

Characterizing local state distinguishability with non-maximal entanglement

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Perfect LOCC (local operations and classical communication) discrimination of locally indistinguishable quantum states usually requires maximally entangled states as a resource. Whether nonmaximally entangled resources can enable perfect LOCC discrimination of locally indistinguishable states remains an interesting area of research. The states in the three-qubit Greenberger-Horne-Zeilinger (GHZ)-SLOCC (stochastic local operation and classical communication) class can display various entanglement distribution patterns, including cases with no bipartite entanglement in any reduced subsystems, or instances where bipartite entanglement occurs across one, two, or all three reduced subsystems. Significant research has explored how such states can be used in entanglement-assisted discrimination tasks. In this paper, we investigate the distinguishability of orthogonal product states using GHZ-SLOCC class entangled states, and we identify two nonmaximal GHZ-SLOCC states that facilitate perfect distinguishability. Additionally, we analyze the relationship between the probability of error and the amount of bipartite and multiparty entanglement. Our analysis focuses on how the distribution of entanglement affects the accuracy of state discrimination.

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I. INTRODUCTION

The intricate relationship between quantum entanglement [1] and quantum nonlocality [2] has long been recognized as a fundamental issue in quantum information theory. While often correlated, quantum nonlocality exhibits its own distinct manifestations, independent of entanglement [3]. In

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fact, when the involved systems are quantum, various counterintuitive phenomena emerge that have no classical analog [4–12]. One notable example is the inability to perfectly extract classical information encoded in the states of a composite quantum system using only local operations and classical communication (LOCC) [13–22], among spatially separated parties [23–25]. A set of orthogonal quantum states that defies such local discrimination is referred to as nonlocal, as their optimal distinguishability necessitates global operations on the composite system [26–29]. In the context of bipartite local state discrimination, the task involves identifying a quantum state secretly chosen from a known set of orthogonal states, using only LOCC protocols [30–33]. This paradigm plays a crucial role in probing the fundamental limitations of LOCC and has revealed that entanglement can often increase the difficulty of such discrimination tasks [12]. Nevertheless, it is now well understood that entanglement is not a necessary condition for exhibiting this form of nonlocality [3,34–41].

In 1999, Bennett *et al.* [3] in their seminal paper first constructed examples of orthonormal product basis in $3 \otimes 3$ that are not perfectly distinguishable under LOCC. The construction was quite striking as it contained only product states and, hence, introduced the phenomenon of quantum nonlocality without entanglement [42–46]. The result shows that entanglement is not an essential feature for nonlocality of quantum states. This motivates us in studying the role of entanglement in a nonlocal set of states. Furthermore, the notion of incomplete bases that exhibit nonlocality without entanglement is captured by the concept of unextendible product bases (UPBs) [47–54]. A UPB is a set of mutually orthogonal product states such that no product state lies in the orthogonal complement of the subspace they span [11]. UPBs cannot be perfectly distinguished by LOCC, and the projector onto their orthogonal complement yields a mixed state exhibiting the phenomenon of bound entanglement. Consequently, such states are of significant importance in quantum information theory [55–59].

Quantum teleportation is one phenomenon where entanglement is used as a resource to send a state altogether from one party to another [60]. Given a state from a locally indistinguishable set of states, one can therefore use entanglement to fetch all particles in one laboratory to perfectly identify the state. However, since entanglement is a valuable resource, finding the minimal amount of entanglement is always an interesting question. In 2008, Cohen [47] first presented a method that uses entanglement more efficiently than teleportation to distinguish certain classes of UPBs, and showed that certain classes of UPBs in $m \otimes n$ ($m \leq n$) can be locally distinguished with the aid of $\lceil \frac{m}{2} \rceil \otimes \lceil \frac{m}{2} \rceil$ maximally entangled state. He also presents a method to locally distinguish a set of five states on a $3 \otimes 3$ bipartite system with only one ebit of entanglement. Whether these states can be distinguished with less than one ebit remained an open problem. Subsequently, the role of entanglement as a resource for local state discrimination in multipartite systems has attracted increasing attention [47,48,50–54,57].

Multipartite entanglement enables tasks that are impossible with only bipartite correlations. It serves as a fundamental resource for quantum repeaters [61,62] and quantum conference key agreement [62], among other applications. Its structure and SLOCC (stochastic local operation and classical communication) classification play a decisive role in determining the power and limitations of multiparty quantum protocols. In three qubits, there are two inequivalent genuinely entangled SLOCC classes [63], namely, GHZ-SLOCC and W-SLOCC classes. In terms of entanglement across reduced bipartitions, a three-qubit pure GHZ-SLOCC class state can be categorized into four broad families. They may exhibit no bipartite entanglement in one, two, or all three reduced bipartitions. While numerous works have explored entanglement-assisted state discrimination, in most scenarios perfect discrimination requires a maximally entangled state.

In this paper, we propose a framework for investigating the local distinguishability of orthogonal product states using a nonmaximally entangled state from the GHZ-SLOCC class. We further analyze how the probability of error depends on both bipartite and multipartite entanglement. These results can help us have a better understanding of the relationship between entanglement classification and nonlocality.

The remaining portion of this paper is arranged as follows. In Sec. II, necessary definitions and other preliminary concepts are presented. In Sec. III, we represent the classification of the three-qubit GHZ class of states. These states, characterized by profound entanglement properties, play a pivotal role in quantum information processing. In Sec. IV, we have demonstrated the distinguishability of a nonlocal set of orthogonal product states by various nongeneric GHZ class states as a resource. In Sec. V, we analyze the probability of error with respect to concurrence and tangle of nongeneric GHZ-SLOCC class states. Finally, the conclusion is drawn in Sec. VI.

II. PRELIMINARIES

In this section, we present basic knowledge and facts that help to realize the whole scenario about this topic.

The resource character of entanglement immediately leads to entanglement quantification, so that we can understand how much of entanglement is required for some information processing task. The introduction of entanglement monotones [64] or entanglement measures [65] is put forward to quantify entanglement.

Definition 1. For two-qubit state ρ_{AB} , concurrence (an entanglement monotone) [66,67] is defined as $C(\rho_{AB}) = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}$, where $\lambda_1, \lambda_2, \lambda_3$, and λ_4 are the square roots of the eigenvalues of the matrix $\rho_{AB}((\sigma_y \otimes \sigma_y)\rho_{AB}^*(\sigma_y \otimes \sigma_y))$ in decreasing order, where σ_y is the Pauli spin matrix and ρ_{AB}^* is the complex conjugate of ρ_{AB} .

For a pure bipartite state, the concurrence can be computed as $C(\rho_{AB}) = 2\sqrt{\det \rho_A}$, where ρ_A is obtained from ρ_{AB} by taking partial trace over qubit B.

Definition 2 [67]. The residual entanglement or tangle is a multipartite entanglement measure defined as $\tau = C^2(\rho_{A|BC}) - C^2(\rho_{AB}) - C^2(\rho_{AC})$.

In quantum theory, measurements on a system of dimension d are represented by a set of positive operator-valued measure (POVM) elements $\{\pi_l\}$, which satisfy the completeness relation

$$\sum_l \pi_l = I_{d \times d}, \quad (1)$$

where $I_{d \times d}$ is the identity matrix of dimension $d \times d$.

Definition 3 [58]. If all the POVM elements of a measurement setup associated with a state discrimination task of a given set of quantum states are proportional to the identity matrix, then such a measurement is not useful for extracting information about this very task and is called a *trivial measurement*.

Conversely, if not all POVM elements of a measurement are proportional to the identity matrix, then the measurement is said to be a *nontrivial measurement*.

Definition 4 [58]. Consider a measurement setup for distinguishing a known set of pairwise orthogonal quantum states. After performing that measurement, if the postmeasurement states remain mutually orthogonal, then such a measurement setup is said to be an *orthogonality-preserving measurement*.

Now, if a set of multipartite orthogonal quantum states is provided, and no party can start with a nontrivial orthogonality-preserving measurement, then it becomes impossible to eliminate any state from the set while ensuring that the remaining postmeasurement states remain orthogonal. This situation actually ensures the nonlocality of the set.

In the next section, we will explore tripartite GHZ-SLOCC class states, focusing on their classification and key characteristics.

III. GHZ STATES: CLASSIFICATION AND CHARACTERISTICS

Genuine tripartite entanglement refers to a form of quantum entanglement that involves three particles. This kind of entanglement extends beyond the traditional bipartite entanglement observed in two-particle systems, as it involves correlations that cannot be fully described by considering the

two subsystems separately. One of the most notable examples of genuine tripartite entanglement is the GHZ state, given by $|\text{GHZ}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$. The GHZ state is the maximally entangled state inside the GHZ-SLOCC class. It is a three-qubit generalization of the Bell state (maximal bipartite entangled state). In contrast to bipartite states, where entanglement properties can be largely captured by focusing on two-qubit subsystems, the GHZ state exhibits correlations that are nonlocal across all three qubits, meaning that the entanglement is truly shared among the entire system, not just between pairs of qubits. This highlights the unique and intricate nature of tripartite entanglement, which plays a critical role in various quantum information protocols. The distinctive properties of such states are crucial for advancing our understanding of quantum systems and exploring new ways of harnessing quantum mechanics for practical applications.

