) and positive in case (2), we now have, in both cases,

$$\begin{aligned} D_{\underline{\alpha}, \alpha_0}(q_{X|H}^{(0)}, r_X^{(1)}, \dots, r_X^{(d)} | q_H) &= \frac{\alpha_0}{\alpha_0 - 1} \log \sum_{h \in \mathcal{H}} q_H(h) \left(\sum_{x \in \mathcal{X}} \left(q_{X|H}^{(0)}(x|h) \prod_{k=1}^d (r_X^{(k)}(x))^{\alpha_k/\alpha_0} \right)^{\alpha_0} \right)^{1/\alpha_0} \\ &\leq \frac{\alpha_0}{\alpha_0 - 1} \log \sum_{g \in \mathcal{G}} p_G(g) \left(\sum_{x \in \mathcal{X}} \left(p_{X|G}^{(0)}(x|g) \prod_{k=1}^d (r_X^{(k)}(x))^{\alpha_k/\alpha_0} \right)^{\alpha_0} \right)^{1/\alpha_0} \\ &= D_{\underline{\alpha}, \alpha_0}(p_{X|G}^{(0)}, r_X^{(1)}, \dots, r_X^{(d)} | p_G), \end{aligned}$$

proving Eq. (6). ■

APPENDIX D: PROOFS OF MAIN RESULTS

1. Proof of Theorem 1

Proof. We simply write $w^{\text{ICE}}(p_X^{(0)}, \vec{O}_X, \vec{B}_X, u_{\underline{R}})$ by w^{ICE} . Since $\sum_{k=1}^d (1 - R_k) = (\alpha_0 - 1)/\alpha_0$, solving from Eq. (10) under Eq. (11) implies

$$\log w^{\text{ICE}} = \frac{\alpha_0}{\alpha_0 - 1} \log \sum_{x \in \mathcal{X}} (b_X^{(1)}(x))^{1-S_1} (C_1 \cdot q_X^{(1)}(x))^{S_1}, \quad (\text{D1})$$

where we introduced numbers $S_1 := (\alpha_0 + \alpha_1)/\alpha_0 \geq 0$,

$$C_1 := \sum_{x \in \mathcal{X}} \left\{ (p_X^{(0)}(x))^{\alpha_0} \left(\frac{1}{o_X^{(1)}(x)} \right)^{\alpha_1} \prod_{k=2}^d (b_X^{(k)}(x) o_X^{(k)}(x))^{-\alpha_k} \right\}^{\frac{1}{\alpha_0 + \alpha_1}}$$

and a function

$$q_X^{(1)} := \frac{1}{C_1} \left\{ (p_X^{(0)})^{\alpha_0} \left(\frac{1}{o_X^{(1)}} \right)^{\alpha_1} \prod_{k=2}^d (b_X^{(k)} o_X^{(k)})^{-\alpha_k} \right\}^{\frac{1}{\alpha_0 + \alpha_1}}$$

on $\mathcal{X}$. Note that the constant C_1 is defined so that $q_X^{(1)}$ is a PMF and also that $q_X^{(1)}$ is independent of $b_X^{(1)}$. Thus, we can rephrase Eq. (D1) by means of the bivariate Rényi divergence [Eq. (4)] as

$$\log w^{\text{ICE}} = \frac{\alpha_1}{\alpha_0 - 1} D_{S_1}(q_X^{(1)} \| b_X^{(1)}) + \frac{\alpha_0 + \alpha_1}{\alpha_0 - 1} \log C_1. \quad (\text{D2})$$

We apply the same argument to the term $\log C_1$ to peel off the PMF $b_X^{(2)}$ and obtain

$$\begin{aligned} \log C_1 &= \log \sum_{x \in \mathcal{X}} (b_X^{(2)}(x))^{1-S_2} (C_2 \cdot q_X^{(2)}(x))^{S_2} \\ &= \frac{\alpha_2}{\alpha_0 + \alpha_1} D_{S_2}(q_X^{(2)} \| b_X^{(2)}) + \frac{\alpha_0 + \alpha_1 + \alpha_2}{\alpha_0 + \alpha_1} \log C_2, \end{aligned}$$

