

Dynamics of reheating in loop quantum cosmology

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In loop quantum cosmology (LQC), the initial singularity is replaced by a quantum bounce, leading to a universal postbounce evolution characterized by three distinct epochs: bouncing, transition, and slow-roll inflation, before the hot big-bang universe starts. While the generic nature of inflation in the presence of an inflaton field in LQC is well established, the subsequent reheating phase—the process that thermalizes the universe and marks the beginning of the hot big bang—has remained unexplored in this quantum gravitational framework. This paper presents the first comprehensive integration of the (generalized) reheating mechanism into the LQC paradigm. Using the power law plateau potential and comparing predictions with the latest Planck 2018 and ACT 2025 data, we demonstrate that the inclusion of a reheating phase with a generic equation of state is fully consistent with the cosmological constraints. In addition, using the observational data for the amplitude and spectral index of the scalar perturbations and the tensor-to-scalar ratio, we also constrain the total number of e-folds from the bounce to the present day and find a lower bound, which is less constrained than that obtained previously from the fitting of the high- l cosmic microwave background temperature power spectrum (TT), the polarization data (TT, TE, EE), and the low- l polarization data (lowP). We also show that at the horizon crossing, there exist six independent algebraic equations, which uniquely determine six physical parameters, once the amplitude and the power spectral index of the pivot mode of the scalar perturbations are provided. Three of the six parameters form a complete set of the initial conditions at the horizon crossing and uniquely determine the entire evolution of the universe, including both the pre- and postbounce phases, through the modified Friedmann and Klein-Gordon equations of LQC. This is sharply in contrast to the case without considering the reheating phase.

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I. INTRODUCTION

The inflationary scenario is a unique and convincing paradigm that serves as one of the cornerstones of contemporary cosmology. A brief phase of rapid expansion that occurred at the initial instant of the universe, just before the radiation-dominated epoch is referred to as cosmological inflation. At first, inflation was used to solve the shortcomings of the big-bang models, including the horizon and flatness problems [1–6]. A successful inflationary model should predict a quasiadiabatic and almost Gaussian primordial spectrum of perturbations, with spectral properties consistent with the cosmic microwave background (CMB) observations [7–13]. However, despite the fact that inflation is an elegant paradigm and solves various problems of the big-bang universe, it fails to address other fundamental issues of the early universe, such as the big-bang singularity and dynamics before the preinflationary phase. In particular, the success of the inflationary scenario depends heavily on our ability to comprehend the ultraviolet physics. For inflationary models that require the number of e-folds to be more than 70, it becomes doubtful whether the underlying quantum field theory on a classical spacetime is reliable or not [14]. This is because these theories treat spacetime as classical when the size of the current universe is smaller than the Planck size at the beginning of inflation and this gives rise to the trans-Planckian problem [15,16]. Furthermore, the initial singularity is still inevitable [17,18].

On the other hand, loop quantum cosmology (LQC) presents a compelling explanation for preinflationary physics that can resolve the issues mentioned above [19–23]. In LQC, the singularity at big bang is replaced by a quantum bounce where the universe evolves from a contracting phase to an expanding phase smoothly without hitting any singularity. LQC now becomes a well-established branch of loop quantum gravity, a quantum gravity theory created to solve the issues of combining quantum mechanics with general relativity (GR), and has the potential to act as a quantum gravity extension for a range of cosmological models, including inflation, ekpyrotic, and matter bounces [21–69]. In the conventional inflationary paradigm, the evolutionary trajectory usually initiates well after the Planck era, marked by curvature and energy density of matter fields in the universe approximately several orders of magnitude lower than the Planck scale, thereby making quantum gravity effects negligible. It is unclear how these preinflationary dynamics will be once quantum gravity effects become important. The lack of knowledge regarding earlier cosmic stages is embedded in the selection of initial conditions at the onset of inflation, encompassing both the background homogeneous geometry and cosmological perturbations. The latter is commonly assumed to adhere to the Bunch-Davies (BD) vacuum state at the onset of inflation. This assumption is crucial, albeit robust. There is considerable interest in extending this scenario backward in time to encompass the Planck era and demonstrating that these initial conditions (or something akin to them) can arise from the preinflationary dynamics when quantum gravity effects become significant. Additionally, in standard inflationary models, the background spacetime exhibits classical singularity, an issue addressed in LQC where quantum gravity effects substitute the big-bang singularity with a non-singular bounce. Similarly, both the ekpyrotic and matter bounce scenarios necessitate a cosmic bounce (see, e.g., Refs. [70–75], and references therein), a requirement for which LQC offers a natural mechanism. However, there is a fundamental difference between LQC and ekpyrotic/matter bounce scenarios: In LQC, inflation is generic in the presence of an inflaton field [76,77], while in ekpyrotic/matter bounce models, an inflationary phase is usually absent [70–75]. As a result, LQC takes the advantage of both classical inflationary models and ekpyrotic/matter bounce-like ones.

LQC serves as a quantum gravity culmination for these cosmological scenarios, emphasizing the additional aspects introduced into observable quantities through this extension. The novel effects arising from quantum gravity act as a gateway to the cosmic Planck era, offering an opportunity to test the concepts outlined here by comparing predictions with observations of CMB. Thus, LQC provides a complete description of the spacetime geometry of the Planck era for the Friedman-Lemaître-Robertson-Walker (FLRW) spacetime. An important question to be addressed is whether the quantum bounce and associated preinflationary physics can leave some interesting imprints on

the current or future cosmological experiments [78,79]. There are primarily several distinct methods for studying preinflationary dynamics and cosmological perturbations [22]. All of these approaches lead to the same set of dynamical equations for the evolution of the background. Therefore, the results we present in this article will be valid in all these approaches. To address the issues of preinflationary physics, various inflationary models have been considered in the framework of LQC [19,23], and it was shown explicitly that the slow-roll inflation is generic in the presence of an inflaton field [76,77,80].

In this work, we shall conduct a thorough analysis of the impact of the reheating phase on LQC. To our best knowledge, this is the first time to study the effects of this phase in the framework of LQC [19,22,23]. The paper is organized as follows. In Sec. II, we provide a concise overview of background evolution in LQC, employing the well-motivated Power Law Plateau (PLP) potential and examining its dynamics through both analytical and numerical methods. Section III is dedicated to a detailed analysis of the PLP model under the slow-roll conditions. In particular, in Sec. III A, using a new method proposed in Appendix 3, we show explicitly how to calculate the six physical quantities

$$(a_*, H_*, \phi_*, \dot{\phi}_*, k_*, V_0), \quad (1)$$

from six independent algebraic equations held at the horizon crossing $k_* = a_* H_*$ for the pivot mode k_* with a given potential $V(\phi, V_0)$ of the inflaton field ϕ and observational data (A_s, n_s) , where the Hubble parameter $H(t)$ is defined as $H = \dot{a}/a$ in terms of the expansion factor $a(t)$, with $\dot{a} \equiv da(t)/dt$, etc. It is remarkable to note that the six quantities given in Eq. (1) *uniquely* determine the entire evolution of the universe, including both the pre- and postbounce phases through the generalized Friedmann and Klein-Gordon equations, given, respectively, by Eqs. (2) and (3). This is sharply in contrast to the results obtained in Refs. [30,76,77], in which it was shown that the pivot mode k_*/a_0 depends on the choice of the initial values of the inflaton ϕ_B at the bounce (see, e.g., Fig. 3 of Ref. [30]). A detailed analysis shows that this is precisely due to the fact that now we take the reheating phase into account, so that the total e-fold N_{re} during the reheating given by Eq. (29), as well as the total e-fold $N_{\text{after}} (\equiv \ln(a_0/a_{\text{re}}))$ between the end of the reheating to the current time given by Eq. (34), is known, while without considering such a phase, both N_{re} and N_{after} are unknown and were taken as free parameters [30,76,77]. Then, a_0 is not uniquely determined, so does k_*/a_0 , even the six parameters of Eq. (1) are known. With such obtained physical quantities, in Sec. III B we fit our model with the current observational data [10–13] and show that the model is consistent with observations. In Sec. IV, we integrate the (generalized) reheating mechanism into the LQC paradigm and demonstrate that the inclusion of a reheating phase with a generic equation of state is fully consistent with the current cosmological constraints. Section V explores the implications of generalized reheating for the total number of e-folds the Universe has undergone from the quantum bounce to the current time when the inflaton field and its potential are exactly specified. We conclude with a summary of our findings and future research directions in Sec. VI.

Additional technical details and derivations are provided in the Appendix, in which we first study the quantum effects of LQC on the power spectra, including the amplitudes, spectral indices, and running of both scalar and tensor perturbations. Then, we consider the slow-roll conditions in Appendix 1 and show explicitly that there exist six independent algebraic equations given, respectively, by Eqs. (A40)–(A44) and (A47) all held at the horizon crossing. Once these six parameters are found from the six independent equations, one can use three of them, that is, $(a_*, \phi_*, \dot{\phi}_*)$, as the initial conditions imposed at the horizon crossing and uniquely determine the entire evolution of the universe through the two field equations [Eqs. (2) and (3)]. These investigations are all carried out in Appendix 2. In Appendix 3, we propose a method to calculate the six physical quantities listed in Eq. (1) at the horizon crossing.

II. BACKGROUND EVOLUTION

The modified Friedmann equation in the LQC can be written as [19,23]

$$H^2 = \frac{8\pi}{3m_{\text{Pl}}^2} \rho \left(1 - \frac{\rho}{\rho_c} \right), \quad (2)$$

where $H \equiv \dot{a}/a$ represents the Hubble parameter and the dot stands for the derivative concerning the cosmic time t . We also use $m_{\text{Pl}} = 1/\sqrt{G}$ for the Planck mass, and ρ and ρ_c denote the energy density and critical energy density, respectively. In our numerical calculations, we always set $G = 1$. The maximal value of the critical energy density is $\rho_c \simeq 0.41m_{\text{Pl}}^4$. The occurrence of a nonsingular quantum bounce, which eliminated the initial singularity in the early stages of the classical universe, is a remarkable prediction of LQC. When $\rho = \rho_c$, the Hubble parameter becomes zero, and energy density takes the maximum value ($\rho(t_B) = \rho_c$), so the quantum bounce becomes unalterable. Extensive studies have been conducted on the background evolution with a bounce phase [47,60,64,81–85]. One of the striking features of the quantum bounce is that the desired slow-roll inflation phase is *generic* [76,77] (also see Refs. [19,35–40,86–90]).