A state belonging to the GHZ-SLOCC class can be expressed, up to local unitary transformations, as [68,69]

$$|\psi(\theta_1, \theta_2, \theta_3, r)\rangle = \frac{1}{\sqrt{k}} g_x^1 \otimes g_x^2 \otimes g_x^3 P_z(|000\rangle + |111\rangle), \quad (2)$$

where $g_x^j = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \cos \theta_j \\ 0 & \frac{1}{\sqrt{2}} \sin \theta_j \end{pmatrix}$, $P_z = \begin{pmatrix} z & 0 \\ 0 & \frac{1}{z} \end{pmatrix}$, $\theta_j \in (0, \frac{\pi}{2}]$, $j = 1, 2, 3$, $z = re^{i\theta} \in \mathbb{C}$ with $r \in (0, 1]$, and $\frac{1}{\sqrt{k}}$ is the normalizing factor,

$$k = \frac{1 + |z|^4 + (z^2 + (z^*)^2) \cos \theta_1 \cos \theta_2 \cos \theta_3}{8|z|^2}. \quad (3)$$

Now, the concurrences of the reduced state of $|\psi(\theta_1, \theta_2, \theta_3, z)\rangle$ are

$$\begin{aligned} C_{12} &= \frac{\sin \theta_1 \sin \theta_2 \cos \theta_3}{4k}, \\ C_{13} &= \frac{\sin \theta_1 \cos \theta_2 \sin \theta_3}{4k}, \\ C_{23} &= \frac{\cos \theta_1 \sin \theta_2 \sin \theta_3}{4k}. \end{aligned} \quad (4)$$

Throughout the section, we establish the classification for the GHZ class of nongeneric pure states. The nongeneric class is of special interest because here at least one reduced bipartite system will not have entanglement, which depends on the parameters θ_1 , θ_2 , and θ_3 , as evident from the concurrences. Including all cases, they all have a nonzero tangle. Nongeneric three-qubit GHZ pure states enable us to consider the parameter z as a real number $r \in (0, 1]$, because a local unitary transformation can be utilized on the qubit with θ_1 or θ_2 or $\theta_3 = \frac{\pi}{2}$ to absorb the phase of z . As there exists a strong correspondence between the parameters and the entanglement of the reduced system, we now characterize all the subclasses of the nongeneric class.

Type I ($\theta_1 = \pi/2$, $\theta_2 = \pi/2$, $\theta_3 = \pi/2$). This is the simplest subclass structure of the nongeneric GHZ class, as there does not exist any kind of bipartite entanglement in the reduced subsystems. Even from the concurrence relation, we can check that $C_{ij} = 0 \forall i \neq j \in \{1, 2, 3\}$. The GHZ state $|\text{GHZ}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$ belongs to this subclass.

Type II ($\theta_1 = \pi/2$, $\theta_2 = \pi/2$, $\theta_3 \neq \pi/2$). In this nongeneric GHZ subclass, there exists bipartite entanglement in only one reduced subsystem. Therefore, the existence of entanglement in the reduced subsystems, as well as the structure of the class, exhibits a notable change from the previous subclass. The concurrences are given by $C_{13} = 0$, $C_{23} = 0$, and $C_{12} = \frac{\cos \theta_3}{4k}$.

Type III ($\theta_1 = \pi/2$, $\theta_2 \neq \pi/2$, $\theta_3 \neq \pi/2$). The complexity of the nongeneric GHZ class in this case gradually increases. In this case, bipartite entanglement exists between two reduced subsystems. The concurrences of the reduced states are given by $C_{12} = \frac{\sin \theta_2 \cos \theta_3}{4k}$, $C_{13} = \frac{\cos \theta_2 \sin \theta_3}{4k}$, and $C_{23} = 0$.

In the upcoming section, we will explore how the use of specific nongeneric GHZ states can significantly influence the process of quantum state discrimination in multipartite systems.

This approach provides a more efficient framework for distinguishing between quantum states, enhancing the accuracy and performance of quantum information protocols within complex multipartite configurations.

IV. QUANTUM STATE DISCRIMINATION IN TRIPARTITE SYSTEMS VIA NONGENERIC GHZ STATES

Consider a tripartite quantum system associated with a Hilbert space $\mathcal{H} = (\mathbb{C}^3)^{\otimes 3}$. In the following, an explicit construction of a small set of pure orthogonal product states $\{|\psi_i\rangle\} \in \mathcal{H} = (\mathbb{C}^3)^{\otimes 3}$ is presented with $i = 1, \dots, 12$. The states are given by

$$\begin{aligned} |\psi_1\rangle &= |0\rangle_A |1\rangle_B |0+1\rangle_C, & |\psi_2\rangle &= |0\rangle_A |1\rangle_B |0-1\rangle_C, \\ |\psi_3\rangle &= |0\rangle_A |2\rangle_B |0+2\rangle_C, & |\psi_4\rangle &= |0\rangle_A |2\rangle_B |0-2\rangle_C, \\ |\psi_5\rangle &= |1\rangle_A |0+1\rangle_B |0\rangle_C, & |\psi_6\rangle &= |1\rangle_A |0-1\rangle_B |0\rangle_C, \\ |\psi_7\rangle &= |2\rangle_A |0+2\rangle_B |0\rangle_C, & |\psi_8\rangle &= |2\rangle_A |0-2\rangle_B |0\rangle_C, \\ |\psi_9\rangle &= |0+1\rangle_A |0\rangle_B |1\rangle_C, & |\psi_{10}\rangle &= |0-1\rangle_A |0\rangle_B |1\rangle_C, \\ |\psi_{11}\rangle &= |0+2\rangle_A |0\rangle_B |2\rangle_C, & |\psi_{12}\rangle &= |0-2\rangle_A |0\rangle_B |2\rangle_C. \end{aligned} \tag{5}$$

For simplicity, we are ignoring the normalizing coefficients for the above set of states. Following, we are stating a proposition involving characteristics of states in Eq. (5).

Proposition 1 [58]. No state from Eq. (5) can be eliminated by performing orthogonality-preserving measurements.

The twisted states [18] play an important role in local distinguishability of orthogonal product states. Using entanglement as a resource to discriminate a set of locally indistinguishable states, we need sufficient ancillary subsystems [47,48]. For the discrimination task with support of a two-qubit Bell state, the dimension of each ancillary subsystem is two, and the number of systems holding ancillary subsystems is also two. To generate a discrimination protocol, sharing a Bell state, respective parties choose appropriate measurements on their systems together with the ancillary subsystems. Whereas for the discrimination task of a tripartite system with support of a three-qubit GHZ state, the dimension of each ancillary subsystem is two, the number of systems holding ancillary subsystems is three. So, the choice of measurement for the Bell-state-assisted discrimination task must be a subset of the choice of measurement for the GHZ-state-assisted discrimination task. We summarize the above discussions by writing the following proposition.

Proposition 2. In a tripartite system, if a set of orthogonal product states is distinguishable by using a two-qubit Bell state shared between any two parties, then the set of states can be distinguished by using a three-qubit GHZ state. But the converse is not always true.

For the $(\mathbb{C}^3)^{\otimes 3}$ system, the set of states (5) can be distinguished by sharing one copy of the Bell state shared between any two parties [58]. So by Proposition 2, it must be distinguished by one copy of the GHZ state.

Corollary 1. One copy of the three-qubit GHZ is sufficient to distinguish the set of states (5).

Proof. First, we assume that a GHZ state shared between three parties, Alice, Bob, and Charlie, is $|\psi\rangle_{abc}$. Then, the initial states shared among them is $|\phi\rangle_{ABC} \otimes |\psi\rangle_{abc}$, where $|\phi\rangle$ is one of the states from Eq. (5).

