

Fluctuations in measurement-based quantum engines and quantum batteries derived from the Kirkwood-Dirac quasiprobability distribution

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In this paper, we consider fluctuations in a three-stroke cycle based on quantum measurement. In the first stroke, the engine is fueled by a nonselective quantum measurement; in the second stroke, an appropriate unitary operation is applied to extract work; and finally, the working medium is thermalized to complete the cycle. This protocol is analogous to charging, then discharging, and finally thermalizing a quantum battery. Using the Kirkwood-Dirac quasiprobability, we derive equations that provide insights into the fluctuations of heat and work, and thus into the charging and discharging processes. We then propose a quantum circuit to measure the cumulants of work and heat using a qubit ancilla. Afterward, we derive important upper and lower bounds on fluctuations in heat; this is thanks to the way we decompose heat fluctuations and the Kadison-Schwarz inequality. We also show, for the first time, that the imaginary part of the second cumulant of work equals the linear response of an average work (to be defined in the main text) or the quantum cross entropy. We further introduce an operator whose average defines the battery capacity and whose variance quantifies the corresponding fluctuations of the battery capacity. Our analytical results are illustrated and discussed through representative examples. We believe that our study will, on the one hand, be useful for understanding fluctuations in energy extracted from and injected into quantum batteries based on quantum measurements and, on the other hand, contribute to the foundations of quantum thermodynamics. More generally, our results can be applied beyond the three-stroke cycle considered here, especially to quantum batteries subject to environmental noise and decoherence.

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I. INTRODUCTION

Quantum engines [1–9] and quantum batteries (QBs) [10–12] are among the most promising emerging quantum technologies [13–15] and are expected to play an important role in future applications. A central goal of quantum thermodynamics [16–21] is to investigate how resources such as energy discreteness, quantum coherence, and *quantum measurement*—the main focus of this paper—can enhance the performance of thermodynamic devices, including clocks [22], refrigerators [2], engines [7], and batteries [12].

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However, fluctuations at the quantum scale are significant; consequently, thermodynamic quantities such as work, heat, efficiency, and entropy production exhibit inherently stochastic behavior [23–35]. Recent studies have shown that these fluctuations are not arbitrary but can be bounded [35–42]. For instance, in the case of the four-stroke quantum cycle, it has been shown that the relative fluctuations of work and heat are lower bounded [35,42–44]. Moreover, it has been demonstrated that the ratio of work to heat fluctuations is constrained from both below and above (see Refs. [36–43]).

A. Motivation

Here, we focus on fluctuations and address an important yet largely unexplored question in the literature, namely, fluctuations in thermodynamic quantities of heat engines and batteries based on (either weak or strong) nonselective quantum measurements [45–48], where it is specified when each measurement type is considered. More precisely, we develop a general theoretical framework, together with bounds on fluctuations, that is applicable to quantum thermodynamic machines fueled by quantum measurements. We believe that our study opens directions that differ from those explored in Refs. [49–75], where the focus was primarily restricted to fluctuations in work (or in power). Strictly speaking, our work will be useful when considering the charging and discharging of QBs using quantum operations beyond unitary ones [45–48].

B. Summary of contributions

In this work, we consider a three-stroke quantum cycle (3SQC) [76,77], where the working medium (WM) is an arbitrary finite-dimensional quantum system. In the first stroke, the engine is fueled by a nonselective quantum measurement [45,48]; in the second stroke, an appropriate unitary operation is applied to extract work; and finally, in the third stroke, the WM is thermalized to close the cycle. This cycle is analogous to charging, then discharging, and finally thermalizing a quantum battery. Owing to this analogy, we use the terms heat (in the context of engines) and charging energy (in the context of batteries) interchangeably.

We employ the Kirkwood-Dirac quasiprobability (KDQ) distribution [78–100] as a tool to compute the cumulants of thermodynamic quantities of the engine, and thus also the cumulants of the charging and discharging processes of the QB. Unlike cumulants derived using the two-point measurement (TPM) scheme [24,101,102], these cumulants account for quantum coherence, thereby capturing genuinely quantum features of the engine. We also propose a quantum circuit [103–105] that allows one to measure the real and imaginary parts of the joint characteristic function (CF), from which the cumulants of heat and work can be extracted.