In the FLRW background, for a potential $V(\phi)$ and single scalar field ϕ the equation of motion in LQC takes the usual form of Klein-Gordon equation as in GR, given by

$$\ddot{\phi} + 3H\dot{\phi} + V_{,\phi} = 0, \quad (3)$$

where $V_{,\phi} = dV(\phi)/d\phi$.

Here, we will study the “preinflationary” and “slow-roll inflationary” scenarios for the PLP potential, which has the following form [91]:

$$V(\phi) = V_0 m_{\text{Pl}}^4 \left(\frac{\phi^2}{\phi^2 + m^2} \right), \quad (4)$$

where ϕ is the inflaton field and V_0 is the scale of the inflation that can be fixed by matching the amplitude of the scalar power spectrum at the pivot scale of the CMB observations. It is clear that if m is super-Planckian, then this model becomes indistinguishable from the large field ϕ'' models. Thus, in our analysis, we consider $m = m_{\text{Pl}}$, whereas V_0 is fixed from the recent observation of CMB (elaborately discussed in Sec. III).

First, we study the background dynamics for inflationary potential given by Eq. (4). By imposing the initial conditions on $a(t)$, $\phi(t)$, and $\dot{\phi}(t)$ at a specific point, one can solve Eqs. (2) and (3) numerically. We set the initial conditions at the bounce $t = t_B$, using

$$\frac{1}{2}\dot{\phi}^2(t_B) + V(\phi(t_B)) = \rho_c \quad \text{and} \quad \dot{a}(t_B) = 0. \quad (5)$$

From now on, the subscript “B” represents the variable values at the point of the bounce. Considering ρ_c to be a constant and following Eq. (5), one can write $\dot{\phi}_B$ in terms of ρ_c and ϕ_B for a given potential. Thus, one needs to choose the specific initial conditions for $a(t_B)$, $\phi(t_B)$, and the sign of $\dot{\phi}_B$ only. For the sake of simplicity, we choose $a(t)$ by setting the scale factor at bounce,

$$a(t_B) = 1. \quad (6)$$

Then, the set of initial conditions reduces to only the value of $\phi(t_B)$ and the sign of $\dot{\phi}_B$. In this analysis, both $\dot{\phi}_B > 0$ and $\dot{\phi}_B < 0$ cases are considered.

To study the preinflationary and inflationary dynamics for the given potential, we first introduce the following background quantities:

- (1) The equation of state ($\omega(\phi)$), defined as

$$\omega(\phi) \equiv \frac{\dot{\phi}^2/2 - V(\phi)}{\dot{\phi}^2/2 + V(\phi)}. \quad (7)$$

During the slow-roll inflation, we have $\omega(\phi) \simeq -1$.

(2) The slow-roll parameters, defined as

$$\epsilon_H \equiv -\frac{\dot{H}}{H^2}, \quad \eta_H \equiv \frac{\ddot{H}}{2\dot{H}H}. \quad (8)$$

Then, we have

$$\frac{\ddot{a}}{a} = H^2(1 - \epsilon_H) = \begin{cases} >0, & \epsilon_H < 1, \\ =0, & \epsilon_H = 1, \\ <0, & \epsilon_H > 1. \end{cases} \quad (9)$$

During the slow-roll inflation, both ϵ_H and η_H are required to be very small. In standard inflationary scenarios, in order to satisfy the current observation bound on tensor-to-scalar ratio, it is required that $\epsilon_H \lesssim 10^{-3}$.

(3) The duration of the slow-roll inflation is given in terms of the number of e-folds (N_{inf}), which represents the amount of the expansion the universe goes from the onset of inflation to the end of it. It is defined as

$$N_{\text{inf}} \equiv \ln \left(\frac{a_{\text{end}}}{a_i} \right). \quad (10)$$

When $\ddot{a}(t_i) = 0$ (or $\epsilon_H(t_i) = 1$), that is, when $\ddot{a}(t)$ first changes its sign right after the bouncing phase, it defines the onset of the inflation, whereas the inflationary phase ends at $t = t_{\text{end}}$; here, t_{end} denotes the first time when

$$\epsilon_H(t_{\text{end}}) = 1, \quad (11)$$

again after t_i , so that $\ddot{a} > 0$ for $t \in (t_i, t_{\text{end}})$.

A. Analytical evolution of the background

Background evolution in LQC can be divided universally into three different phases: bouncing, transition, and slow-roll inflationary phase [60,81–85]. First, we will present the analytic solutions of the scale factor ($a(t)$) near the bouncing phase. Near the bounce phase, if the kinetic energy is dominant, the potential term can be ignored, and then Eq. (2) can be rewritten as

$$H^2 = \frac{8\pi}{3m_{\text{Pl}}^2} \frac{1}{2} \dot{\phi}^2 \left(1 - \frac{\dot{\phi}^2}{2\rho_c} \right), \quad (12)$$

while the field evolution [Eq. (3)] now becomes

$$\ddot{\phi} + 3H\dot{\phi} = 0. \quad (13)$$

Solving the above two equations analytically gives

$$\dot{\phi}(t) = \pm \sqrt{2\rho_c} \left(\frac{a_{\text{B}}}{a(t)} \right)^3. \quad (14)$$

Finally, using Eqs. (12) and (14), the expression of the scale factor ($a(t)$) can be calculated to be

$$a(t) = a_{\text{B}} \left(1 + \gamma_{\text{B}} \frac{t^2}{t_{\text{Pl}}^2} \right)^{1/6}, \quad (15)$$

where $\gamma_{\text{B}} \equiv 24\pi \rho_c / m_{\text{Pl}}^4 \simeq 30.9$ is a dimensionless constant.

B. Numerical evolution of the background

In this section, we will present the numerical evolution of the background quantities for the different initial field values (ϕ_{B}). As already mentioned, we are setting initial conditions at the time of bounce. The difference in the analytical and numerical evolutions can be seen clearly in Fig. 1(a).

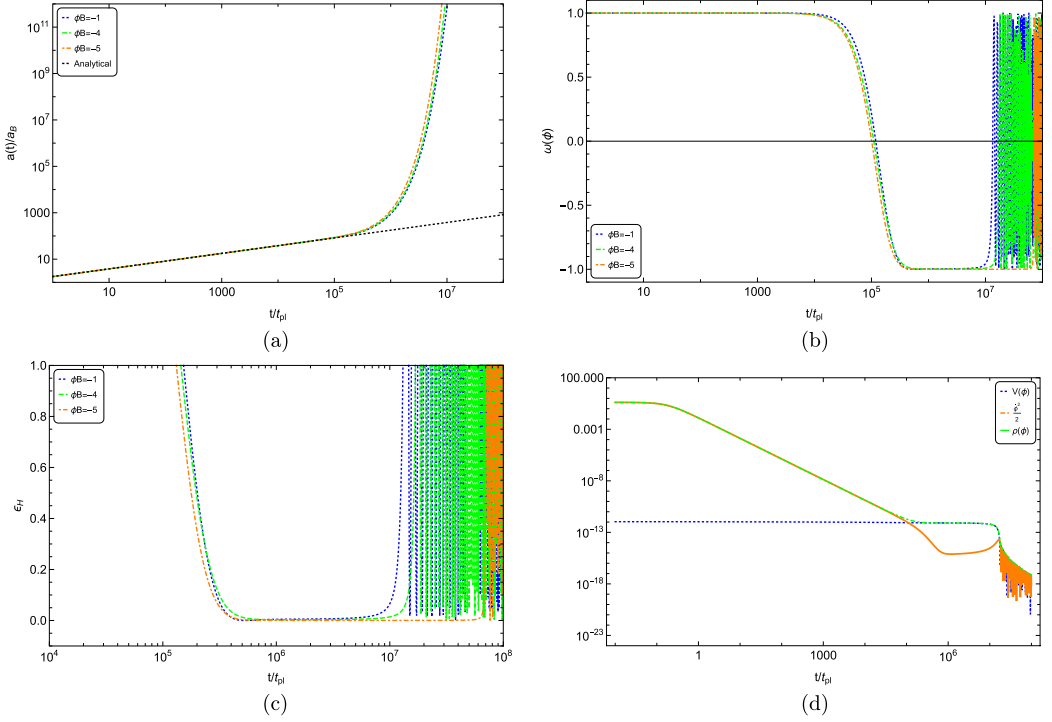


FIG. 1. Numerical evolution of the background quantities for the potential (4), for the kinetic energy dominated at bounce with $\dot{\phi} > 0$. Panel (a) represents the evolution of the scale factor $a(t)$ for the different initial values of the field ϕ at the bounce. The black dotted line is the representative of the analytical solution. Panel (b) shows the evolution of the equation of state ($\omega(\phi)$) parameter for the different initial values of ϕ . In panel (c) we show the solution for the first slow-roll parameter (ϵ_H), and panel (d) shows the comparison between the potential $V(\phi)$, kinetic energy density $\dot{\phi}^2/2$, and the total energy density $\rho = \dot{\phi}^2/2 + V(\phi)$. Here, we take the initial field value to be $\phi_B = -4 m_{\text{pl}}$ keeping $\dot{\phi}_B > 0$.

We carried out the numerical evolution of the background quantities for two different cases, (1) $\phi_B < 0$ and $\dot{\phi}_B > 0$ and (2) $\phi_B > 0$ and $\dot{\phi}_B < 0$. Interestingly, due to the intrinsic symmetry of the PLP potential, both cases end up with similar results.