Step 1. Alice performs the following measurement:

$$\mathcal{A} \equiv \{M := \mathbb{P}[(|1\rangle, |2\rangle)_A; |1\rangle_a] + \mathbb{P}[|0\rangle_A; |0\rangle_a], \bar{M} := \mathbb{I} - M\}.$$

Suppose the outcomes correspond to M clicks, then the resulting postmeasurement states are

$$\begin{aligned}
 |\psi_1, \psi_2\rangle &\rightarrow |0\rangle_A |1\rangle_B |0 \pm 1\rangle_C |000\rangle_{abc}, \\
 |\psi_3, \psi_4\rangle &\rightarrow |0\rangle_A |2\rangle_B |0 \pm 2\rangle_C |000\rangle_{abc}, \\
 |\psi_5, \psi_6\rangle &\rightarrow |1\rangle_A |0 \pm 1\rangle_B |0\rangle_C |111\rangle_{abc}, \\
 |\psi_7, \psi_8\rangle &\rightarrow |2\rangle_A |0 \pm 2\rangle_B |0\rangle_C |111\rangle_{abc}, \\
 |\psi_9, \psi_{10}\rangle &\rightarrow |0\rangle_A |0\rangle_B |1\rangle_C |000\rangle_{abc} \pm |1\rangle_A |0\rangle_B |1\rangle_C |111\rangle_{abc}, \\
 |\psi_{11}, \psi_{12}\rangle &\rightarrow |0\rangle_A |0\rangle_B |2\rangle_C |000\rangle_{abc} \pm |2\rangle_A |0\rangle_B |2\rangle_C |111\rangle_{abc}.
 \end{aligned}$$

Step 2. Bob performs the following measurement:

$$\mathcal{B} \equiv \{N_1 := \mathbb{P}[(|1\rangle, |2\rangle)_B; |0\rangle_b], N_2 := \mathbb{P}[|0\rangle_B; |1\rangle_b], \bar{N} := \mathbb{I} - N_1 - N_2\}.$$

If the outcomes correspond to N_1 and N_2 clicks, then the respective resulting postmeasurement states are

$$\begin{aligned}
 &|0\rangle_A |1\rangle_B |0 \pm 1\rangle_C |000\rangle_{abc}, \\
 &|0\rangle_A |2\rangle_B |0 \pm 2\rangle_C |000\rangle_{abc},
 \end{aligned}$$

which can be easily distinguished by Charlie by projecting onto $|0 \pm 1\rangle_C$ and $|1 \pm 2\rangle_C$. If the outcome corresponds to $\bar{N}$ clicks, then the remaining eight states are isolated.

Step 3. Charlie performs a measurement

$$\begin{aligned}
 \mathcal{C} &\equiv \{Q_1 := \mathbb{P}[|1\rangle_C; |0\rangle_c] + \mathbb{P}[|1\rangle_C; |1\rangle_c], \\
 Q_2 &:= \mathbb{P}[|2\rangle_C; |0\rangle_c] + \mathbb{P}[|2\rangle_C; |1\rangle_c], \\
 \bar{Q} &:= \mathbb{I} - Q_1 - Q_2\}.
 \end{aligned}$$

There are only two states due to each outcome, Q_1 and Q_2 . For each case, by Walgate *et al.*'s result [6], the states are distinguishable. If the outcome corresponds to $\bar{Q}$ clicks, then the remaining four states are isolated. After that, Alice makes two outcome rank 1 measurements $|1\rangle_A |1\rangle_a$ and $|2\rangle_A |1\rangle_a$. There are only two states due to each outcome, and they can be easily distinguished by Bob. Hence, those states are perfectly distinguished. ■

In the next proposition, we study the role of nonmaximal biparty entanglement for the discrimination of the set of states (5) and explore the interconnection between the amount of entanglement and the corresponding discrimination error.

Proposition 3. If an ancillary entangled state is shared between Alice and Bob, such as $|\chi\rangle = \frac{r}{\sqrt{1+r^4}}(r|00\rangle_{ab} + \frac{1}{r}|11\rangle_{ab})$, $r \in (0, 1]$, then the set of state (5) can be locally distinguished with probability of error $p_e = \frac{1-r^4}{1+r^4}\xi_e$, $0 < \xi_e < 1$.

Proof. Suppose an ancillary nonmaximally entangled state is shared between Alice and Bob, such as $|\chi\rangle = \frac{r}{\sqrt{1+r^4}}(r|00\rangle_{ab} + \frac{1}{r}|11\rangle_{ab})$. Then, the initial states are transformed into

$$|\psi'_i\rangle = |\psi_i\rangle \otimes \left[\frac{r}{\sqrt{1+r^4}} \left(r|00\rangle_{ab} + \frac{1}{r}|11\rangle_{ab} \right) \right],$$

where the concrete expression of $|\psi_i\rangle$ is presented in Eq. (5). Then, Alice performs a three-outcome measurement, each outcome corresponding to an operator as

$$\begin{aligned}
 M_1 &= |0\rangle_A \langle 0| \otimes |0\rangle_a \langle 0| + r^2(|1\rangle_A \langle 1| + |2\rangle_A \langle 2|) \otimes |1\rangle_a \langle 1|, \\
 M_2 &= r^2|0\rangle_A \langle 0| \otimes |1\rangle_a \langle 1| + (|1\rangle_A \langle 1| + |2\rangle_A \langle 2|) \otimes |0\rangle_a \langle 0|, \\
 M_3 &= \sqrt{1-r^4}I_A \otimes |1\rangle_a \langle 1|,
 \end{aligned}$$

satisfying the completeness equation $\sum_{i=1}^3 M_i^\dagger M_i = I_{Aa}$, where I_{Aa} is the identity operator of the parts A and a . Alice will obtain the outcome M_t with probability $\gamma_t = \langle \psi'_i | (M_t^\dagger M_t \otimes I_{Bb}) | \psi'_i \rangle$, and the state of the system after the measurement is $\frac{(M_t \otimes I_{Bb}) | \psi'_{ik} \rangle}{\sqrt{\langle \psi'_{ik} | (M_t^\dagger M_t \otimes I_{Bb}) | \psi'_{ik} \rangle}}$, where $t = 1, 2, 3$ and I_{Bb} is the identity operator of the parts B and b . The postmeasurement reduced states corresponding to the outcome M_1 are

$$\begin{aligned}
 |\psi_1, \psi_2\rangle &\rightarrow |0\rangle_A |1\rangle_B |0 \pm 1\rangle_C |00\rangle_{ab}, \\
 |\psi_3, \psi_4\rangle &\rightarrow |0\rangle_A |2\rangle_B |0 \pm 2\rangle_C |00\rangle_{ab}, \\
 |\psi_5, \psi_6\rangle &\rightarrow |1\rangle_A |0 \pm 1\rangle_B |0\rangle_C |11\rangle_{ab}, \\
 |\psi_7, \psi_8\rangle &\rightarrow |2\rangle_A |0 \pm 2\rangle_B |0\rangle_C |11\rangle_{ab}, \\
 |\psi_9, \psi_{10}\rangle &\rightarrow |0\rangle_A |0\rangle_B |1\rangle_C |00\rangle_{ab} \pm |1\rangle_A |0\rangle_B |1\rangle_C |11\rangle_{ab}, \\
 |\psi_{11}, \psi_{12}\rangle &\rightarrow |0\rangle_A |0\rangle_B |2\rangle_C |00\rangle_{ab} \pm |2\rangle_A |0\rangle_B |2\rangle_C |11\rangle_{ab},
 \end{aligned} \tag{6}$$

and the outcome M_1 occurs with probability $\mathcal{P}_1 = \frac{r^4}{1+r^4}$.

After that, Bob performs the following measurement:

$$\mathcal{B} \equiv \{N_1 := \mathbb{P}[|1\rangle_B; |0\rangle_b], N_2 := \mathbb{P}[|2\rangle_B; |0\rangle_b], \bar{N} := \mathbb{I} - N_1 - N_2\}.$$

If the outcomes correspond to N_1 and N_2 clicks, then the respective resulting postmeasurement states are

$$\begin{aligned}
 &|0\rangle_A |1\rangle_B |0 \pm 1\rangle_C |00\rangle_{ab}, \\
 &|0\rangle_A |2\rangle_B |0 \pm 2\rangle_C |00\rangle_{ab},
 \end{aligned}$$

which can be easily distinguished by Charlie by projecting onto $|0 \pm 1\rangle_C$ and $|1 \pm 2\rangle_C$. If the outcome corresponds to $\bar{N}$ clicks, then the remaining eight states are isolated, which can be easily distinguished by Charlie and Alice.

Alice obtains the second outcome M_2 with probability $\mathcal{P}_2 = \frac{r^4}{1+r^4}$, then this creates new states that differ from the states (6) only by ancillary systems $|00\rangle_{ab} \rightarrow |11\rangle_{ab}$ and $|11\rangle_{ab} \rightarrow |00\rangle_{ab}$.