where $S_2 := (\alpha_0 + \alpha_1 + \alpha_2)/(\alpha_0 + \alpha_1) \geq 0$,

$$C_2 := \sum_{x \in \mathcal{X}} \left\{ (p_X^{(0)}(x))^{\alpha_0} \left(\frac{1}{o_X^{(1)}(x)} \right)^{\alpha_1} \left(\frac{1}{o_X^{(2)}(x)} \right)^{\alpha_2} \prod_{k=3}^d (b_X^{(k)} o_X^{(k)})^{-\alpha_k} \right\}^{\frac{1}{\alpha_0 + \alpha_1 + \alpha_2}}$$

are constants and

$$q_X^{(2)} := \frac{1}{C_2} \sum_{x \in \mathcal{X}} \left\{ (p_X^{(0)})^{\alpha_0} \left(\frac{1}{o_X^{(1)}} \right)^{\alpha_1} \left(\frac{1}{o_X^{(2)}} \right)^{\alpha_2} \prod_{k=3}^d (b_X^{(k)} o_X^{(k)})^{-\alpha_k} \right\}^{\frac{1}{\alpha_0 + \alpha_1 + \alpha_2}}$$

is a PMF. Applying the same argument and peeling off the bet functions $b_X^{(k)}$ one by one, we can reduce Eq. (D2) to

$$\log w^{\text{ICE}} = \frac{1}{\alpha_0 - 1} \left(\sum_{k=1}^d \alpha_k D_{S_k}(q_X^{(k)} \| b_X^{(k)}) + \log C_d \right), \quad (\text{D3})$$

with constants

$$S_k := \frac{\alpha_0 + \dots + \alpha_k}{\alpha_0 + \dots + \alpha_{k-1}},$$

$$C_k := \sum_{x \in \mathcal{X}} \left\{ (p_X^{(0)}(x))^{\alpha_0} \prod_{l=1}^k \left(\frac{1}{o_X^{(l)}(x)} \right)^{\alpha_l} \prod_{m=k+1}^d (b_X^{(m)}(x) o_X^{(m)}(x))^{-\alpha_m} \right\}^{\frac{1}{\alpha_0 + \dots + \alpha_k}},$$

and PMFs

$$q_X^{(k)} := \frac{1}{C_k} \left\{ (p_X^{(0)})^{\alpha_0} \prod_{l=1}^k \left(\frac{1}{o_X^{(l)}} \right)^{\alpha_l} \prod_{m=k+1}^d (b_X^{(m)} o_X^{(m)})^{-\alpha_m} \right\}^{\frac{1}{\alpha_0 + \dots + \alpha_k}}$$

for $k = 1, \dots, d$. In particular,

$$\begin{aligned} \log C_d &= \log \sum_{x \in \mathcal{X}} (p_X^{(0)}(x))^{\alpha_0} \prod_{k=1}^d \left(\frac{1}{o_X^{(k)}(x)} \right)^{\alpha_k} \\ &= \sum_{k=1}^d \alpha_k \log F^{(k)} + (\alpha_0 - 1) D_{\underline{\alpha}}(p_X^{(0)}, p_X^{(1)}, \dots, p_X^{(d)}) \end{aligned}$$

holds, and substituting this in Eq. (D3), we obtain Eq. (14). $\blacksquare$

The continuous counterpart of the d -commodity betting game can be described in the same way as Sec. IV. Now, each bet $b_X^{(k)}$ is a probability measure and odds function $o_X^{(k)}$ is a measurable function on X . Moreover, we assume that there are probability measures $p_X^{(k)}$ and $F^{(k)} > 0$, $k = 1, \dots, d$, such that $(o_X^{(k)})^{-1} = F^{(k)} dp_X^{(k)} / dp_X^{(0)}$. Setting again $p_X^{(0)}$ as the dominating measure, we can develop the same proof as Theorem 1 with formally replacing $\sum_x \rightarrow \int dp_X^{(0)}$ and $b_X^{(k)} \rightarrow db_X^{(k)} / dp_X^{(0)}$ [also $(p_X^{(0)}(x))^{\alpha_0} \rightarrow (dp_X^{(0)} / dp_X^{(0)})^{\alpha_0} = 1$]. It can be finally proved that there are probability measures $q_X^{(1)}, \dots, q_X^{(d)} \ll p_X^{(0)}$ and $S_1, \dots, S_d \geq 0$ such that we have for the isoelastic certainty equivalent w^{ICE} in the utility function [Eq. (11)] and the statistical experiment $\vec{P}_X := (p_X^{(0)}, \dots, p_X^{(d)})$:

$$\begin{aligned} \log w^{\text{ICE}} &= D_{\underline{\alpha}}(\vec{P}_X) + \frac{1}{\alpha_0 - 1} \sum_{k=1}^d \alpha_k \{ D_{S_k}(q_X^{(k)} \| b_X^{(k)}) + \log F^{(k)} \} \\ &\leq D_{\underline{\alpha}}(\vec{P}_X) + \frac{1}{\alpha_0 - 1} \sum_{k=1}^d \alpha_k \log F^{(k)}, \end{aligned}$$

where the upper bound is reached by optimizing over the betting strategies.

2. Proof of Theorem 2

Proof. We write $w^{\text{ICE}}(p_{XG}^{(0)}, \vec{O}_X, \vec{B}_{X|G}, u_{\vec{R}})$ by w^{ICE} . Proceeding as earlier, we have

$$\begin{aligned} \log w^{\text{ICE}} &= \frac{\alpha_0}{\alpha_0 - 1} \log \sum_{(x,g) \in \mathcal{X} \times \mathcal{G}} p_{XG}^{(0)}(x, g) \left[\prod_{k=1}^d (b_{X|G}^{(k)}(x|g) o_X^{(k)}(x))^{-\frac{\alpha_k}{\alpha_0}} \right] \\ &= \frac{\alpha_0}{\alpha_0 - 1} \log \sum_{x,g} (b_{X|G}^{(1)}(x|g))^{1-S_1} (c_1(g) \cdot q_{X|G}^{(1)}(x|g))^{S_1}, \end{aligned} \quad (\text{D4})$$

where we introduced numbers $S_1 := (\alpha_0 + \alpha_1)/\alpha_0 \geq 0$,

$$c_1(g) := \sum_{x \in \mathcal{X}} \left\{ (p_{XG}^{(0)}(x, g))^{\alpha_0} \left(\frac{1}{o_X^{(1)}(x)} \right)^{\alpha_1} \prod_{k=2}^d (b_{X|G}^{(k)}(x|g) o_X^{(k)}(x))^{-\alpha_k} \right\}^{\frac{1}{\alpha_0 + \alpha_1}},$$

a conditional PMF $q_{X|G}^{(1)}$ by

$$q_{X|G}^{(1)}(x|g) := \frac{1}{c_1(g)} \left\{ (p_{XG}^{(0)}(x, g))^{\alpha_0} \left(\frac{1}{o_X^{(1)}(x)} \right)^{\alpha_1} \prod_{k=2}^d (b_{X|G}^{(k)}(x|g) o_X^{(k)}(x))^{-\alpha_k} \right\}^{\frac{1}{\alpha_0 + \alpha_1}}$$

that produces a PMF on $\mathcal{X}$ for each $g \in \mathcal{G}$. We in addition define a constant

$$C_1 = \sum_{g \in \mathcal{G}} (c_1(g))^{S_1} = \sum_g \left[\sum_x \left\{ (p_{XG}^{(0)}(x, g))^{\alpha_0} \left(\frac{1}{o_X^{(1)}(x)} \right)^{\alpha_1} \prod_{k=2}^d (b_{X|G}^{(k)}(x|g) o_X^{(k)}(x))^{-\alpha_k} \right\}^{\frac{1}{\alpha_0 + \alpha_1}} \right]^{S_1},$$