The numerical evolution shown in Fig. 1 for $\dot{\phi}_B > 0$ and different background quantities allows us to extend our understanding of the three different phases: bouncing, transition, and slow-roll inflation. From Figs. 1(a) and 1(b), one can see the behavior of the scale factor and the equation of state parameter from the time of the bounce to the inflation. The exponential expansion required for the slow-roll inflation is explicitly demonstrated. Around the bounce point, kinetic energy dominates, leading to the equation of state, $\omega = 1$. However, once the kinetic energy dominates at the bounce, it will dominate for a long time [$t \in (0, 10^5 t_{\text{pl}})$] and after that, it starts to decrease dramatically and quickly reaches the point where $\omega = -1$, whereby the slow-roll inflation begins. The end of the inflationary phase is achieved when $\omega \simeq -1/3$ at which we have $\epsilon_H \simeq 1$, as shown in Fig. 1(c). The brief period for ω from 1 to -1 is known as the transition phase. On the other hand, from Fig. 1(d) we can see that the potential energy $V(\phi)$ remains almost constant during the bouncing, transition, and slow-roll inflation phases. The slow-roll inflation does not start until the kinetic energy of the inflaton decreases below the potential energy. After a period of the rapidly exponential expansion, the potential energy drops dramatically, and the inflation ends when it is about equal to the kinetic energy of the inflaton. The e-folds of the expansion of the universe during this epoch depends on the initial conditions ϕ_B and the signs of $\dot{\phi}_B$, as shown in Fig. 2. In this figure,

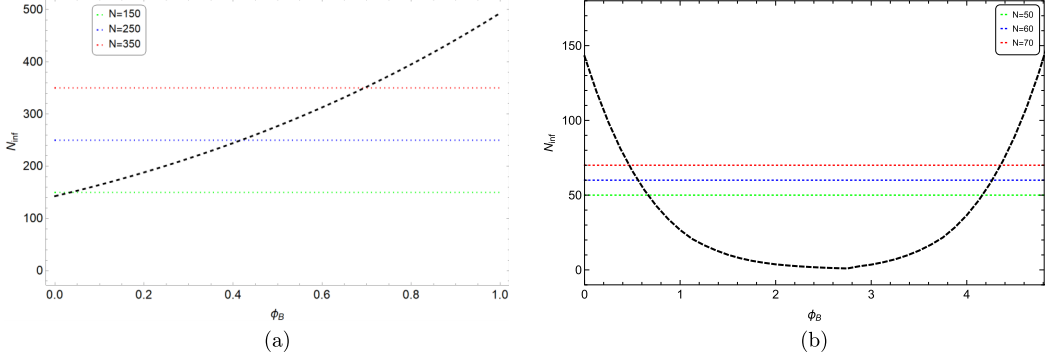


FIG. 2. Evolution of e-folds for the slow-roll inflation against the different choices of ϕ_B . In the left panel (a), we have $\dot{\phi}_B > 0$, whereas in the right panel (b) we have $\dot{\phi}_B < 0$. In the left panel, the three horizontal dotted lines green, blue, and red denote $N_{\text{inf}} = 150, 250$, and 350 , respectively, while in the right panel they denote $N_{\text{inf}} = 50, 60$, and 70 , respectively.

we have focused on two distinct cases with $\phi_B > 0$ and $\dot{\phi}_B > 0$ in the left panel and $\phi_B > 0$ and $\dot{\phi}_B < 0$ in the right panel. Since when $\dot{\phi}_B > 0$, the inflaton can easily climb up the right wing of the potential and result in a larger initial value of the inflaton field at the onset of inflation. This leads to much larger inflationary e-folds in the case of $\dot{\phi}_B > 0$ as compared with the case of $\dot{\phi}_B < 0$. Moreover, in each case, the number of the inflationary e-folds is expected to rapidly increase with an increasing ϕ_B when $\phi_B \geq 4$ due to the same reason as stated above; that is, a larger initial ϕ_B at the bounce can lead to a large value of the inflaton field at the onset of inflation, which according to Eq. (21) yields a larger number of the inflationary e-folds.

In Fig. 3, we show the numerical solution of the background quantities for $\dot{\phi}_B < 0$. One can notice that the results are similar to the case with $\dot{\phi}_B > 0$. This is attributed to the symmetry in the potential around the minima. A common feature that can be noticed from Figs. 1 to 3 is that the potential energy remains almost constant during the bounce, transition, and inflationary phases. On the other hand, once the evolution of the universe at the bounce is dominated by kinetic energy, it will dominate the evolution for a long period. Then, at $t \approx 10^5 t_{\text{Pl}}$, it drops dramatically, whereby the slow-roll inflationary epoch soon starts.

III. THE SLOW-ROLL INFLATIONARY EPOCH

After the bounce, followed by the transition period, the slow-roll inflationary phase begins as $\omega \approx -1$. In this period, the potential energy starts to dominate over the kinetic energy. At this time, sufficiently far away from the bounce, all LQC effects become negligible and the dynamical equations reduce to those given in GR [6]:

$$H^2 \simeq \frac{8\pi}{3m_{\text{Pl}}^2} V(\phi), \quad (16)$$

$$3H\dot{\phi} + \frac{dV(\phi)}{d\phi} \simeq 0, \quad (17)$$

with the assumptions $\frac{1}{2}\dot{\phi}^2 \ll V(\phi)$ and $|\ddot{\phi}| \ll |H\dot{\phi}|$. Using the Friedmann equation for the potential $V(\phi)$, one gets

$$a(t) \propto e^{H_{\text{inf}} t}, \quad (18)$$

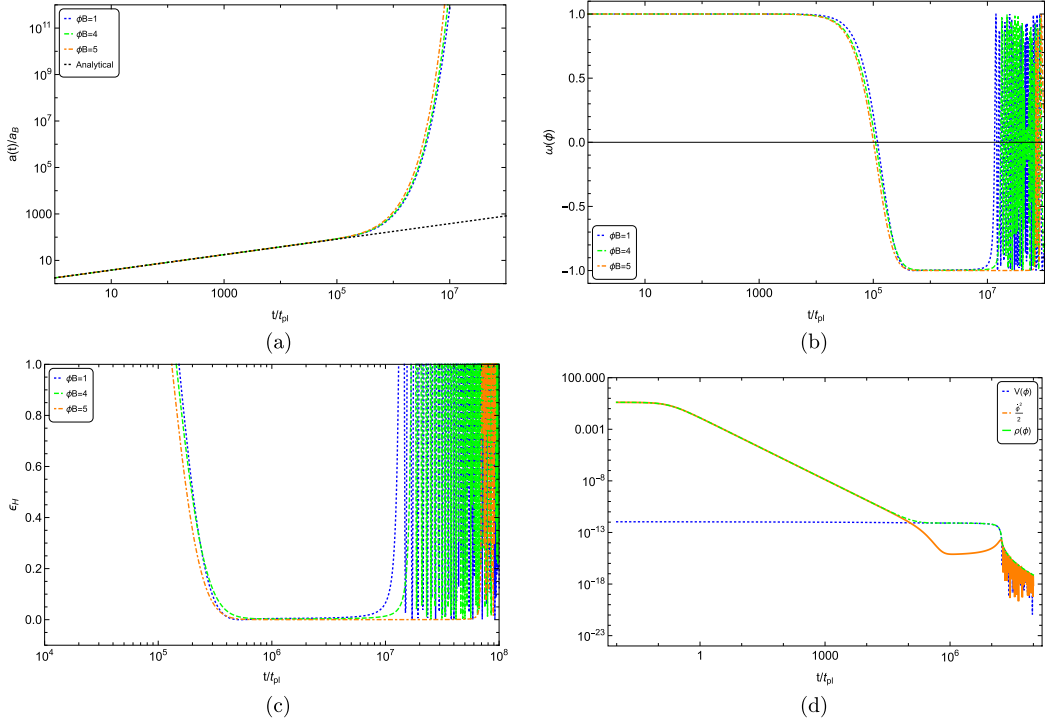


FIG. 3. Numerical evolution of the background quantities for the potential (4), for the kinetic energy dominated at bounce with $\dot{\phi}_B < 0$. Panel (a) represents the evolution of the scale factor $a(t)$ for the different initial values of the field ϕ . The black dotted is for the analytical solution. Panel (b) shows the evolution of the equation of state ($\omega(\phi)$) parameter for the different initial values of the ϕ . In panel (c), we show the solution for the first slow-roll parameter (ϵ_H) and panel (d) shows the comparison between the potential $V(\phi)$, kinetic energy density $\dot{\phi}^2/2$, and the total energy density $\rho = \dot{\phi}^2/2 + V(\phi)$. Here, we take the initial field value to be $\phi_B = 4$ keeping $\dot{\phi} < 0$.

where H_{inf} represents the Hubble parameter during the slow-roll inflation. Introducing the two parameters ϵ_V and η_V via the relations [6,92,93]

$$\epsilon_V = \frac{m_{\text{Pl}}^2}{16\pi} \left(\frac{V_{,\phi}(\phi)}{V(\phi)} \right)^2, \quad \eta_V = \frac{m_{\text{Pl}}^2}{8\pi} \left(\frac{V_{,\phi\phi}(\phi)}{V(\phi)} \right), \quad (19)$$

as shown in the Appendix, the slow-roll inflation conditions ϵ_H , $|\eta_H| \ll 1$, lead to

$$\epsilon_H \simeq \epsilon_V, \quad \eta_H \simeq \eta_V - \epsilon_V. \quad (20)$$

For details, see Appendix. The explicit expressions for the slow-roll parameters and e-folds for our current model are given (setting $m_{\text{Pl}} = 1$, which will be adopted in the rest of this paper) by

$$\epsilon_V = \frac{1}{4\pi(\phi + \phi^3)^2}, \quad \eta_V = \frac{(1 - 3\phi^2)}{4\pi(\phi + \phi^3)^2}, \quad N_{\text{inf}} \simeq 8\pi \int_{\phi_{\text{end}}}^{\phi_i} \frac{1}{2} \phi(1 + \phi^2) d\phi. \quad (21)$$

A. Calculations of physical quantities of the pivot mode k_* at horizon crossing

In LQC, using the scaling invariance of the homogeneous and isotropic background of the universe, one often chooses the expansion factor equal to 1 at the quantum bounce, that is, $a_B \equiv a(t_B) = 1$, as we did in the last section. Once this is chosen, the expansion factor can be

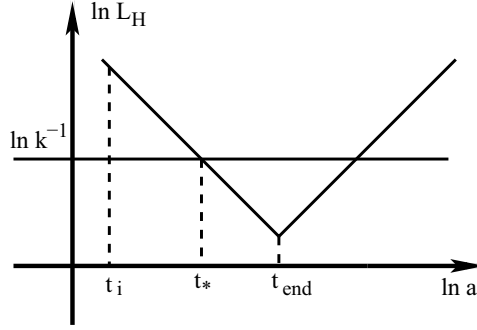


FIG. 4. Horizon crossing (exit) for a given mode k at t_* . The line of $\ln L_H \equiv \ln(1/aH)$ is the Hubble horizon. At the exit, we have $k = a(t_*)H(t_*)$. The pivot mode k_* used by Planck 2018 is $k_*/a_0 = 0.05/\text{Mpc}$ [10].

no longer set to 1 at the current time, so we cannot identify the comoving modes k to the current observational (physical) modes k^{ob} . Instead, now k^{ob} is given by $k^{\text{ob}} = k/a_0$, where a_0 denotes the current expansion factor and k the comoving wave number. In particular, the pivot mode k_*^{ob} is equal to $k_*^{\text{ob}} = k_*/a_0$. In the Appendix, we present a detailed analysis on how to calculate the power spectra after quantum geometric effects are taken into account and obtain the complete set of equations that the physical quantities must satisfy. One of them is the nonlocal condition

$$a_B(a_*, \phi_*, \dot{\phi}_*) = 1, \quad (22)$$

where $(a_*, \phi_*, \dot{\phi}_*)$ are three of the six physical quantities

$$(a_*, \phi_*, \dot{\phi}_*, H_*, k_*, V_0), \quad (23)$$

of the pivot mode k_*^{ob} , which are completely determined at the horizon crossing $k_* = a_*H_*$ through the six algebraic equations (A40)–(A44) and (22), once (n_s, A_s) are read off from observations. It should be noted that once $(a_*, \phi_*, \dot{\phi}_*)$ are known, the whole evolution of the homogeneous and isotropic universe, from the contracting phase to the end of inflation, is uniquely determined from the dynamical equations (2) and (3) for a given potential $V(\phi)$.