Alice obtains the third outcome M_3 with probability $\mathcal{P}_3 = \frac{1-r^4}{1+r^4}$, and the postmeasurement states are

$$\begin{aligned}
 |\psi_1, \psi_2\rangle &\rightarrow |0\rangle_A |1\rangle_B |0 \pm 1\rangle_C |11\rangle_{ab}, \\
 |\psi_3, \psi_4\rangle &\rightarrow |0\rangle_A |2\rangle_B |0 \pm 2\rangle_C |11\rangle_{ab}, \\
 |\psi_5, \psi_6\rangle &\rightarrow |1\rangle_A |0 \pm 1\rangle_B |0\rangle_C |11\rangle_{ab}, \\
 |\psi_7, \psi_8\rangle &\rightarrow |2\rangle_A |0 \pm 2\rangle_B |0\rangle_C |11\rangle_{ab}, \\
 |\psi_9, \psi_{10}\rangle &\rightarrow |0 \pm 1\rangle_A |0\rangle_B |1\rangle_C |11\rangle_{ab}, \\
 |\psi_{11}, \psi_{12}\rangle &\rightarrow |0 \pm 2\rangle_A |0\rangle_B |2\rangle_C |11\rangle_{ab}.
 \end{aligned}$$

It is to be noted that the postmeasurement reduced states have not been changed from Eq. (5) by omitting the ancillary parts $|11\rangle_{ab}$ due to the outcome M_3 . So, we can continue to do some measurements to distinguish them without an ancillary entanglement system in probability, where an ambiguous discrimination strategy can always be taken, or we can simply guess which state is prepared in the system. We denote by ξ_e the optimal minimum-error probability for this conditional ensemble, obtained by optimizing over all admissible POVMs, defined by

$$\xi_e = 1 - \max_{\{\Pi_k\} \in \text{LOCC}} \frac{1}{12} \sum_{k=1}^{12} \text{Tr}(\Pi_k \rho_k^{(3)}), \tag{7}$$

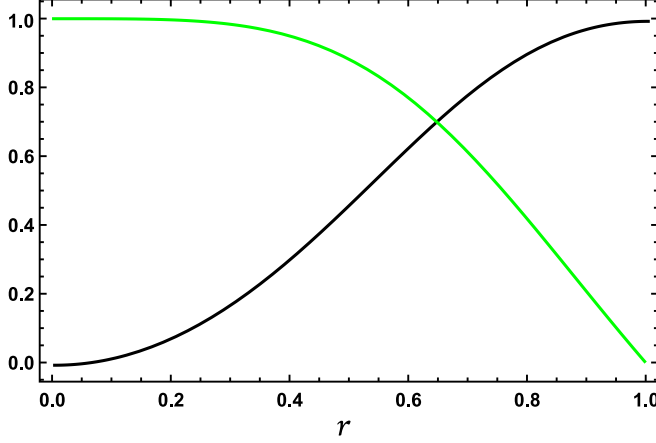


FIG. 1. C_{AB} (black line) and p_e (green line) with respect to r for the state $|\chi\rangle = \frac{r}{\sqrt{1+r^4}}(r|00\rangle_{ab} + \frac{1}{r}|11\rangle_{ab})$.

where $\rho_k^{(3)} = |\psi_k\rangle\langle\psi_k|$ and the optimization is over LOCC measurements. Thus, ξ_e is a well-defined conditional error probability rather than an unoptimized parameter.

To distinguish these states, we first let Alice make a measurement. If Alice obtains the first outcome M_1 or the second outcome M_2 , these states after the measurement can be perfectly distinguished by LOCC. If Alice obtains the third outcome M_3 , we can continue to distinguish them without entanglement. Thus, the final probability of error is $p_e = \mathcal{P}_1 \times 0 + \mathcal{P}_2 \times 0 + \mathcal{P}_3 \times \xi_e = \frac{1-r^4}{1+r^4} \xi_e$, $0 < \xi_e < 1$. This completes the proof. ■

We now explore the relationship between the probability of error in state discrimination and the quantity of entanglement, using a two-qubit nonmaximally entangled state. Consider the state $|\chi\rangle = \frac{r}{\sqrt{1+r^4}}(r|00\rangle_{ab} + \frac{1}{r}|11\rangle_{ab})$, where $r \in (0, 1]$. The degree of entanglement in this state can be quantified using concurrence, which is a widely used measure of entanglement for two-qubit systems. The concurrence $C_{AB}(\chi)$ for the given state is given by the formula $C_{AB}(\chi) = \frac{2r^2}{1+r^4}$. This expression shows how the concurrence, and thus the entanglement, depends on the parameter r . When $r = 1$, the state becomes maximally entangled, and the concurrence reaches its maximum value of 1. As r decreases, the concurrence decreases as well, indicating a reduction in entanglement. This relationship between r and the concurrence will be key in analyzing how entanglement influences the probability of error in various quantum tasks, such as state discrimination. According to Proposition 3, the probability of error in distinguishing the orthogonal states can be expressed as $p_e = \frac{1-r^4}{1+r^4} \xi_e$, $0 < \xi_e < 1$, where ξ_e is a parameter that modulates the error probability. The exact behavior of these functions (see Ref. [70]) is illustrated in Fig. 1. From the figure, it is evident that as the amount of entanglement shared between Alice and Bob decreases, the probability of error in locally distinguishing the orthogonal states described by Eq. (5) increases. Notably, when $r = 1$, the entanglement reaches its maximum value of one ebit, and the probability of error becomes zero. This demonstrates that maximal bipartite entanglement enables error-free discrimination, while nonmaximal entanglement increases the probability of error in the discrimination process. Next, we will analyze state discrimination capability of nonmaximally type-I GHZ states.

Corollary 2. If $|\chi\rangle = |\psi(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, r)\rangle$ with $\tau_{ABC}(|\chi\rangle) = \frac{4r^4}{(1+r^4)^2}$ and $C_{AB}(|\chi\rangle) = C_{AC}(|\chi\rangle) = C_{BC}(|\chi\rangle) = 0$, the set of states (5) can be distinguished with probability of error $p_e = \frac{1-r^4}{1+r^4} \xi_e$, $0 < \xi_e < 1$.

Proof. In this context, $|\chi\rangle = \frac{r}{\sqrt{1+r^4}}(r|000\rangle_{abc} + \frac{1}{r}|111\rangle_{abc})$ and the initial states are transformed into

$$|\psi'_i\rangle = |\psi_i\rangle \otimes \left[\frac{r}{\sqrt{1+r^4}} \left(r|000\rangle_{abc} + \frac{1}{r}|111\rangle_{abc} \right) \right],$$

where the concrete expression of $|\psi_i\rangle$ is presented in Eq. (5). Then, Alice performs a three-outcome measurement, each outcome corresponding to an operator as

$$\begin{aligned} M_1 &= |0\rangle_A \langle 0| \otimes |0\rangle_a \langle 0| + r^2(|1\rangle_A \langle 1| + |2\rangle_A \langle 2|) \otimes |1\rangle_a \langle 1|, \\ M_2 &= r^2|0\rangle_A \langle 0| \otimes |1\rangle_a \langle 1| + (|1\rangle_A \langle 1| + |2\rangle_A \langle 2|) \otimes |0\rangle_a \langle 0|, \\ M_3 &= \sqrt{1-r^4}I_A \otimes |1\rangle_a \langle 1|, \end{aligned}$$

satisfying the completeness equation $\sum_{i=1}^3 M_i^\dagger M_i = I_{Aa}$. Now the postmeasurement reduced states corresponding to the outcome M_1 are

$$\begin{aligned} |\psi_1, \psi_2\rangle &\rightarrow |0\rangle_A |1\rangle_B |0 \pm 1\rangle_C |000\rangle_{abc}, \\ |\psi_3, \psi_4\rangle &\rightarrow |0\rangle_A |2\rangle_B |0 \pm 2\rangle_C |000\rangle_{abc}, \\ |\psi_5, \psi_6\rangle &\rightarrow |1\rangle_A |0 \pm 1\rangle_B |0\rangle_C |111\rangle_{abc}, \\ |\psi_7, \psi_8\rangle &\rightarrow |2\rangle_A |0 \pm 2\rangle_B |0\rangle_C |111\rangle_{abc}, \\ |\psi_9, \psi_{10}\rangle &\rightarrow |0\rangle_A |0\rangle_B |1\rangle_C |000\rangle_{abc} \pm |1\rangle_A |0\rangle_B |1\rangle_C |111\rangle_{abc}, \\ |\psi_{11}, \psi_{12}\rangle &\rightarrow |0\rangle_A |0\rangle_B |2\rangle_C |000\rangle_{abc} \pm |2\rangle_A |0\rangle_B |2\rangle_C |111\rangle_{abc}, \end{aligned} \tag{8}$$

and the outcome M_1 occurs with probability $\mathcal{P}_1 = \frac{r^4}{1+r^4}$.