and a PMF $q_G^{(1)}$ on $\mathcal{G}$ by

$$q_G^{(1)}(g) = \frac{(c_1(g))^{S_1}}{C_1}.$$

It enables us to rewrite Eq. (D4) as

$$\begin{aligned} \log w^{\text{ICE}} &= \frac{\alpha_0}{\alpha_0 - 1} \log \sum_g q_G^{(1)}(g) \sum_x (b_{X|G}^{(1)}(x|g))^{1-S_1} (q_{X|G}^{(1)}(x|g))^{S_1} + \frac{\alpha_0}{\alpha_0 - 1} \log C_1 \\ &= \frac{\alpha_0}{\alpha_0 - 1} \log \sum_{g,x} (q_G^{(1)}(g) b_{X|G}^{(1)}(x|g))^{1-S_1} (q_G^{(1)}(g) q_{X|G}^{(1)}(x|g))^{S_1} + \frac{\alpha_0}{\alpha_0 - 1} \log C_1 \\ &= \frac{\alpha_1}{\alpha_0 - 1} D_{S_1}(q_{XG}^{(1)} \| r_{XG}^{(1)}) + \frac{\alpha_0}{\alpha_0 - 1} \log C_1, \end{aligned} \quad (\text{D5})$$

where $q_{XG}^{(1)} := q_G^{(1)} q_{X|G}^{(1)}$ and $r_{XG}^{(1)} := q_G^{(1)}(g) b_{X|G}^{(1)}$ are joint PMFs on $\mathcal{X} \times \mathcal{G}$ and $D_{S_1}(q_{XG}^{(1)} \| r_{XG}^{(1)})$ is their bivariate Rényi divergence. We note that by choosing $b_{X|G}^{(1)} = q_{X|G}^{(1)}$, we can make the first term vanish. Let us analyze the final term in the above equation [see Eq. (D4)] when the parameters $\underline{\alpha}$ satisfy condition (1) of Eq. (2) as

$$\begin{aligned} \log C_1 &= \log \sum_g \left[\sum_x \left\{ (p_{XG}^{(0)}(x, g))^{\alpha_0} \left(\frac{1}{o_X^{(1)}(x)} \right)^{\alpha_1} \prod_{k=2}^d (b_{X|G}^{(k)}(x|g) o_X^{(k)}(x))^{-\alpha_k} \right\}^{\frac{1}{\alpha_0 + \alpha_1}} \right]^{S_1} \\ &= \log \sum_g p_G^{(0)}(g) \left[\sum_x \left\{ (p_{X|G}^{(0)}(x|g))^{\alpha_0} \left(\frac{1}{o_X^{(1)}(x)} \right)^{\alpha_1} \prod_{k=2}^d (b_{X|G}^{(k)}(x|g) o_X^{(k)}(x))^{-\alpha_k} \right\}^{\frac{1}{\alpha_0 + \alpha_1}} \right]^{S_1} \\ &= \log \sum_g p_G^{(0)}(g) \left[\sum_x (b_{X|G}^{(2)}(x|g))^{1-S_2} (c_2(g) \cdot q_{X|G}^{(2)}(x|g))^{S_2} \right]^{S_1}, \end{aligned}$$

where we introduced numbers $S_2 := (\alpha_0 + \alpha_1 + \alpha_2)/(\alpha_0 + \alpha_1) \geq 0$,

$$c_2(g) := \sum_{x \in \mathcal{X}} \left\{ (p_{X|G}^{(0)}(x|g))^{\alpha_0} \left(\frac{1}{o_X^{(1)}(x)} \right)^{\alpha_1} \left(\frac{1}{o_X^{(2)}(x)} \right)^{\alpha_2} \prod_{k=3}^d (b_{X|G}^{(k)}(x|g) o_X^{(k)}(x))^{-\alpha_k} \right\}^{\frac{1}{\alpha_0 + \alpha_1 + \alpha_2}}, \quad (\text{D6})$$

a conditional PMF $q_{X|G}^{(2)}$ by

$$q_{X|G}^{(2)}(x|g) := \frac{1}{c_2(g)} \left\{ (p_{X|G}^{(0)}(x|g))^{\alpha_0} \left(\frac{1}{o_X^{(1)}(x)} \right)^{\alpha_1} \left(\frac{1}{o_X^{(2)}(x)} \right)^{\alpha_2} \prod_{k=3}^d (b_{X|G}^{(k)}(\cdot|g) o_X^{(k)}(x))^{-\alpha_k} \right\}^{\frac{1}{\alpha_0 + \alpha_1 + \alpha_2}}.$$