It must be also noted that for the observable mode k in the CMB, there always exists a time t_* , at which the mode k crosses the Hubble horizon at the time t_* , where $k = a(t_*)H(t_*)$, as shown in Fig. 4. In the rest of this subsection, we shall focus ourselves only on the pivot mode k_* ($\equiv a_0k_*^{\text{ob}}$) at the horizon crossing, where $k_*^{\text{ob}} = 0.05/\text{Mpc}$ was used by Planck 2018 and ACT 2025 [10,12]. Since $a_0 = a_B e^{N_T}$, where N_T is the total e-fold of the expansion of the Universe from the quantum bounce to the current time, so we have $a_0/a_B = e^{N_T} \gg 1$. As a result, the comoving pivot mode k_* at the quantum bounce reads

$$\frac{k_*}{a_B} = e^{N_T} k_*^{\text{ob}} \gtrsim \mathcal{O}(m_{\text{Pl}}), \quad (24)$$

that is, the pivot mode measured at the quantum bounce is at least of order of the Planck size, which is expected [94]. For the slow-roll inflation, in Appendix 3 we summarize the steps for calculating the six physical quantities listed in Eq. (23).

In addition, the quantum geometric effects in Eqs. (A40)–(A44) and (22) are coded in the function $\mathcal{F}(k)$ defined by [60]

$$\mathcal{F}(k) \equiv 1 + \delta_{\text{Pl}}(k) = 1 + \left[1 + \cos\left(\frac{\pi}{\sqrt{3}}\right) \right] \text{csch}^2\left(\frac{\pi k}{\sqrt{6}k_B}\right), \quad k_B \equiv a_B \sqrt{\frac{8\pi\rho_c}{m_{\text{Pl}}^2}}. \quad (25)$$

As shown in Fig. 5, for the relevant modes we have $|\mathcal{F}(k) - 1| \approx \mathcal{O}(10^{-5})$. Therefore, taking the accuracy of the current observations into account, we can safely set $\mathcal{F}(k) = 1$ as the leading-order

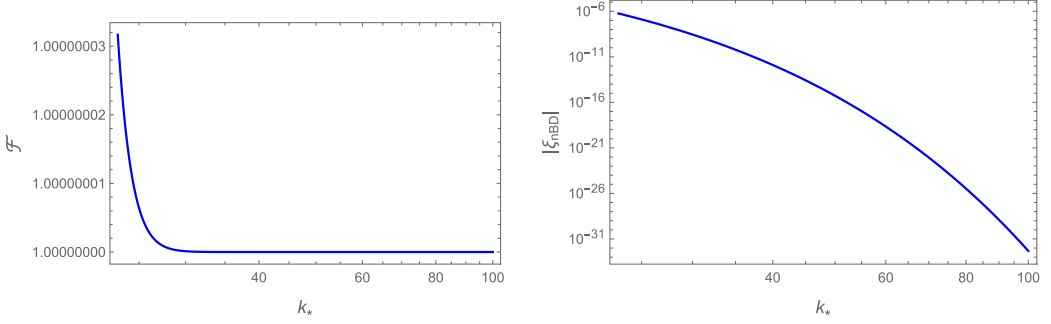


FIG. 5. The functions $\mathcal{F}(k)$ and $\xi_{\text{nBD}}(k)$ defined, respectively, by Eqs. (A46) and (A29).

approximations when calculating the six parameters of Eq. (23) using Eqs. (A40)–(A44) and (22). Without further notice, in the rest of this paper, all the above six quantities are calculated to this order.

B. Fitting the model with observational data

To fit the LQC model with observational data, we first note that, in comparing with that of GR, the state at the onset of inflationary is no longer the BD vacuum, due to the particle creations during the preinflationary phase. As a result, the parameter β_k appearing in the mode function given in Eq. (A11) does not vanish [22]. Then, the power spectra and indexes of the scalar and tensor modes are given by

$$P_{\mathcal{R}}(k) = 2\pi G \left(\frac{H^2}{k^3 \epsilon_H} \right) \mathcal{F}(k), \quad P_h(k) = 16\pi G \left(\frac{H^2}{k^3} \right) \mathcal{F}(k), \quad (26)$$

where $\mathcal{F}(k)$ is given by Eq. (25). Then, introducing the spectral indexes n_s and n_t and their running through Eqs. (A21) and (A22), we find that

$$n_s(k_*) - 1 = \xi_{\text{nBD}}(k_*) - 2(3\epsilon_V^* - \eta_V^*), \quad n_t(k_*) \simeq \xi_{\text{nBD}}(k_*) - 2\epsilon_V^*, \quad (27)$$

where $\xi_{\text{nBD}}(k_*) \equiv d \ln \mathcal{F}(k) / d \ln k|_{k=k_*}$, and

$$\epsilon_V^* = \frac{1}{4\pi(\phi_* + \phi_*^3)^2}, \quad \eta_V^* = \frac{(1 - 3\phi_*^2)}{4\pi(\phi_* + \phi_*^3)^2}. \quad (28)$$

From Planck BICEP/Keck Array 2018 and ACT 2025 observations [10–13], we find that $A_s(k_*^{ph}) \approx 2.09 \times 10^{-9}$ for $k_*/a_0 = 0.05 \text{ Mpc}^{-1}$, which allows us to fix the potential parameter $V_0 \approx 1.63 \times 10^{-12}$. Using this value of V_0 along with the relation between N_{inf} and ϕ_B as depicted in Fig. 2, we can establish a relation between n_s and r_* , where $r_* = 16\epsilon_V^*$ as given by Eq. (A20), which holds even after the quantum geometric effects are taken into account. In Fig. 6, we plot the contours in the (r, n_s) plane with the recent CMB observations of Planck BICEP/Keck Array 2018 and ACT 2025 [10–13]. From the contours, it can be seen that our model is consistent with these observations. In particular, the results with $N_* = 50$ are well inside the 1σ bound, and the ones with $N_* = 60$ are consistent with the 2σ bound, where $N_* \equiv \ln(a_{\text{end}}/a_*)$ [cf. Eq. (31)].

IV. REHEATING ANALYSIS

One of the inevitable epochs to follow the inflationary paradigm is reheating [95–97], where the inflaton field transfers its energy to other degrees of freedom and resurrects the universe from supercooled state to a hot thermal bath of relativistic particles. In the standard inflationary scenario, numerous methods of reheating have been proposed in the literature such as perturbative decay

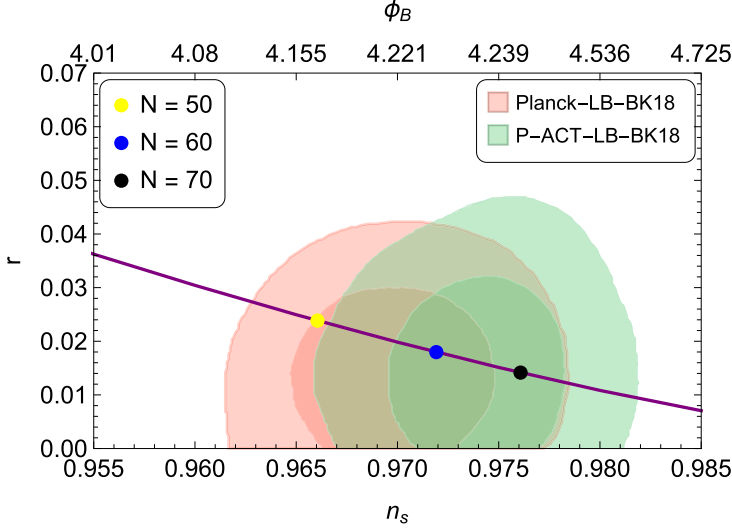


FIG. 6. Figure for $(r - n_s)$ with different numbers of e-fold during the inflation, along with the initial field value ϕ_B at the bounce. The two light and dark pink (green) contours are for Planck BICEP/Keck Array 2018 (ACT 2025 + Planck BICEP/Keck Array 2018) [10–13], respectively, for 1σ and 2σ bounds. Here, the e-fold N is defined as $N \equiv N_* = \ln(a_{\text{end}}/a_*)$, as defined in Eq. (31).

where the inflaton field reaches the bottom of the potential and starts to decay to other elementary particles [98–101]. The produced particles interact with each other and reach an equilibrium at a temperature known as reheating temperature (T_{re}). There are other theoretically interesting models that were introduced later, such as tachyonic instability [102] and parametric resonance [103–107]. The latter is nonperturbative and often referred to as preheating, an initial stage of the reheating [106–110]. Preheating is more efficient as compared to perturbative reheating. In preheating, the decay happens exponentially, generating the high number of particles.