After that, Bob performs the following measurement:

$$\mathcal{B} \equiv \{N_1 := \mathbb{P}[|1\rangle_B; |0\rangle_b], N_2 := \mathbb{P}[|2\rangle_B; |0\rangle_b], \bar{N} := \mathbb{I} - N_1 - N_2\}.$$

If the outcomes correspond to N_1 and N_2 clicks, then the respective resulting postmeasurement states are

$$\begin{aligned} &|0\rangle_A |1\rangle_B |0 \pm 1\rangle_C |000\rangle_{abc}, \\ &|0\rangle_A |2\rangle_B |0 \pm 2\rangle_C |000\rangle_{abc}, \end{aligned}$$

which can be easily distinguished by projecting onto $|0 \pm 1\rangle_C |0\rangle_c$ and $|1 \pm 2\rangle_C |0\rangle_c$. If the outcome corresponds to $\bar{N}$ clicks, then the remaining eight states can be easily distinguished by Charlie and Alice.

Alice obtains the second outcome M_2 with probability $\mathcal{P}_2 = \frac{r^4}{1+r^4}$; then, this creates new states that differ from the states (8) only by ancillary systems $|000\rangle_{abc} \rightarrow |111\rangle_{abc}$ and $|111\rangle_{abc} \rightarrow |000\rangle_{abc}$.

Alice obtains the third outcome M_3 with probability $\mathcal{P}_3 = \frac{1-r^4}{1+r^4}$, and the postmeasurement states are

$$\begin{aligned} |\psi_1, \psi_2\rangle &\rightarrow |0\rangle_A |1\rangle_B |0 \pm 1\rangle_C |111\rangle_{abc}, \\ |\psi_3, \psi_4\rangle &\rightarrow |0\rangle_A |2\rangle_B |0 \pm 2\rangle_C |111\rangle_{abc}, \\ |\psi_5, \psi_6\rangle &\rightarrow |1\rangle_A |0 \pm 1\rangle_B |0\rangle_C |111\rangle_{abc}, \\ |\psi_7, \psi_8\rangle &\rightarrow |2\rangle_A |0 \pm 2\rangle_B |0\rangle_C |111\rangle_{abc}, \\ |\psi_9, \psi_{10}\rangle &\rightarrow |0 \pm 1\rangle_A |0\rangle_B |1\rangle_C |111\rangle_{abc}, \\ |\psi_{11}, \psi_{12}\rangle &\rightarrow |0 \pm 2\rangle_A |0\rangle_B |2\rangle_C |111\rangle_{abc}. \end{aligned}$$

Thus, the final probability of error is $p_e = \mathcal{P}_1 \times 0 + \mathcal{P}_2 \times 0 + \mathcal{P}_3 \times \xi_e = \frac{1-r^4}{1+r^4} \xi_e$, $0 < \xi_e < 1$, where ξ_e is defined in Eq. (7). This completes the proof. ■

The nongeneric type-II and type-III GHZ class refers to a special category of GHZ states. In this class, while the state may exhibit bipartite entanglement between certain pairs of reduced subsystems, at least one pair shows no entanglement at all. This creates an asymmetry in the distribution of entanglement, where the bipartite entanglement between reduced subsystems is not equally shared. The complexity of the system's quantum correlations gradually increases as parameters θ_1 , θ_2 , and θ_3 evolve, reflecting more intricate quantum behavior. Hereafter, we will deal with a three-outcome measurement setup denoted as $\mathcal{M}(\mathcal{X}, \mathcal{Y})$ and defined by

$$\mathcal{M}(\mathcal{X}, \mathcal{Y}) \equiv \begin{cases} M_1 := \mathcal{X}\mathbb{P}[|0\rangle_A; |0\rangle_a] + \mathcal{Y}\mathbb{P}[|1\rangle, |2\rangle)_A; |1\rangle_a], \\ M_2 := \mathcal{Y}\mathbb{P}[|0\rangle_A; |1\rangle_a] + \mathcal{X}\mathbb{P}[|1\rangle, |2\rangle)_A; |0\rangle_a], \\ M_3 := I_A \otimes (\sqrt{1 - \mathcal{X}^2}\mathbb{P}[|0\rangle_a] + \sqrt{1 - \mathcal{Y}^2}\mathbb{P}[|1\rangle_a]), \end{cases}$$

satisfying the completeness equation $\sum_{i=1}^3 M_i^\dagger M_i = I_{Aa}$, where I_{Aa} is the identity operator of the parts A and a .

Proposition 4. If $|\chi\rangle = |\psi(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{4}, r)\rangle$, $r \in (0, 1]$, with $\mathcal{T}_{ABC}(|\chi\rangle) = \frac{\sqrt{2}r^4}{(1+r^4)^2}$, and $\mathcal{C}_{AB}(|\chi\rangle) = \frac{\sqrt{2}r^2}{1+r^4}$, $\mathcal{C}_{AC}(|\chi\rangle) = \mathcal{C}_{BC}(|\chi\rangle) = 0$, then the set of states (5) can be distinguished by using $|\chi\rangle$ with probability of error

$$p_e = \begin{cases} \frac{1}{2(1+r^4)}(1 - 2r^4)\xi_e, & r^4 \leq \frac{1}{2}, \\ \frac{r^4}{1+r^4}(1 - \frac{1}{2r^4})\xi_e, & r^4 > \frac{1}{2}, \end{cases}$$

where $0 < \xi_e < 1$.

Proof. Suppose an ancillary entangled state is shared between Alice, Bob, and Charlie. Then, the initial states are transformed into

$$|\psi'_i\rangle = |\psi_i\rangle \otimes \left| \psi\left(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{4}, r\right) \right\rangle,$$

where the concrete expression of $|\psi_i\rangle$ is presented in Eq. (5). Then, Alice performs a three-outcome measurement

$$\mathcal{A} = \begin{cases} \mathcal{M}(1, \sqrt{2}r^2), & r^4 \leq \frac{1}{2}, \\ \mathcal{M}(\frac{1}{\sqrt{2}r^2}, 1), & r^4 > \frac{1}{2}. \end{cases}$$

The postmeasurement states corresponding to the outcome M_1 are

$$\begin{aligned} |\psi_1, \psi_2\rangle &\rightarrow |0\rangle_A |1\rangle_B |0 \pm 1\rangle_C |000\rangle_{abc}, \\ |\psi_3, \psi_4\rangle &\rightarrow |0\rangle_A |2\rangle_B |0 \pm 2\rangle_C |000\rangle_{abc}, \\ |\psi_5, \psi_6\rangle &\rightarrow |1\rangle_A |0 \pm 1\rangle_B |0\rangle_C |11\rangle_{ab} |0 \pm 1\rangle_c, \\ |\psi_7, \psi_8\rangle &\rightarrow |2\rangle_A |0 \pm 2\rangle_B |0\rangle_C |11\rangle_{ab} |0 \pm 1\rangle_c, \\ |\psi_9, \psi_{10}\rangle &\rightarrow [|001\rangle|000\rangle \pm |101\rangle|110\rangle \pm |101\rangle|111\rangle]_{ABC abc}, \\ |\psi_{11}, \psi_{12}\rangle &\rightarrow [|002\rangle|000\rangle \pm |202\rangle|110\rangle \pm |202\rangle|111\rangle]_{ABC abc}, \end{aligned}$$

and the outcome M_1 occurs with probability

$$\mathcal{P}_1 = \begin{cases} \frac{r^4}{1+r^4}, & r^4 \leq \frac{1}{2}, \\ \frac{1}{2(1+r^4)}, & r^4 > \frac{1}{2}. \end{cases}$$

After that, Bob and Charlie perform a sequence of measurements that is used in Proposition 3.

The impact of M_2 on Eq. (5) gives the postmeasurement states as

$$\begin{aligned} |\psi_1, \psi_2\rangle &\rightarrow |0\rangle_A |1\rangle_B |0 \pm 1\rangle_C |11\rangle_{ab} |0 + 1\rangle_c, \\ |\psi_3, \psi_4\rangle &\rightarrow |0\rangle_A |2\rangle_B |0 \pm 2\rangle_C |11\rangle_{ab} |0 + 1\rangle_c, \\ |\psi_5, \psi_6\rangle &\rightarrow |1\rangle_A |0 \pm 1\rangle_B |0\rangle_C |000\rangle_{abc}, \\ |\psi_7, \psi_8\rangle &\rightarrow |2\rangle_A |0 \pm 2\rangle_B |0\rangle_C |000\rangle_{abc}, \\ |\psi_9, \psi_{10}\rangle &\rightarrow (|001\rangle |11\rangle |0 + 1\rangle \pm |101\rangle |000\rangle)_{ABC abc}, \\ |\psi_{11}, \psi_{12}\rangle &\rightarrow (|002\rangle |11\rangle |0 + 1\rangle \pm |202\rangle |000\rangle)_{ABC abc}, \end{aligned}$$

and it occurs with probability

$$\mathcal{P}_2 = \begin{cases} \frac{r^4}{1+r^4}, & r^4 \leq \frac{1}{2}, \\ \frac{1}{2(1+r^4)}, & r^4 > \frac{1}{2}. \end{cases}$$

Next, Bob and Charlie perform a sequence of measurements that is followed by the previous theorem.