With a constant

$$C_2 = \sum_g (c_2(g))^{S_1 S_2} p_G^{(0)}(g)$$

and a PMF

$$q_G^{(2)} = \frac{(c_2(g))^{S_1 S_2} p_G^{(0)}}{C_2}, \quad (\text{D7})$$

on G , it follows that

$$\begin{aligned} \log C_1 &= \log \sum_g q_G^{(2)}(g) \left[\sum_x (b_{X|G}^{(2)}(x|g))^{1-S_2} (q_{X|G}^{(2)}(x|g))^{S_2} \right]^{S_1} + \log C_2 \\ &\geq \log \left[\sum_{g,x} q_G^{(2)}(g) (b_{X|G}^{(2)}(x|g))^{1-S_2} (q_{X|G}^{(2)}(x|g))^{S_2} \right]^{S_1} + \log C_2 \\ &= \log \left[\sum_{g,x} (q_G^{(2)}(g) b_{X|G}^{(2)}(x|g))^{1-S_2} (q_G^{(2)}(g) q_{X|G}^{(2)}(x|g))^{S_2} \right]^{S_1} + \log C_2 \\ &= S_1 (S_2 - 1) D_{S_2}(q_{XG}^{(2)} \| r_{XG}^{(2)}) + \log C_2, \end{aligned}$$

where $q_{XG}^{(2)} := q_G^{(2)} q_{X|G}^{(2)}$ and $r_{XG}^{(2)} := q_G^{(2)}(g) b_{X|G}^{(2)}$ are joint PMFs, and the inequality is due to Jensen's inequality and the fact that $S_1 \geq 1$ in case (1) of Eq. (2). If we set $b_{X|G}^{(2)} = q_{X|G}^{(2)}$, we clearly reach the above lower bound. Since $S_1 \leq 1$ in case (2) of Eq. (2), the inequality is reversed in this case. Noting that $\alpha_0/(\alpha_0 - 1) < 0$ in case (1) and $\alpha_0/(\alpha_0 - 1) \geq 0$ in case (2), we now have

$$\begin{aligned} \frac{\alpha_0}{\alpha_0 - 1} \log C_1 &\leq \frac{\alpha_0 S_1}{\alpha_0 - 1} (S_2 - 1) D_{S_2}(q_{XG}^{(2)} \| r_{XG}^{(2)}) + \frac{\alpha_0}{\alpha_0 - 1} \log C_2 \\ &= \frac{\alpha_2}{\alpha_0 - 1} D_{S_2}(q_{XG}^{(2)} \| r_{XG}^{(2)}) + \frac{\alpha_0}{\alpha_0 - 1} \log C_2, \end{aligned}$$

and thus from Eq. (D5)

$$\log w^{\text{ICE}} \leq \frac{\alpha_1}{\alpha_0 - 1} D_{S_1}(q_{XG}^{(1)} \| r_{XG}^{(1)}) + \frac{\alpha_2}{\alpha_0 - 1} D_{S_2}(q_{XG}^{(2)} \| r_{XG}^{(2)}) + \frac{\alpha_0}{\alpha_0 - 1} \log C_2.$$

Remember that when we choose $b_{X|G}^{(k)}$ so that $b_{X|G}^{(k)} = q_{X|G}^{(k)}$ for $k = 1, 2$, the above inequality is saturated and the two bivariate Rényi divergences vanish. We proceed in the same way using Jensen's inequality and notice that

$$\frac{\alpha_0}{\alpha_0 - 1} \log C_{k-1} \leq \frac{\alpha_k}{\alpha_0 - 1} D_{S_k}(q_{XG}^{(k)} \| r_{XG}^{(k)}) + \frac{\alpha_0}{\alpha_0 - 1} \log C_k \quad (\text{D8})$$

for $k = 2, \dots, d$ in both cases (1) and (2) of Eq. (2), where the constants S_k and C_k , and the PMFs $q_{XG}^{(k)}$ and $r_{XG}^{(k)}$ are defined as in the step above. Explicitly, they are given as [see Eqs. (D6)]

and (D7)]