In Ref. [111], using the Uniform Asymptotical Approximation method [43–45, 62, 63, 111–125], the parametric resonance was studied analytically and the analytical solutions of the mode function were found for some given effective potentials. However, the epoch of reheating can be studied in great detail without delving deep into the microphysical dynamics of this phase [126]. As there is no direct observational bound on the reheating temperature, analysis of this era in an indirect approach can be extremely useful. In this approach, one can give a bound on the thermalization temperature through the inflationary observables. Moreover, such approaches can be used as a new way to constrain different inflationary models. Along with the reheating temperature (T_{re}) and the equation of state parameter (ω_{re}), another important physical quantity is the duration of reheating dubbed as the number of e-folds (N_{re}). N_{re} quantifies the expansion of the universe from the end of inflation to the end of the thermalization period. Following Refs. [126–130] reheating temperature (T_{re}) and duration of the reheating (N_{re}) can be written in terms of the effective equation of state (ω_{re}) as

$$N_{\text{re}} = -\frac{4}{(1-3w_{\text{re}})} \left[\frac{1}{4} \ln \left(\frac{3^2 \times 5}{\pi^2 g_{\text{re}}} \right) + \frac{1}{3} \ln \left(\frac{11g_{\text{re}}}{43} \right) + \ln \left(\frac{k_*}{a_0 T_0} \right) + \ln \left(\frac{V_{\text{end}}^{1/4}}{H_*} \right) + N_* \right], \quad (29)$$

$$T_{\text{re}} = \left[\left(\frac{43}{11g_{\text{re}}} \right)^{\frac{1}{3}} \left(\frac{a_0}{k_*} \right) T_0 H_* e^{-N_*} \left(\frac{3^2 \times 5V_{\text{end}}}{\pi^2 g_{\text{re}}} \right)^{-\frac{1}{3(1+w_{\text{re}})}} \right]^{\frac{3(1+w_{\text{re}})}{3w_{\text{re}}-1}}. \quad (30)$$

Here, g_{re} is the relativistic degrees of freedom, which is of order $g_{\text{re}} \simeq \mathcal{O}(100)$ [126], and T_0 denotes the present value of temperature, while k_* signifies Planck's pivot scale, as noted above, and H_* represents the Hubble parameter measured at the horizon crossing of this pivot mode, and V_{end} is the potential at the end of inflation. The number of e-folds N_* is defined as

$$N_* \equiv \ln \left(\frac{a_{\text{end}}}{a_*} \right) = \int_{t_*}^{t_{\text{end}}} H(t) dt = \int_{\phi_*}^{\phi_{\text{end}}} \frac{H}{\dot{\phi}} d\phi \simeq \frac{8\pi}{m_{\text{Pl}}^2} \int_{\phi_{\text{end}}}^{\phi_*} \frac{V}{V_{,\phi}} d\phi, \quad (31)$$

where ϕ_* and ϕ_{end} denote the field values at the horizon crossing and at the end of inflation, respectively. Note that in writing the last expression of the above equation, we had used the slow-roll condition. In addition, from Eqs. (A20) and (A21) we find that the Hubble parameter can be written in terms of inflationary observables as

$$H_* = \left(\frac{\pi r_* A_s(k_*)}{16\mathcal{F}(k_*)} \right)^{1/2} m_{\text{PL}}. \quad (32)$$

Then, maintaining $A_s(k_0) = 2.09 \times 10^{-9}$ (variation of A_s has a negligible effect on T_{re} and N_{re}), the rest of the calculations are carried straightforward. It is evident from Eqs. (29) and (30) that both equations depend on H_* . From Eq. (32), it is clear that H_* is a function of the tensor-to-scalar ratio (r_*) and k_* . The latter was calculated in the last section (Sec. III A). We write r_* in terms of scalar spectral index (n_s) that would help us to constrain T_{re} and N_{re} more accurately. From $\epsilon_H = 1$ at the end of inflation, we can calculate V_{end} ; equipped with all the preliminaries, now we can calculate T_{re} and N_{re} for different equations of state (ω_{re}). The results for the reheating analysis can be found in Fig. 7. We have considered four different values of ($\omega_{\text{re}} = -1/3, 0, 2/3, 1$); however, one can easily extend the calculations for other values of ω_{re} . The solid lines with, respectively, red, green, blue, and black colors signify the different ω_{re} as mentioned in Fig. 7. The dark and light pink (green) shaded strips represent Planck BICEP/Keck Array 2018 (ACT 2025 + Planck BICEP/Keck Array 2018) data, respectively, for 2σ and 1σ bounds. In Fig. 7, we have plotted T_{re} from the scale of instantaneous reheating to the big-bang nucleosynthesis (BBN), which is around 10 MeV. The point where all the curves in N_{re} and T_{re} merge represents the scale of instantaneous reheating ($N_{\text{re}} = 0$). From this figure, it can be seen clearly that the reheating phase can be smoothly integrated into the LQC framework, and the model is well consistent with current observations. It must be noted that in drawing the above figure, we have taken $g_{\text{re}} = 226$, a value obtained from supersymmetric theories, as our inflationary model, PLP, has an origination from such theories. However, we find that our results do not sensitively depend on the specific value of g_{re} , as long as it is on the order of $\mathcal{O}(10^2)$, which is expected even without the supersymmetric considerations.

V. REHEATING AND EXPANSION HISTORY

The total e-fold (N_T) of the expansion of the universe from the bounce till today can be expressed in terms of the scale factor in the form

$$N_T = \ln \left(\frac{a_0}{a_B} \right) = \ln \left(\frac{a_*}{a_B} \times \frac{a_{\text{end}}}{a_*} \times \frac{a_{\text{re}}}{a_{\text{end}}} \times \frac{a_0}{a_{\text{re}}} \right) = \ln \left(\frac{a_*}{a_B} \right) + N_* + N_{\text{re}} + \ln \left(\frac{a_0}{a_{\text{re}}} \right), \quad (33)$$

where N_{re} and N_* are given, respectively, by Eqs. (29) and (31), and

$$\frac{a_0}{a_{\text{re}}} = \left(\frac{11g_{\text{re}}}{43} \right)^{\frac{1}{3}} \frac{T_{\text{re}}}{T_0} = \left(\frac{11g_{\text{re}}}{43} \right)^{\frac{1}{3}} \left[\left(\frac{43}{11g_{\text{re}}} \right)^{\frac{1}{3}} \left(\frac{a_0}{k_*} \right) T_0 H_* e^{-N_*} \left(\frac{3^2 \times 5 V_{\text{end}}}{\pi^2 g_{\text{re}}} \right)^{-\frac{1}{3(1+w_{\text{re}})}} \right]^{\frac{3(1+w_{\text{re}})}{3w_{\text{re}}-1}} \frac{1}{T_0}. \quad (34)$$

They are all functions of H_* , ϕ_* , and ϕ_{end} for any given T_0 , g_{re} , and w_{re} . However, ϕ_{end} is fixed by $\epsilon_V(t_{\text{end}}) = 1$. Then, in the framework of the slow-roll inflation, at the horizon crossing t_* we have six equations [Eqs. (A54)–(A59)], for the six unknown parameters, (a_* , H_* , ϕ_* , $\dot{\phi}_*$, k_* , V_0). Based

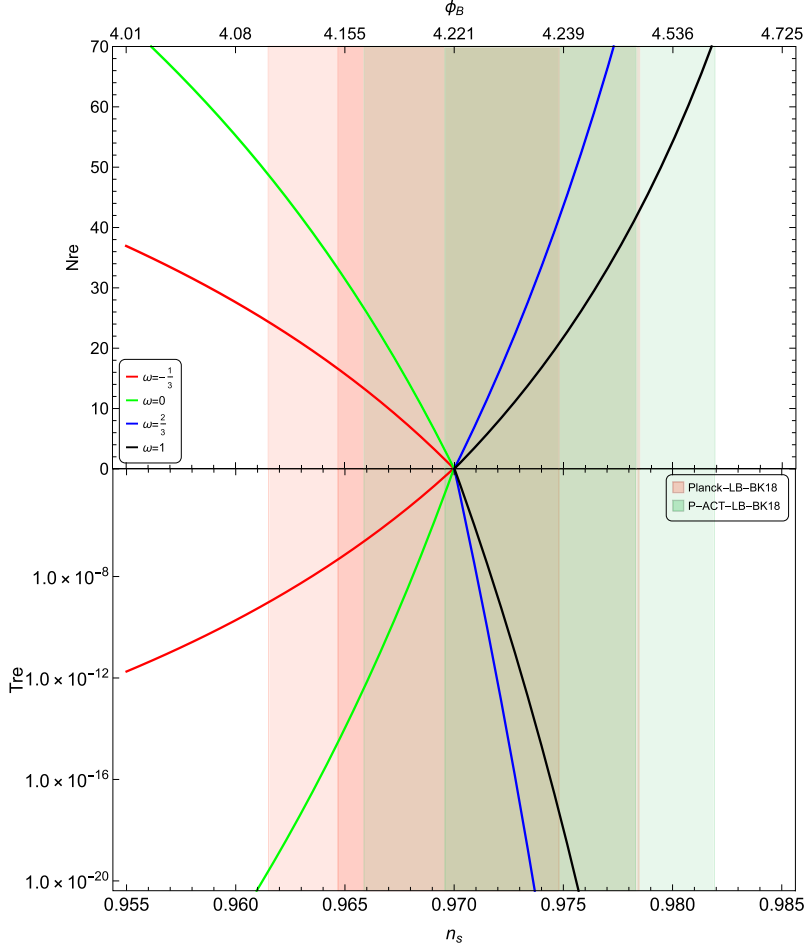


FIG. 7. Plot for the number of e-folds N_{re} and reheating temperature T_{re} against n_s (ϕ_B). The two light and dark pink (green) strips are for Planck BICEP/Keck Array 2018 data (ACT 2025 + Planck BICEP/Keck Array 2018 data) [10–13], respectively, for 1σ and 2σ bounds. We have considered four different cases with the values of the equation of state ($\omega_{\text{re}} = -1/3, 0, 2/3, 1$), which are represented, respectively, by the red, green, blue, and black curves.

on the analysis carried out in Sec. III A and details in Appendix 3, we can proceed to computing the values of the relevant physical quantities at the horizon crossing, given the forms of the inflationary potentials.

Substituting Eqs. (34) and (29) into Eq. (33), one can write the total number N_T of e-folds in terms of a_* , H_* , ϕ_* , ϕ_{end} , ω_{re} , g_{re} , T_0 . It is well known that one needs N_* to be in the range 50–65, in order to solve the problems of the big-bang cosmology. In our analysis, we are not assuming $N_{\text{re}} = 0$ (instantaneous reheating scenario), rather allowing the reheating to be the general one. Furthermore, in our analysis N_{re} and T_{re} both can vary to a wide range from instantaneous to the BBN temperature depending on the values of ω_{re} and n_s . This allows us to compute N_T for different values of ω_{re} and n_s .