Alice obtains the third outcome M_3 with probability

$$\mathcal{P}_3 = \begin{cases} \frac{1}{2(1+r^4)}(1 - 2r^4), & r^4 \leq \frac{1}{2}, \\ \frac{r^4}{1+r^4}(1 - \frac{1}{2r^4}), & r^4 > \frac{1}{2}, \end{cases}$$

and the postmeasurement states due to M_3 become

$$\begin{aligned} |\psi_1, \psi_2\rangle &\rightarrow |0\rangle_A |1\rangle_B |0 \pm 1\rangle_C |111\rangle_{abc}, \\ |\psi_3, \psi_4\rangle &\rightarrow |0\rangle_A |2\rangle_B |0 \pm 2\rangle_C |111\rangle_{abc}, \\ |\psi_5, \psi_6\rangle &\rightarrow |1\rangle_A |0 \pm 1\rangle_B |0\rangle_C |111\rangle_{abc}, \\ |\psi_7, \psi_8\rangle &\rightarrow |2\rangle_A |0 \pm 2\rangle_B |0\rangle_C |111\rangle_{abc}, \\ |\psi_9, \psi_{10}\rangle &\rightarrow |0 \pm 1\rangle_A |0\rangle_B |1\rangle_C |111\rangle_{abc}, \\ |\psi_{11}, \psi_{12}\rangle &\rightarrow |0 \pm 2\rangle_A |0\rangle_B |2\rangle_C |111\rangle_{abc}. \end{aligned}$$

Thus, the final probability of error is $p_e = \mathcal{P}_1 \times 0 + \mathcal{P}_2 \times 0 + \mathcal{P}_3 \times \xi_e$ ($0 < \xi_e < 1$), where ξ_e is defined in Eq. (7). This completes the proof. $\blacksquare$

We will now investigate how the probability of error correlates with the level of entanglement. With the three-qubit nonmaximally entangled state $|\chi\rangle = |\psi(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{4}, r)\rangle$, $r \in (0, 1]$. The concurrences can be calculated by the formula $\mathcal{C}_{AB}(|\chi\rangle) = \frac{\sqrt{2}r^2}{1+r^4}$, $\mathcal{C}_{AC}(|\chi\rangle) = \mathcal{C}_{BC}(|\chi\rangle) = 0$. On the other hand, the tangle is $\mathcal{T}_{ABC}(|\chi\rangle) = \frac{\sqrt{2}r^4}{(1+r^4)^2}$.

According to Proposition 4, we know that the probability of error can be calculated by

$$p_e = \begin{cases} \frac{1}{2(1+r^4)}(1 - 2r^4)\xi_e, & r^4 \leq \frac{1}{2}, \\ \frac{r^4}{1+r^4}(1 - \frac{1}{2r^4})\xi_e, & r^4 > \frac{1}{2}, \end{cases}$$

where $0 < \xi_e < 1$. From Fig. 3 in the region $r \leq (\frac{1}{2})^{\frac{1}{4}}$, we can see that the less bipartite entanglement (black line) shared between Alice and Bob, the more the probability of error (green line) to locally distinguish the orthogonal set of states (5). However, for the region $r > (\frac{1}{2})^{\frac{1}{4}}$, we observe that the probability of error (green line) increases with an increase in bipartite entanglement (black line). A similar pattern appears between the amount of multipartite entanglement (red line) and the probability of error (green line). Moreover, with $r = (\frac{1}{2})^{\frac{1}{4}}$, the probability of error is zero, and the quantity of entanglement distribution is as follows: $\mathcal{C}_{AB}(|\chi\rangle) = \frac{2}{3}$, $\mathcal{C}_{AC}(|\chi\rangle) = \mathcal{C}_{BC}(|\chi\rangle) = 0$, and $\mathcal{T}_{ABC}(|\chi\rangle) = \frac{2\sqrt{2}}{9}$.

Proposition 5. If $|\chi\rangle = |\psi(\frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{4}, r)\rangle$ with $\mathcal{T}_{ABC}(|\chi\rangle) = \frac{r^4}{(1+r^4)^2}$ and $\mathcal{C}_{AB}(|\chi\rangle) = \frac{r^2}{1+r^4}$, $\mathcal{C}_{AC}(|\chi\rangle) = \frac{r^2}{1+r^4}$, $\mathcal{C}_{BC}(|\chi\rangle) = 0$, the set of states (5) can be distinguished by using $|\chi\rangle$ with probability of error

$$p_e = \begin{cases} \frac{1}{4(1+r^4)}(1 - 4r^4)\xi_e, & r^2 \leq \frac{1}{2}, \\ \frac{r^4}{1+r^4}(1 - \frac{1}{4r^4})\xi_e, & r^2 > \frac{1}{2}, \end{cases}$$

where $0 < \xi_e < 1$.

Proof. Without loss of generality, let us suppose that an ancillary entangled state $|\psi(\frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{4}, r)\rangle$ is shared between Alice, Bob, and Charlie. Then, the initial states are transformed into

$$|\psi'_i\rangle = |\psi_i\rangle \otimes \left| \psi\left(\frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{4}, r\right) \right\rangle,$$

where the concrete expression of $|\psi_i\rangle$ is presented in Eq. (5). Alice can perform a three-outcome measurement,

$$\mathcal{A} = \begin{cases} \mathcal{M}(1, 2r^2), & r^2 \leq \frac{1}{2}, \\ \mathcal{M}(\frac{1}{2r^2}, 1), & r^2 > \frac{1}{2}, \end{cases}$$

satisfying the completeness equation $\sum_{i=1}^3 M_i^\dagger M_i = I_{Aa}$, where I_{Aa} is the identity operator of the parts A and a . After that, Bob and Charlie do some sequence of measurements as described in Proposition 4. Thus, the final probability of error is $p_e = \mathcal{P}_1 \times 0 + \mathcal{P}_2 \times 0 + \mathcal{P}_3 \times \xi_e$ ($0 < \xi_e < 1$), where ξ_e is defined in Eq. (7). This completes the proof. ■

Now, we analyze the relationship between the probability of error and the amount of entanglement. For the three-qubit nonmaximally entangled state $|\chi\rangle = |\psi(\frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{4}, r)\rangle$, $r \in (0, 1]$, the tangle is $\mathcal{T}_{ABC}(|\chi\rangle) = \frac{r^4}{(1+r^4)^2}$, and the concurrences are $\mathcal{C}_{AB}(|\chi\rangle) = \frac{r^2}{1+r^4}$, $\mathcal{C}_{AC}(|\chi\rangle) = \frac{r^2}{1+r^4}$, and $\mathcal{C}_{BC}(|\chi\rangle) = 0$. According to Proposition 5, we know that the probability of error can be calculated by

$$p_e = \begin{cases} \frac{1}{4(1+r^4)}(1 - 4r^4)\xi_e, & r^2 \leq \frac{1}{2}, \\ \frac{r^4}{1+r^4}(1 - \frac{1}{4r^4})\xi_e, & r^2 > \frac{1}{2}, \end{cases}$$

where $0 < \xi_e < 1$. From Fig. 4 in the region $r \leq (\frac{1}{2})^{\frac{1}{2}}$, we can see that as the bipartite entanglement (black line) shared between Alice and Bob decreases, the probability of error (green line) to locally distinguish the orthogonal set of states (5) increases. However, for the region $r > (\frac{1}{2})^{\frac{1}{2}}$, we observe that the probability of error (green line) increases with an increase in bipartite entanglement (black line). A similar pattern appears between the amount of multipartite entanglement (red line) and the probability of error (green line). Moreover, with $r = (\frac{1}{2})^{\frac{1}{2}}$, the probability of error is zero, and the quantity of entanglement distribution is as follows: $\mathcal{T}_{ABC}(|\chi\rangle) = \frac{4}{25}$, $\mathcal{C}_{AB}(|\chi\rangle) = \mathcal{C}_{AC}(|\chi\rangle) = \frac{2}{5}$, and $\mathcal{C}_{BC}(|\chi\rangle) = 0$.

In Ref. [58], Halder *et al.* showed that the set of states (5) is initially locally indistinguishable, and to distinguish it perfectly, one copy of the Bell state shared between any two parties is sufficient as a resource. By Proposition 2, it is clear that one copy of the three-qubit GHZ state is also sufficient to distinguish perfectly the set of states (5). Therefore, for such a discrimination task, we are unable to differentiate between the advantages of multiparty correlation and bipartite correlation when both are at their maximum level. But dilution of both resources makes the gap in advantages visible. In the bipartite case, nonmaximally entangled states cannot perfectly distinguish the set of states in Eq. (5). In contrast, multiparty scenarios allow for nonmaximally entangled states that enable perfect discrimination of the same set of states (5). Next, we will state a theorem that characterizes state transformations between nongeneric GHZ class states.