$$\begin{aligned}
 S_k &= \frac{\alpha_0 + \dots + \alpha_k}{\alpha_0 + \dots + \alpha_{k-1}} \geq 0, \\
 c_k(g) &:= \sum_{x \in \mathcal{X}} \left\{ (p_{X|G}^{(0)}(x|g))^{\alpha_0} \prod_{l=1}^k \left(\frac{1}{o_X^{(l)}(x)} \right)^{\alpha_l} \prod_{m=k+1}^d (b_{X|G}^{(m)}(x|g) o_X^{(m)}(x))^{-\alpha_m} \right\}^{\frac{1}{\alpha_0 + \dots + \alpha_k}}, \\
 C_k &= \sum_{g \in \mathcal{G}} (c_k(g))^{S_1 \dots S_k} p_G^{(0)}(g) = \sum_{g \in \mathcal{G}} (c_k(g))^{\frac{\alpha_0 + \dots + \alpha_k}{\alpha_0}} p_G^{(0)}(g), \\
 q_{X|G}^{(k)}(x|g) &:= \frac{1}{c_k(g)} \left\{ (p_{X|G}^{(0)}(x|g))^{\alpha_0} \prod_{l=1}^k \left(\frac{1}{o_X^{(l)}(x)} \right)^{\alpha_l} \prod_{m=k+1}^d (b_{X|G}^{(m)}(x|g) o_X^{(m)}(x))^{-\alpha_m} \right\}^{\frac{1}{\alpha_0 + \dots + \alpha_k}}, \\
 q_G^{(k)}(g) &= \frac{(c_k(g))^{S_1 \dots S_k} p_G^{(0)}(g)}{C_k} = \frac{(c_k(g))^{\frac{\alpha_0 + \dots + \alpha_k}{\alpha_0}} p_G^{(0)}(g)}{C_k}, \\
 q_{XG}^{(k)}(x, g) &:= q_G^{(k)}(g) q_{X|G}^{(k)}(x|g), \quad r_{XG}^{(k)}(x, g) := q_G^{(k)}(g) b_{X|G}^{(k)}(x|g).
 \end{aligned}$$

Again, if choosing $b_{X|G}^{(k)} = q_{X|G}^{(k)}$, the bivariate Rényi divergence vanishes saturating the bound in Eq. (D8). This finally gives us

$$\log w^{\text{ICE}} \leq \frac{1}{\alpha_0 - 1} \sum_{k=1}^d \alpha_k D_{S_k}(q_{XG}^{(k)} \| r_{XG}^{(k)}) + \frac{\alpha_0}{\alpha_0 - 1} \log C_d \quad (\text{D9})$$

$$\leq \frac{\alpha_0}{\alpha_0 - 1} \log C_d, \quad (\text{D10})$$

where the final upper bound can be reached. Let us investigate the final term $\log C_d$. Recalling $p_X^{(k)} := \frac{1}{F^{(k)} o_X^{(k)}}$ and $F^{(k)} := \sum_x \frac{1}{o_X^{(k)}(x)}$, we now have

$$\begin{aligned}
 \frac{\alpha_0}{\alpha_0 - 1} \log C_d &= \frac{\alpha_0}{\alpha_0 - 1} \log \sum_g p_G^{(0)}(g) (c_d(g))^{\frac{1}{\alpha_0}} \\
 &= \frac{\alpha_0}{\alpha_0 - 1} \log \sum_g p_G^{(0)}(g) \left(\sum_x (p_{X|G}^{(0)}(x|g))^{\alpha_0} \prod_{k=1}^d \left(\frac{1}{o_X^{(k)}(x)} \right)^{\alpha_k} \right)^{1/\alpha_0} \\
 &= \frac{\alpha_0}{\alpha_0 - 1} \log \sum_g p_G^{(0)}(g) \left(\sum_x (p_{X|G}^{(0)}(x|g))^{\alpha_0} \prod_{k=1}^d (F^{(k)} p_X^{(k)}(x))^{\alpha_k} \right)^{1/\alpha_0} \\
 &= \frac{1}{\alpha_0 - 1} \sum_{k=1}^d \alpha_k \log F^{(k)} \\
 &\quad + \frac{\alpha_0}{\alpha_0 - 1} \log \sum_g p_G^{(0)}(g) \left(\sum_x (p_{X|G}^{(0)}(x|g))^{\alpha_0} \prod_{k=1}^d (p_X^{(k)}(x))^{\alpha_k} \right)^{1/\alpha_0} \\
 &= \frac{1}{\alpha_0 - 1} \sum_{k=1}^d \alpha_k \log F^{(k)} + D_{\underline{\alpha}, \alpha_0}(p_{X|G}^{(0)}, p_X^{(1)}, \dots, p_X^{(d)} | p_G^{(0)}).
 \end{aligned}$$