In Fig. 8, we plot N_T versus n_s , from which we find that

$$N_T \gtrsim N_T^{\text{min}} \approx 130, \quad (35)$$

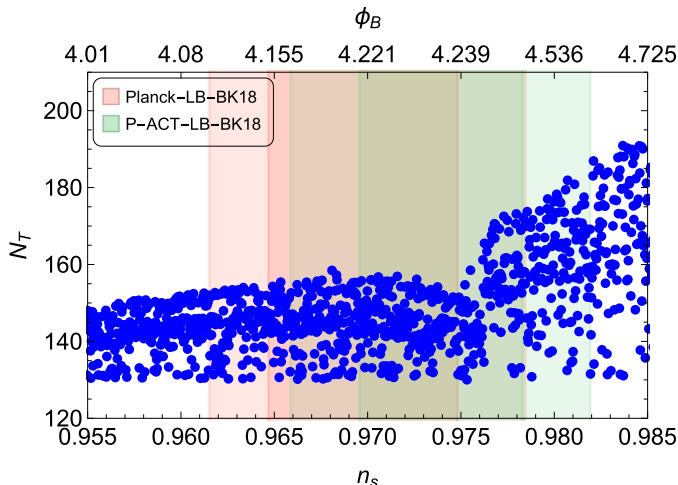


FIG. 8. Plot of N_T for the total number of e-folds against n_s . The two light and dark pink (green) strips are for Planck BICEP/Keck Array 2018 data (ACT 2025 + Planck BICEP/Keck Array 2018 data) [10–13], respectively, for 1σ and 2σ bounds.

which is lower than the bound, $N_T \gtrsim 141$, obtained previously [60]. A detailed analysis shows that these results do not contradict. In particular, the low bound obtained in Ref. [60] came from the fitting of the model with the high- l CMB temperature power spectrum (TT), the polarization data (TT, TE, EE), and the low- l polarization data (lowP) from Planck 2015 [9], while in this paper we mainly consider the constraints from the current observational data of (A_s, n_s, r) [10–13]. Therefore, the above different results just show that the former imposes more stringent constraints than the latter. Finally, it should be noted that the results presented here are specific to the PLP potential (4) with the mass fixed to $m = 1$. It will be interesting to investigate the effects of the shape of the potential as well as the mass of the inflaton on N_T .

VI. CONCLUSION

This work has presented the first analysis of the reheating epoch within the framework of loop quantum cosmology, thereby completing the cosmic history from the nonsingular quantum bounce to the present day. By integrating a generalized reheating phase into the well-established LQC background evolution—comprising the bounce, transition, and slow-roll inflation—we have bridged a critical gap in the theoretical investigation of embedding the reheating phase into the LQC model. Our key findings and implications are as follows:

(1) *A unified picture.* This research bridges a critical gap in the LQC narrative. The theory has long provided a robust, singularity-free description of the universe from the quantum bounce through the slow-roll inflation. Our work now extends this coherent picture to include the subsequent transition to a hot, thermal universe via reheating, thereby connecting the quantum gravity regime directly to the standard hot big-bang cosmology.

(2) *Initial conditions at horizon crossing.* A by-product of the above investigations is that at the horizon crossing there exist six independent algebraic equations, given, respectively, by Eqs. (A40)–(A44) and (A47), which uniquely determine the six parameters

$$(a_*, H_*, \phi_*, \dot{\phi}_*, k_*, V_0), \quad (36)$$

once the amplitude and the power spectral index (A_s, n_s) of the pivot mode of the scalar perturbations are provided. Three of the six parameters, that is, $a_*, \phi_*, \dot{\phi}_*$, form a complete set of the initial conditions at the horizon crossing and uniquely determine the entire evolution of the universe,

including both the pre- and postbounce phases, through the modified Friedmann and Klein-Gordon equations of LQC, given, respectively, by Eqs. (2) and (3).

Note in Refs. [30,76,77], the pivot mode k_* was calculated given the choice of the initial values of the inflaton ϕ_B at the bounce, while in our approach from the reheating phase we fix a_0 once $(a_*, \phi_*, \dot{\phi}_*)$ are given. There is no contradiction here; this simply corresponds to fixing initial conditions in different ways at different times. In particular, now the total e-fold N_{re} during the reheating given by Eq. (29) and the total e-fold $N_{\text{after}} (\equiv \ln(a_0/a_{\text{re}}))$ between the end of the reheating to the current time given by Eq. (34) are all known. As a result, the total e-fold N_T from the bounce to the current time is known, so is $a_0 (= a_B e^{N_T})$, while without considering such a phase, both N_{re} and N_{after} are unknown and were taken as free parameters [30,76,77]. Then, a_0 is not uniquely determined, so does k_*/a_0 , even the six parameters of Eq. (1) are known. Therefore, given the inflaton field and its potential, combining the scalar amplitude A_s and the spectral index n_s with the detailed mechanism of the reheating phase can fix the pivot mode k_*/a_0 .

With the above observations, a method that systematically solves the abovementioned six equations for the six parameters listed in Eq. (36) is proposed [cf. Appendix 3].

(3) *Constraining cosmic history.* By utilizing observational data for the amplitude and spectral index of scalar perturbations and the tensor-to-scalar ratio, we have placed a lower bound on the total number of e-folds from the bounce to the present day when the inflaton field is chosen to be Eq. (4) with the mass set to unity, finding

$$N_T \gtrsim 130. \quad (37)$$

This bound, derived from the (A_s, n_s, r) dataset, is less stringent than that obtained from a direct fit to the full CMB power spectrum [60], highlighting how different observational priors can refine our understanding of the preinflationary epoch.

(4) *Consistency with observations.* We have demonstrated that the inclusion of a reheating phase with a generic equation of state is fully consistent with the latest cosmological constraints from Planck 2018 and ACT 2025 data [10–13]. This establishes the viability of the LQC paradigm through the entire cosmic evolution—from the quantum bounce through inflation and into the hot big bang.

In conclusion, this study not only reinforces the robustness of LQC as a compelling scenario for the origin of our universe but also opens up new avenues for testing quantum gravitational effects through their imprints on the thermal history of the cosmos [22]. Future work will involve extending this analysis to other well-motivated inflationary potentials and loop quantum gravity related models, such as mLQC-I and mLQC-II [131], and exploring the implications of these findings for the production of primordial black holes and stochastic gravitational waves produced during the reheating phase [98–100].

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX: POWER SPECTRA AND SLOW-ROLL PARAMETERS WITH GENERAL INITIAL CONDITIONS

In the framework of LQC, the inflationary phase generically happens in the presence of an inflaton field after the quantum bounce when the energy density falls to the order of $\rho/\rho_c \simeq 10^{-12}$ [19,22,23]. In such a scale, the field equations can be well approximated by the classical ones

$$H^2 \approx \frac{8\pi G}{3} \rho, \quad (\text{A1})$$

$$\frac{\ddot{a}}{a} \approx -\frac{4\pi G}{3}(\rho + 3P), \quad (\text{A2})$$

where ρ and P denote, respectively, the energy density and pressure of the perfect fluid, $\dot{a} \equiv da(t)/dt$, and so on. For a single scalar field ϕ with a potential $V(\phi)$, we have

$$\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi), \quad P = \frac{1}{2}\dot{\phi}^2 - V(\phi). \quad (\text{A3})$$

Taking the derivative of Eq. (A1) with respect to t and then using Eqs. (A2) and (A3), we obtain

$$\ddot{\phi} + 3H\dot{\phi} + V_{,\phi}(\phi) = 0, \quad (\text{A4})$$

which is nothing but the familiar Klein-Gordon equation.

To study inflationary universe, two sets of parameters, the so-called slow-roll parameters, are often introduced: the one (ϵ_H, η_H) defined in terms of the Hubble parameter H and its derivatives, given by Eq. (8), and the one (ϵ_V, η_V) defined in terms of the scalar field potential $V(\phi)$ and its derivatives, given by Eq. (8), which are [132]

$$\epsilon_H \equiv -\frac{\dot{H}}{H^2}, \quad \eta_H \equiv \frac{\ddot{H}}{2\dot{H}H}, \quad \epsilon_V = \frac{m_{\text{Pl}}^2}{16\pi} \left(\frac{V_{,\phi}(\phi)}{V(\phi)} \right)^2, \quad \eta_V = \frac{m_{\text{Pl}}^2}{8\pi} \left(\frac{V_{,\phi\phi}(\phi)}{V(\phi)} \right). \quad (\text{A5})$$

However, the two sets of parameters are connected through the field equations (A1) and (A2). To show their relations, in the following we first consider the general case and then restrict ourselves to the slow-roll case.

To the above purposes, let us first note that

$$\frac{\ddot{a}}{a} = H^2(1 - \epsilon_H) = -4\pi G\dot{\phi}^2 + H^2, \quad (\text{A6})$$

so that we have

$$\epsilon_H = \frac{4\pi G\dot{\phi}^2}{H^2} \quad \text{or} \quad \frac{H^2}{\dot{\phi}^2} = \frac{4\pi G}{\epsilon_H}. \quad (\text{A7})$$

In addition, we also have

$$\dot{H} = \frac{\ddot{a}}{a} - H^2 = -4\pi G\dot{\phi}^2, \quad \ddot{H} = -8\pi G\dot{\phi}\ddot{\phi}, \quad (\text{A8})$$

so that

$$\eta_H = \frac{\ddot{H}}{2H\dot{H}} = \frac{\ddot{\phi}}{H\dot{\phi}} = \frac{1}{H} \frac{d}{dt}(\ln |\dot{\phi}|). \quad (\text{A9})$$

On the other hand, when the background is de Sitter spacetime, both of the scalar and tensor perturbations satisfy the Mukhanov-Sasaki (MS) equation [6]

$$v_k'' + \left(k^2 - \frac{2}{\eta^2} \right) v_k = 0, \quad (\text{A10})$$

which has the general solution

$$v_k = \frac{1}{\sqrt{2k}} \left\{ \alpha_k e^{-ik\eta} \left(1 - \frac{i}{k\eta} \right) + \beta_k e^{ik\eta} \left(1 + \frac{i}{k\eta} \right) \right\}, \quad (\text{A11})$$

where α_k and β_k are two integration constants subjected to the Wronskian condition

$$|\alpha_k|^2 - |\beta_k|^2 = 1. \quad (\text{A12})$$

The BD initial conditions correspond to $\alpha_k = 1$ and $\beta_k = 0$. However, in LQC the initial conditions are usually imposed in the remote contracting phase [19,22]. Therefore, even if initially the system is in the vacuum state, β_k will be different from zero at the onset of the inflation, due to particle creations in the preinflationary phase. Then, we find that two-point correlation functions $P_{\mathcal{R}}(k)$ and $P_h(k)$ are given by

$$\langle \mathcal{R}_{\vec{k}} \mathcal{R}_{\vec{k}'} \rangle = (2\pi)^3 \delta(\vec{k} + \vec{k}') \frac{H^2}{\phi^2} \left| \frac{v_k}{a} \right|^2 \equiv (2\pi)^3 \delta(\vec{k} + \vec{k}') P_{\mathcal{R}}(k), \quad (\text{A13})$$

$$\langle h_{\vec{k}} h_{\vec{k}'} \rangle = (2\pi)^3 \delta(\vec{k} + \vec{k}') 32\pi G \left| \frac{v_k}{a} \right|^2 \equiv (2\pi)^3 \delta(\vec{k} + \vec{k}') P_h(k), \quad (\text{A14})$$

where the comoving curvature perturbation $\mathcal{R}$ of the scalar modes and the spin-2 tensor perturbation h_k are given, respectively, by

$$\mathcal{R} = \frac{H}{a\dot{\phi}} v_k, \quad h_k = \sqrt{16\pi G} \frac{v_k}{a}. \quad (\text{A15})$$