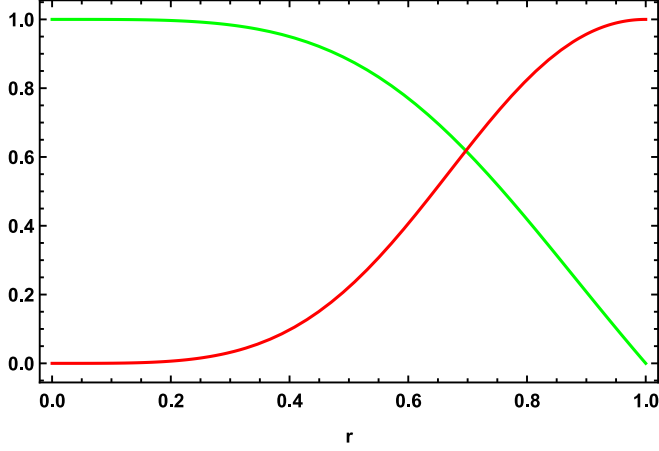


FIG. 2. τ_{ABC} (redline) and p_e (green line) with respect to r for the tripartite state $|\chi\rangle = |\psi(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, r)\rangle$.

Theorem 1. State transformation via LOCC from a state $|\psi(\alpha_1, \alpha_2, \alpha_3, z)\rangle$ to $|\psi'(\theta_1, \theta_2, \theta_3, z')\rangle$, where $\alpha_i \in (0, \frac{\pi}{2}]$ is arbitrary, at least one $\theta_i \in (0, \frac{\pi}{2}]$ is equal to $\frac{\pi}{2}$, and $|z|, |z'| \in (0, 1]$, is possible iff (1) $\alpha_i \geq \theta_i$ and (2) $|z| \geq |z'|$ [71].

According to Theorem 1, we can reach $|\psi(\frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{4}, \sqrt{0.5})\rangle$ from the type-II GHZ states $|\psi(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{4}, r)\rangle$ with $r \in (\sqrt{0.5}, 1] \setminus \{\sqrt[4]{0.5}\}$ via LOCC. As a result, also $|\psi(\frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{4}, \sqrt{0.5})\rangle$ have less amount of entanglement than the states $|\psi(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{4}, r)\rangle$ with $r \in (\sqrt{0.5}, 1] \setminus \{\sqrt[4]{0.5}\}$, which is also evident from Figs. 3 and 4. Now from Fig. 4 and Proposition 5 we see that for $r' = \sqrt{0.5}$ we can distinguish the set of states (5) without any error via LOCC with the help of the type-III GHZ state $|\psi(\frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{4}, r')\rangle$. But perfect distinguishability of set of states (5) is not possible with the type-II GHZ states $|\psi(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{4}, r)\rangle$ when $r \in (\sqrt{0.5}, 1] \setminus \{\sqrt[4]{0.5}\}$ (see Fig. 3). Similarly the type-II GHZ state $|\psi(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{4}, \sqrt[4]{0.5})\rangle$ can be reached from type-I GHZ states $|\psi(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, r)\rangle$ with $r \in (\sqrt[4]{0.5}, 1)$ via LOCC, but these type-I GHZ states are unable to facilitate perfect state discrimination (see Fig. 2) of set of states (5). This is a counterintuitive result because some states with less

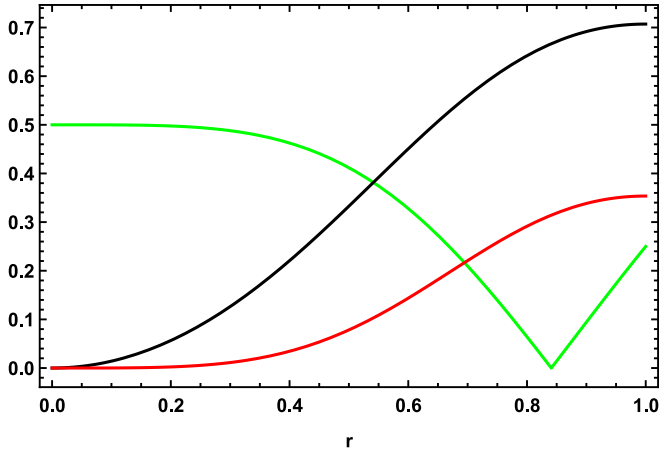


FIG. 3. τ_{ABC} (redline), C_{AB} (black line), and p_e (green line) with respect to r for the tripartite state $|\chi\rangle = |\psi(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{4}, r)\rangle$.

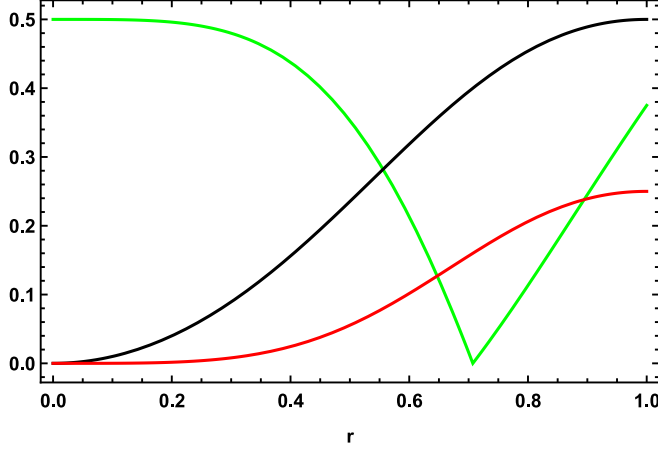


FIG. 4. τ_{ABC} (redline), $\mathcal{C}_{AB}$ (black line), and p_e (green line) with respect to r for the tripartite state $|\chi\rangle = |\psi(\frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{4}, r)\rangle$.

amount of entanglement can distinguish set of states (5) perfectly, while states with much higher entanglement cannot. In the bipartite case, we have already shown that perfect state discrimination is possible only with maximal entangled states and any lower entangled state results in a error. But this is actually possible for three-party scenario.

V. THE IMPACT OF THE NONGENERIC CLASS ON PROBABILISTIC DISTINGUISHABILITY IN QUANTUM SYSTEMS

In the study of tripartite entangled states, it is crucial to examine the correlations and entanglement structures within the subsystems. A particular class of nongeneric GHZ-like states, such as the resource state $|\psi(\frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{4}, r)\rangle$, represents a genuinely tripartite entangled state, where bipartite entanglement is only present between reduced subsystems A-B, with no entanglement detected between either reduced subsystem pair B-C or A-C. This configuration places the state in type II, where entanglement is restricted to a single pair of subsystems. On the other hand, the resource state $|\psi(\frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{4}, r)\rangle$ serves as a prime example of a tripartite entangled state exhibiting a less correlated structure. In this case, the entanglement distribution is such that there is bipartite entanglement between reduced subsystems A-B and A-C, while no bipartite entanglement exists between reduced subsystems B-C. This configuration aligns with what is referred to as type III in entanglement classification, where specific pairs of subsystems exhibit entanglement while others do not. These examples highlight the rich diversity in the structure of tripartite entanglement and the importance of understanding the specific ways in which entanglement manifests in such systems, particularly in terms of reduced subsystems and their interdependency.

For the case of $|\chi\rangle = |\psi(\frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{4}, r)\rangle$, $r \in (0, 1]$, the quantity of concurrence between A-B is given by $\mathcal{C}_{AB}(|\chi\rangle) = \frac{\sqrt{2}r^2}{1+r^4}$, and the probability of error can be calculated by

$$p_e = \begin{cases} \frac{1}{2(1+r^4)}(1 - 2r^4)\xi_e, & r^4 \leq \frac{1}{2}, \\ \frac{r^4}{1+r^4}(1 - \frac{1}{2r^4})\xi_e, & r^4 > \frac{1}{2}, \end{cases}$$

where $0 < \xi_e < 1$.

Now, representing the probability of error p_e in terms of tangle (τ) and concurrence of the reduced system A-B (C_{AB}), we get the following equations, respectively:

$$p_e = \begin{cases} \frac{\tau}{2(1-\sqrt{1-2\tau})} \left(1 - \frac{2-2\tau-2\sqrt{1-2\tau}}{\tau}\right) \xi_e, & \tau \leq \frac{4}{9}, \\ \frac{1-\tau-\sqrt{1-2\tau}}{1-\sqrt{1-2\tau}} \left(1 - \frac{\tau}{2-2\tau-2\sqrt{1-2\tau}}\right) \xi_e, & \tau > \frac{4}{9}, \end{cases} \quad (9)$$

$$p_e = \begin{cases} \frac{1 - \frac{(1-\sqrt{1-2C_{AB}})^2}{C_{AB}^2}}{2\left\{1 + \left(1 + \frac{1-\sqrt{1-2C_{AB}}}{\sqrt{2}C_{AB}}\right)^2\right\}} \xi_e, & C_{AB} \leq \frac{2}{3}, \\ \frac{1 - \frac{C_{AB}^2}{(1-\sqrt{1-2C_{AB}})^2}}{1 + \frac{2C_{AB}^2}{(1-\sqrt{1-2C_{AB}})^2}} \xi_e, & C_{AB} > \frac{2}{3}, \end{cases} \quad (10)$$

where $0 < \xi_e < 1$.