Substituting this into Eqs. (D9) and (D10) gives Theorem 2. We remark that the continuous counterpart of the current theorem can be obtained by the same procedure as explained in Appendix D 1. ■

3. Proof of Theorem 4

Proof. To prove Eq. (28), choose an uninformative measurement $\mathbf{N} = \{n_a\}_{a \in \mathcal{B}}$, where $n_b = \eta_B(b)u$, with the unit effect u and a PMF η_B on $\mathcal{B}$. We introduce a PMF

$$q_{XB}^{(0)}[\mathbf{N}, \Lambda](x, b) = p_X^{(0)}(x)\langle n_b, \omega_x \rangle = p_X^{(0)}(x)\eta_B(b),$$

which particularly satisfies $q_{X|B}^{(0)} = q_X^{(0)} = p_X^{(0)}$. It is obvious that the conditional multivariate divergence $D_{\underline{\alpha}, \alpha_0}(q_{X|B}^{(0)}, p_X^{(1)}, \dots, p_X^{(d)} | q_B^{(0)}[\mathbf{N}, \Lambda])$ reduces to an unconditional multivariate divergence:

$$\begin{aligned} D_{\underline{\alpha}, \alpha_0}(q_{X|B}^{(0)}, p_X^{(1)}, \dots, p_X^{(d)} | q_B^{(0)}[\mathbf{N}, \Lambda]) \\ &= \frac{\alpha_0}{\alpha_0 - 1} \log \sum_b q_B(b) \left(\sum_x (q_{X|B}^{(0)}(x|b))^{\alpha_0} \prod_{k=1}^d (p_X^{(k)}(x))^{\alpha_k} \right)^{1/\alpha_0} \\ &= \frac{1}{\alpha_0 - 1} \log \sum_x \prod_{k=0}^d (p_X^{(k)}(x))^{\alpha_k} = D_{\underline{\alpha}}(p_X^{(0)}, p_X^{(1)}, \dots, p_X^{(d)}). \end{aligned}$$

Then, Theorem 3 (or Corollary 7) implies

$$\begin{aligned} &\log \frac{\max_{\vec{B}_{X|G}, s_{G|A}} w^{\text{ICE}}(\mathbf{M}, \Lambda, s_{G|A}, \vec{O}_X, \vec{B}_{X|G}, u_R)}{\max_{\vec{B}_{X|G}, s_{G|B}} w^{\text{ICE}}(\mathbf{N}, \Lambda, s_{G|B}, \vec{O}_X, \vec{B}_{X|G}, u_R)} \\ &= D_{\underline{\alpha}, \alpha_0}(p_{X|A}^{(0)}, p_X^{(1)}, \dots, p_X^{(d)} | p_A^{(0)}[\mathbf{M}, \Lambda]) - D_{\underline{\alpha}, \alpha_0}(q_{X|B}^{(0)}, p_X^{(1)}, \dots, p_X^{(d)} | q_B^{(0)}[\mathbf{N}, \Lambda]) \\ &= D_{\underline{\alpha}, \alpha_0}(p_{X|A}^{(0)}, p_X^{(1)}, \dots, p_X^{(d)} | p_A^{(0)}[\mathbf{M}, \Lambda]) - D_{\underline{\alpha}}(p_X^{(0)}, p_X^{(1)}, \dots, p_X^{(d)}). \end{aligned}$$

This holds independently of the uninformative measurement $\mathbf{N}$ and thus proves Eq. (28). The resource-monotonic property is a direct implication of the data processing inequality (Proposition 2). ■

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