In the de Sitter background, we have $a = 1/(H\eta)$. Then, from Eq. (A11) we find

$$\begin{aligned} \left| \frac{v_k}{a} \right|^2 &= \frac{H^2}{2k^3} \{ (|\alpha_k|^2 + |\beta_k|^2)(k^2\eta^2 + 1) + \alpha_k \beta_k^* e^{-2ik\eta} (k^2\eta^2 - 2ik\eta - 1) \\ &\quad + \alpha_k^* \beta_k e^{2ik\eta} (k^2\eta^2 + 2ik\eta - 1) \}. \end{aligned} \quad (\text{A16})$$

On the superhorizon scales, we have $|k\eta| \ll 1$ [6], so the above expression reduces to

$$\left| \frac{v_k}{a} \right|^2 = \frac{H^2}{2k^3} \mathcal{F}(k), \quad \mathcal{F}(k) \equiv |\alpha_k|^2 + |\beta_k|^2 - (\alpha_k \beta_k^* + \alpha_k^* \beta_k). \quad (\text{A17})$$

Inserting it into Eqs. (A13) and (A14), we find

$$P_{\mathcal{R}}(k) = 2\pi G \left(\frac{H^2}{k^3 \epsilon_H} \right) \mathcal{F}(k), \quad (\text{A18})$$

$$P_h(k) = 16\pi G \left(\frac{H^2}{k^3} \right) \mathcal{F}(k). \quad (\text{A19})$$

Note that in writing down Eq. (A18) we had used Eq. (A7). Then, we find that

$$r \equiv \frac{P_t(k)}{P_{\mathcal{R}}(k)} = \frac{2P_h(k)}{P_{\mathcal{R}}(k)} = 16\epsilon_H. \quad (\text{A20})$$

To connect the power spectra to observations, let us introduce the spectrum indices n_s and n_t and their runnings via the relations

$$\Delta_{\mathcal{R}}^2(k) \equiv \frac{k^3}{2\pi^2} P_{\mathcal{R}}(k) = \frac{H^2}{\pi m_{\text{PL}}^2 \epsilon_H} \mathcal{F}(k) \equiv A_s \left(\frac{k_*}{a_0} \right) \left(\frac{k}{k_*} \right)^{n_s(k_*) - 1 + \frac{\alpha_s}{2} \ln(k/k_*) + \frac{\beta_s}{6} \ln^2(k/k_*) + \dots}, \quad (\text{A21})$$

$$\Delta_t^2(k) \equiv \frac{k^3}{2\pi^2} P_t(k) = \frac{16H^2}{\pi m_{\text{PL}}^2} \mathcal{F}(k) \equiv A_t \left(\frac{k_*}{a_0} \right) \left(\frac{k}{k_*} \right)^{n_t(k_*) + \frac{\alpha_t}{2} \ln(k/k_*) + \frac{\beta_t}{6} \ln^2(k/k_*) + \dots}, \quad (\text{A22})$$

where k_* denotes the pivot mode, a_0 is the current value of the expansion factor, and

$$\begin{aligned} n_s(k_*) &\equiv 1 + \left. \frac{d \ln \Delta_{\mathcal{R}}^2(k)}{d \ln k} \right|_{k=k_*}, & \alpha_s(k_*) &\equiv \left. \frac{d^2 \ln \Delta_{\mathcal{R}}^2(k)}{d \ln k^2} \right|_{k=k_*}, & \beta_s(k_*) &\equiv \left. \frac{d^3 \ln \Delta_{\mathcal{R}}^2(k)}{d \ln k^3} \right|_{k=k_*}, \\ n_t(k_*) &\equiv \left. \frac{d \ln \Delta_t^2(k)}{d \ln k} \right|_{k=k_*}, & \alpha_t(k_*) &\equiv \left. \frac{d^2 \ln \Delta_t^2(k)}{d \ln k^2} \right|_{k=k_*}, & \beta_t(k_*) &\equiv \left. \frac{d^3 \ln \Delta_t^2(k)}{d \ln k^3} \right|_{k=k_*}. \end{aligned} \quad (\text{A23})$$

Note that

$$\eta_H = \frac{\ddot{H}}{2H\dot{H}} = \frac{1}{2H} \frac{d}{dt} (\ln |\dot{H}|) = \frac{1}{2H} \frac{d}{dt} \ln(H^2 |\epsilon_H|) = -\epsilon_H + \frac{1}{2} \frac{d}{dN} (\ln |\epsilon_H|), \quad (\text{A24})$$

where $dN \equiv H dt$. Then, we find that

$$\frac{d \ln |\epsilon_H|}{dN} = 2(\epsilon_H + \eta_H). \quad (\text{A25})$$

On the other hand, at the horizon crossing we have $k = aH$, that is, $\ln k = N + \ln H$, from which we obtain

$$\left. \frac{dN}{d \ln k} \right|_{k=k_*} = \left(\frac{d \ln k}{dN} \right)^{-1} \Big|_{k=k_*} = \frac{1}{1 - \epsilon_H}, \quad (\text{A26})$$

where $\epsilon_H^* \equiv \epsilon_H(k_*)$. Hence, we find

$$\begin{aligned} \left. \frac{d \ln \Delta_{\mathcal{R}}^2(k)}{d \ln k} \right|_{k=k_*} &= \left. \frac{d}{d \ln k} [2 \ln H - \ln |\epsilon_H| + \ln \mathcal{F}(k)] \right|_{k=k_*} \\ &= \xi_{\text{nBD}}(k_*) + \frac{dN}{d \ln k} \left(-2\epsilon_H - \frac{d \ln |\epsilon_H|}{dN} \right) \Big|_{k=k_*} \\ &= \xi_{\text{nBD}}(k_*) - \frac{2(2\epsilon_H^* + \eta_H^*)}{1 - \epsilon_H^*}, \end{aligned} \quad (\text{A27})$$

where $\xi_{\text{nBD}}(k_*)$ denotes the non-BD vacuum effects, defined as

$$\xi_{\text{nBD}}(k_*) \equiv \left. \frac{d \ln \mathcal{F}(k)}{d \ln k} \right|_{k=k_*}. \quad (\text{A28})$$

If the BD vacuum is chosen, we have $\alpha_k = 1$ and $\beta_k = 1$, so that $\mathcal{F}(k) = 1$ and $\xi_{\text{nBD}}(k_*) = 0$. Inserting the above expressions into Eq. (A23), we obtain

$$n_s(k_*) - 1 = \xi_{\text{nBD}}(k_*) - \frac{2(2\epsilon_H^* + \eta_H^*)}{1 - \epsilon_H^*}, \quad (\text{A29})$$

$$n_t(k_*) = \xi_{\text{nBD}}(k_*) - \frac{2\epsilon_H^*}{1 - \epsilon_H^*}. \quad (\text{A30})$$

It is important to note that all the equations presented so far in this Appendix hold for any inflation as long as its background is quasi-de Sitter.

1. Slow-roll inflation

The slow-roll inflation requires

$$\epsilon_H, |\eta_H| \ll 1. \quad (\text{A31})$$

Then, from Eq. (A9) we can see that the second condition of the above equation is equivalent to require $|\dot{\phi}| \ll |H\dot{\phi}|$. As a result, Eq. (A4) yields

$$\dot{\phi} \simeq -\frac{V_{,\phi}}{3H}. \quad (\text{A32})$$

On the other hand, the first condition of Eq. (A31) implies $H^2 \gg 4\pi G\dot{\phi}^2$, so that

$$H^2 = \frac{8\pi G}{3} \left(\frac{1}{2} \dot{\phi}^2 + V(\phi) \right) \simeq \frac{8\pi G}{3} V(\phi). \quad (\text{A33})$$

Hence, we have

$$\epsilon_H = -\frac{\dot{H}}{H^2} \simeq -4\pi G\dot{\phi}^2 \frac{3}{8\pi GV(\phi)} \simeq \frac{1}{16\pi G} \left(\frac{V_{,\phi}}{V} \right)^2 = \epsilon_V. \quad (\text{A34})$$

On the other hand, from Eq. (A9) we find that

$$\begin{aligned} \eta_H &= \frac{1}{H} \frac{d}{dt} \ln |\dot{\phi}| \simeq \frac{1}{H} \frac{d}{dt} \left[\ln \left(-\frac{V_{,\phi}}{3H} \right) \right] \simeq \epsilon_H - \frac{1}{H} \left(-\frac{V_{,\phi}}{3H} \right) \frac{V_{,\phi\phi}}{V_{,\phi}} \\ &\simeq \epsilon_H - \frac{1}{8\pi G} \left(\frac{V_{,\phi\phi}}{V} \right) = \epsilon_V - \eta_V. \end{aligned} \quad (\text{A35})$$

Therefore, with the slow-roll conditions (A31), the two sets of parameters are related via the relations

$$\epsilon_H \simeq \epsilon_V, \quad \eta_H \simeq \epsilon_V - \eta_V. \quad (\text{A36})$$

Then, Eq. (A29) becomes

$$n_s(k_*) - 1 \simeq \xi_{\text{nBD}}(k_*) - 2(3\epsilon_V^* - \eta_V^*), \quad (\text{A37})$$

$$n_t(k_*) \simeq \xi_{\text{nBD}}(k_*) - 2\epsilon_V^*. \quad (\text{A38})$$

Planck 2018 chose $k_*/a_0 = 0.05/\text{Mpc}$ and obtained the following constraints for the scalar modes [10]:

$$n_s = 0.9587 \pm 0.0056, \quad \alpha_s = 0.013 \pm 0.012, \quad \beta_s = 0.022 \pm 0.012, \quad (\text{A39})$$

from the TT + lowE + lensing data all at 68% CL.