For the case of $|\chi\rangle = |\psi(\frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{4}, r)\rangle$, $r \in (0, 1]$, the quantity of concurrence between A-B is given by $C_{AB}(|\chi\rangle) = \frac{r^2}{1+r^4}$, and the probability of error can be calculated by

$$p_e = \begin{cases} \frac{1}{4(1+r^4)} (1 - 4r^4) \xi_e, & r^2 \leq \frac{1}{2}, \\ \frac{r^4}{1+r^4} \left(1 - \frac{1}{4r^4}\right) \xi_e, & r^2 > \frac{1}{2}, \end{cases}$$

where $0 < \xi_e < 1$.

Again, representing the probability of error p_e in terms of tangle (τ) and concurrence of the reduced system A-B (C_{AB}), we get the following equations, respectively:

$$p_e = \begin{cases} \frac{\tau}{2(1-2\sqrt{1-4\tau})} \left(1 - \frac{2-4\tau-4\sqrt{1-4\tau}}{\tau}\right) \xi_e, & \tau \leq \frac{4}{25}, \\ \frac{1-2\tau-2\sqrt{1-4\tau}}{1-2\sqrt{1-4\tau}} \left(1 - \frac{\tau}{2-4\tau-4\sqrt{1-4\tau}}\right) \xi_e, & \tau > \frac{4}{25}, \end{cases} \quad (11)$$

$$p_e = \begin{cases} \frac{1 - \frac{(1-\sqrt{1-4C_{AB}})^2}{C_{AB}^2}}{4\left\{1 + \left(1 + \frac{1-\sqrt{1-4C_{AB}}}{2C_{AB}}\right)^2\right\}} \xi_e, & C_{AB} \leq \frac{2}{5}, \\ \frac{1 - \frac{C_{AB}^2}{(1-\sqrt{1-4C_{AB}})^2}}{1 + \frac{4C_{AB}^2}{(1-\sqrt{1-4C_{AB}})^2}} \xi_e, & C_{AB} > \frac{2}{5}, \end{cases} \quad (12)$$

where $0 < \xi_e < 1$.

From Figs. 5 and 6, we observe that the state $|\psi(\frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{4}, r)\rangle$ offers a greater advantage for the distinguishing task outlined in Eq. (5) compared to the state $|\psi(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{4}, r)\rangle$. Specifically, this advantage becomes particularly evident when the value of tangle τ is less than or equal to approximately 0.245 (Fig. 5) and the C_{AB} are less than or equal to approximately 0.495 (Fig. 6). This suggests that the type-III states exhibit superior performance in terms of probabilistic distinguishability under these conditions, making it a more effective resource for such tasks in the specified parameter ranges. The visual representation in the figures helps to clearly delineate these regions where one state outperforms the other in terms of its ability to distinguish between different quantum states.

Also, from Figs. 5 and 6, it can be seen that for the type-II state, the probability of error is zero if the values of tangle and concurrence equal $\frac{4}{9}$ and $\frac{2}{3}$, respectively. However, for the case of type III, the probability of error becomes zero when the values of tangle and concurrence equal $\frac{4}{25}$ and $\frac{2}{5}$, respectively. Now, it is well known that the type-III state is less entangled than the type II. This is because one can always convert a type-II state into a type-III state using LOCC, whereas the reverse conversion is impossible [72]. So, it is evident that a less entangled state can perfectly distinguish the set of states (5) like a more entangled state. It is also interesting that for the case of type-I state, the probability of error is zero only when it is a GHZ state, which is the maximally entangled state

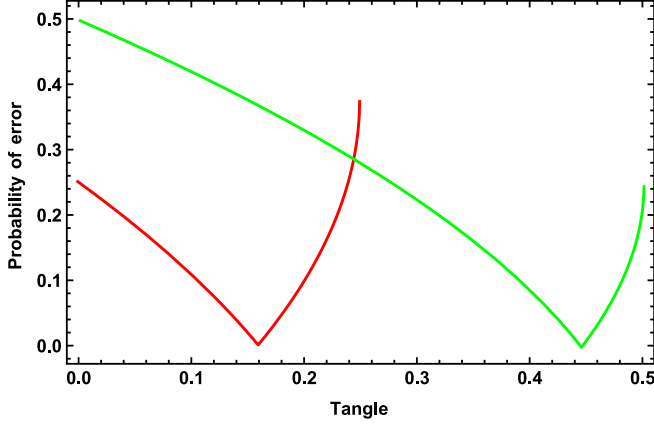


FIG. 5. Graph of tangle τ vs probability of error p_e for $|\psi(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{4}, r)\rangle$ (green line) and $|\psi(\frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{4}, r)\rangle$ (red line), $r \in (0, 1]$.

in the GHZ-SLOCC class. Type-III state has two nonzero bipartite entangled reduced subsystems, whereas type-II and GHZ states have one and zero reduced subsystems with bipartite entanglement.

VI. DISCUSSION

In this paper, we address a notable gap in the current understanding of quantum state discrimination by investigating the potential of three-qubit GHZ-SLOCC class states in the discrimination task, with a focus on their structural characteristics. In Ref. [58], the authors showed that to distinguish a set of states (5) perfectly, one copy of the Bell state is needed. In this work, we demonstrated that achieving perfect distinguishability of the set of states (5) is not possible with a nonmaximal bipartite entangled state. Additionally, we identified three distinct genuine entangled states within the GHZ-SLOCC class that can serve as resources for the perfect discrimination of states in Eq. (5), one of which is the GHZ state, the maximally entangled state of the GHZ-SLOCC class, but the other two $|\psi(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{4}, \sqrt[4]{0.5})\rangle$ and $|\psi(\frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{4}, \sqrt[4]{0.5})\rangle$ are nonmaximally entangled state. Additionally, from Theorem 1 it is clear there exists states from which we can reach $|\psi(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{4}, \sqrt[4]{0.5})\rangle$ and

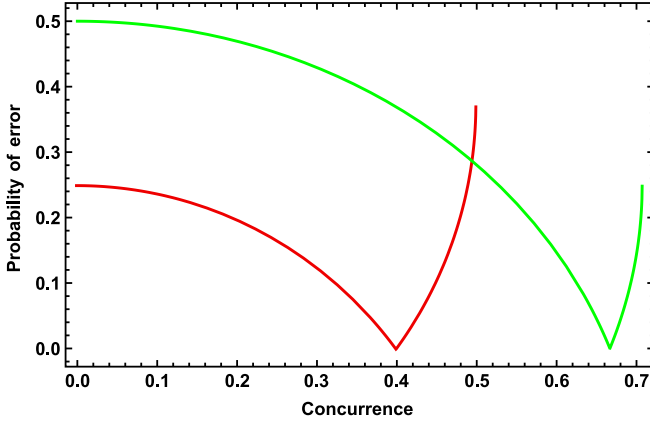


FIG. 6. Graph of concurrence C_{AB} vs probability of error p_e for $|\psi(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{4}, r)\rangle$ (green line) and $|\psi(\frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{4}, r)\rangle$ (red line), $r \in (0, 1]$.

$|\psi(\frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{4}, \sqrt{0.5})\rangle$ via LOCC. But from Figs. 2–4, it is clear that those states can not be helpful in perfect discrimination. Therefore, we can conclude that a nonmaximally genuine multiparty entangled state can become more powerful than a more resourceful multipartite entangled state for this discrimination task only due to its unique entanglement distribution.

Here, we also present a detailed quantitative analysis of the relationship between the probability of error (p_e) in discrimination and the amount of multipartite entanglement (τ), along with the bipartite entanglement (C_{AB}) present in the reduced subsystem AB . This provides further insight into how different forms of entanglement contribute to quantum discrimination tasks. The unique entanglement distribution of nongeneric GHZ states enables us to distinguish between states in probabilistic settings with fewer resources. From Figs. 5 and 6, we can say that to discriminate (probabilistic or perfect) set of states (5), type-III GHZ states are sufficient. The inherent qualities of these states make them particularly effective in overcoming the limitations typically encountered in distinguishing such states, offering a valuable resource for improving quantum discrimination techniques.

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

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