2. Calculation of the pivot mode k_*

To calculate the pivot mode $k_* = k_*^{\text{ob}} a_0$, where $k_*^{\text{ob}} = 0.05 \text{ Mpc}^{-1}$ for the Planck 2018 observations, let us first note that at the horizon crossing we have the following algebraic equations:

$$H_*^2 = \pi m_{\text{PL}}^2 \frac{\epsilon_V^* A_s(k_*)}{\mathcal{F}(k_*)}, \quad (\text{A40})$$

$$\xi_{\text{nBD}}(k_*) - 2(3\epsilon_V^* - \eta_V^*) = n_s(k_*) - 1, \quad (\text{A41})$$

$$k_* = a_* H_*, \quad (\text{A42})$$

$$\dot{\phi}_* \simeq -\frac{V_{,\phi}(V_0, \phi_*)}{3H_*}, \quad (\text{A43})$$

$$H_*^2 \simeq \frac{8\pi G}{3} V(V_0, \phi_*), \quad (\text{A44})$$

$$k_* = a_0 k_*^{\text{ob}}, \quad (\text{A45})$$

where in LQC we have [60]

$$\mathcal{F}(k) = 1 + \delta_{\text{PL}}(k) = 1 + \left[1 + \cos \left(\frac{\pi}{\sqrt{3}} \right) \right] \text{csch}^2 \left(\frac{\pi k}{\sqrt{6} k_B} \right), \quad k_B \equiv a_B \sqrt{\frac{8\pi \rho_c}{m_{\text{Pl}}^2}}, \quad (\text{A46})$$

with $\text{csch } x \equiv (\sinh x)^{-1}$. In addition, because of the rescaling symmetry, $a(t) \rightarrow a(t)/L$, of the dynamical equations (2) and (3), where L is a real constant, without loss of the generality, we can always set

$$a_B(H_*, \phi_*, \dot{\phi}_*) = 1. \quad (\text{A47})$$

Note that, once $(H_*, \phi_*, \dot{\phi}_*)$ are given, the dynamical equations (2) and (3) will uniquely determine $(a_B, \phi_B, \dot{\phi}_B)$. So, Eq. (A47) represents a nonlocal constraint of the three parameters $(H_*, \phi_*, \dot{\phi}_*)$ through the field equations.

On the other hand, once a_B is fixed, we can calculate a_0 via the relation

$$a_0 = a_B e^{N_T} = a_* e^{\hat{N}_T}, \quad (\text{A48})$$

where

$$\begin{aligned} N_T &\equiv \ln \left(\frac{a_0}{a_B} \right) = \ln \left(\frac{a_*}{a_B} \times \frac{a_0}{a_*} \right) = \ln \left(\frac{a_*}{a_B} \right) + \hat{N}_T, \\ \hat{N}_T &\equiv \ln \left(\frac{a_0}{a_*} \right) = \ln \left(\frac{a_{\text{end}}}{a_*} \times \frac{a_{\text{re}}}{a_{\text{end}}} \times \frac{a_0}{a_{\text{re}}} \right) = N_* + N_{\text{re}} + \ln \left(\frac{a_0}{a_{\text{re}}} \right). \end{aligned} \quad (\text{A49})$$

Here, N_* , N_{re} , and a_0/a_{re} , given, respectively, by Eqs. (30), (31), and (34), are functions of $(\phi_*, \phi_{\text{end}}, H_*)$. However, from the slow-roll conditions and the definition of the end of inflation, we have

$$\epsilon_V(\phi_{\text{end}}) = 1, \quad (\text{A50})$$

which uniquely determine ϕ_{end} . For example, for the potential given by Eq. (4), we have

$$4\pi\phi_{\text{end}}^2(1 + \phi_{\text{end}}^2)^2 = 1, \quad (\text{A51})$$

as can be seen from Eq. (21). Then, from Eqs. (A48)–(A50) we find that $N_T = N_T(a_*, H_*, \phi_*)$ for any given $(w_{\text{re}}, g_{\text{re}}, T_0, k_*^{\text{ob}})$. Hence, we have

$$a_0 = a_B e^{N_T} = a_* e^{\hat{N}_T} = a_0(a_*, \phi_*, H_*). \quad (\text{A52})$$

In summary, for any given $(w_{\text{re}}, g_{\text{re}}, T_0, k_*^{\text{ob}})$, we have seven equations [Eqs. (A40)–(A45) and (A47)] for the six unknown

$$(a_*, H_*, \phi_*, \dot{\phi}_*, k_*, V_0), \quad (\text{A53})$$

once (A_s, n_s) are provided by observations. Thus, the system, consisting of the seven equations, seems overdetermined. However, we find that Eq. (A45) is not independent and is satisfied once Eqs. (A40)–(A44) and (A50) hold. Therefore, the six equations [Eqs. (A40)–(A44) and (A47)] uniquely determine the six unknown (A53), and the problem becomes deterministic [133].

The above analysis provides a routine for computing the values of the relevant physical quantities of the pivot mode k_* at the horizon crossing $t = t_*$, given the form of the inflationary potential.

3. Quantum geometric effects

To understand the quantum geometric effects, let us first consider the functions $\mathcal{F}(k)$ and $\xi_{\text{nBD}}(k)$ defined, respectively, by Eqs. (A46) and (A29), which are plotted in Fig. 5. From this figure, it can be seen that the quantum effects are very small (except for the case $k \ll k_B$). Therefore, to the first-order approximation, we can set $\mathcal{F}(k) = 1$ and $\xi_{\text{nBD}}(k) = 0$. Then, we find

$$H_*^2 = \pi m_{\text{PL}}^2 \epsilon_V^* A_s(k_*), \quad (\text{A54})$$

$$2(\eta_V^* - 3\epsilon_V^*) = n_s(k_*) - 1, \quad (\text{A55})$$

$$k_* = a_* H_*, \quad (\text{A56})$$

$$\dot{\phi}_* \simeq -\frac{V_{,\phi}(V_0, \phi_*)}{3H_*}, \quad (\text{A57})$$

$$H_*^2 \simeq \frac{8\pi G}{3} V(V_0, \phi_*), \quad (\text{A58})$$

$$a_B(a_*, \phi_*, \dot{\phi}_*) = 1. \quad (\text{A59})$$

To solve the above equations, our strategy is as follows:

(1) Since ϵ_V^* and η_V^* are functions of ϕ_* only, then for any given observational values of (A_s, n_s) , from Eq. (A55) we find ϕ_* . Then, from Eq. (A54) we find $H_* = \sqrt{\pi m_{\text{PL}}^2 \epsilon_V^* A_s(k_*)}$. Then, inserting them into Eq. (A58) we find V_0 . With such obtained (H_*, ϕ_*, V_0) , Eq. (A57) will give $\dot{\phi}_*$, which has two solutions, corresponding, respectively, to the two different choices of $\epsilon_{\pm}$. Therefore, now we find the following four quantities:

$$(H_*, \phi_*, \dot{\phi}_*, V_0). \quad (\text{A60})$$

(2) With the values of $(\phi_*, \dot{\phi}_*)$ obtained above, we choose a value of a_* , say, $\hat{a}_*$, and then evolve the dynamical equations (2) and (3) back to the bounce time to find $(a_B, \phi_B, \dot{\phi}_B)$. Clearly, such obtained $a_B(\hat{a}_*)$ is usually not equal to 1. We shall keep trying different values of $\hat{a}_*$, until we find that $a_B = 1$. This value of $\hat{a}_*$ is the value that we look for, which will be denoted by a_* , so we have $a_B(a_*, \phi_*, \dot{\phi}_*) = 1$.

It is interesting to note that in the above process, we find that such obtained $(\phi_B, \dot{\phi}_B)$ do not depend on the value of $\hat{a}_*$. This is expected, as the dynamical equations (2) and (3) have the scaling symmetry, $\hat{a}_* \rightarrow \hat{a}_*/L$, as mentioned above. The above observation also leads to a more efficient way to find a_* : Choosing a value of $\hat{a}_*$ together with the values of $(\phi_*, \dot{\phi}_*, V_0)$ found in Eq. (A60) to evolve the dynamical equations (2) and (3) until the bounce to obtain

$$(\phi_B, \dot{\phi}_B). \quad (\text{A61})$$

Then, taking $a_B = 1$ and $(\phi_B, \dot{\phi}_B)$ given by Eq. (A61) as the new initial conditions but now specified at the bounce, evolve the dynamical equations (2) and (3) from t_B to t_* to find a_* . Clearly, such obtained a_* will satisfy Eq. (A59). To make sure that this indeed corresponds to the horizon cross of the pivot mode k_* , we can calculate $(H_*, \phi_*, \dot{\phi}_*)$ and then compare them with those given by Eq. (A60). The consistency of these values implies that they are indeed corresponding to the pivot mode.

(3) Once we find the quantities

$$(a_*, H_*, \phi_*, \dot{\phi}_*, V_0), \quad (\text{A62})$$

from Eq. (A56) we can find k_* .

(4) On the other hand, from Eqs. (A42) and (A45) we obtain

$$H_* = e^{\hat{N}_T(H_*, \phi_*)} k_*^{\text{ob}}, \quad (\text{A63})$$

where $\hat{N}_T$ is given by Eq. (A49). Recall that for Planck 2018 the pivot mode $k_*^{\text{ob}} \equiv k_*/a_0$ is chosen as $k_*^{\text{ob}} = 0.05/\text{Mpc}$. Inserting the values of $(a_*, k_*, H_*, \phi_*, \dot{\phi}_*, V_0)$ obtained in the above steps into Eq. (A63), we check the consistency of the system.

(5) Now, let us consider the quantum effects for which we have $\mathcal{F}(k) \neq 1$. Considering the fact that $\mathcal{F}(k)$ is not significantly different from one, as shown by Fig. 5, we can consider the solutions obtained in the previous steps as the first-order approximation, denoted by $\mathbf{A}_*^{(0)}$, and then expand them as

$$\mathbf{A}_* = \mathbf{A}_*^{(0)} + \delta \mathbf{A}_*, \quad (\text{A64})$$

to find δA_* by repeating the above steps, where $A_* = \{\mathcal{F}(k_*), a_*, H_*, \phi_*, \dot{\phi}_*, k_*\}$.

(6) Certainly, if we need higher accuracy, we can expand A_* to high orders, and repeat the above steps to obtain the six physical quantities $(a_*, H_*, \phi_*, \dot{\phi}_*, k_*, V_0)$ at the horizon crossing. These quantities uniquely determine the whole evolution of the universe, from the contracting phase to the current time through the dynamical equations (2) and (3).

It should be noted that when the reheating phase is not taken into account, the e-fold $\hat{N}_T$ becomes a free parameter, so is a_0 . Then, the quantity k_*/a_0 is not fixed, as shown explicitly [30,77]. Therefore, it is the reheating phase that uniquely fixes the quantity k_*/a_0 .

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