Since the cumulants of work are derived from the KDQ distribution, the second cumulant generally possesses an imaginary part. We provide physical interpretations of this imaginary part [106,107] in our setting. More precisely, we show that the imaginary part can be interpreted as the linear response [108] of (1) an average work, to be defined below, and (2) the quantum cross entropy [109].

We compare fluctuations in heat obtained from either the TPM scheme or the KDQ distribution with those derived from the *heat operator*, where the latter is defined analogously to the work operator [110,111]. We show that while the average of the heat operator correctly reproduces the average heat, its variance fails to capture all terms in heat fluctuation. Nevertheless, using the Kadison-Schwarz inequality [112], we prove that the variance of the heat operator provides a lower bound on heat fluctuations. Overall, we establish two lower bounds and two upper bounds on heat fluctuations.

We show that, in the ergotropy regime, the marginal quasiprobability distribution from which work fluctuations emerge becomes a genuine probability distribution (PD). More precisely, we demonstrate that, in this regime, the TPM scheme can be used to estimate work fluctuations even when the initial state does not commute with the initial Hamiltonian. This is possible because the unitary that extracts ergotropy, when acting on the pre-work-extraction state, produces a state that commutes with the battery Hamiltonian.

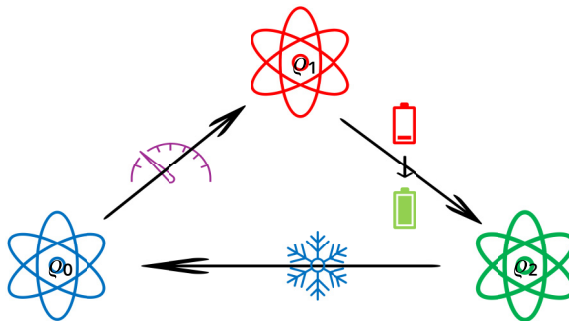


FIG. 1. Schematic representation of the 3SQC. From ρ_0 (blue atom) to ρ_1 (red atom), the WM (represented by an atom) is fueled by a nonselective quantum measurement, which may be strong (i.e., projective) or weak, with the type specified in each case. From ρ_1 to ρ_2 (green atom), work is extracted from the WM to charge, e.g., a battery, where the battery sketched in red color is uncharged while the green one is charged. However, of course, one can also consider adding work to the WM, i.e., energy injection. Finally, from ρ_2 to ρ_0 , the WM returns to its initial state via thermalization. Note that the working medium itself can also be regarded as a quantum battery. See the main text for details.

We further introduce an operator whose average defines the battery capacity [113] and whose variance quantifies the corresponding fluctuations. We show that this operator can be expressed as the difference between two work operators [110]: one whose average yields the ergotropy [114], and another whose average yields the antiergotropy [113]. We prove that the variance of the battery capacity operator can be interpreted as a measure of the work fluctuations associated with discharging (charging) the QB from its active (passive) state to its passive (active) state. We believe that the variance of the battery capacity operator provides a valuable extension to the set of tools available for characterizing the performance of quantum batteries.

C. Paper flow

The flow of this paper is as follows. In Sec. II, we define the 3SQC and demonstrate its analogy with charging, discharging, and thermalizing a quantum battery. In Sec. III, we present our analytical results and their limits of applicability, along with their interpretations and usefulness. Further results are given in the Appendixes. Finally, we conclude in Sec. IV.

Throughout this paper, we set $\hbar = k_B = 1$. The natural logarithm is denoted by $\ln$. Furthermore, for a given average and variance, we define the reliability as the ratio of the average to the square root of the variance; the square of the reliability is also referred to as the signal-to-noise ratio [32]. This definition applies to both scalar quantities and operators.

II. MEASUREMENT-BASED THREE-STROKE QUANTUM CYCLE

A. Three-stroke quantum cycle

We consider a 3SQC (Fig. 1) in which the WM is a quantum system with Hamiltonian H acting on a finite-dimensional Hilbert space $\mathcal{H}_{d_b}$ of dimension d_b . The Hamiltonian H can be spectrally decomposed as

$$H = \sum_{i=1}^{d_b} E_i \Pi_{E_i}, \quad (1)$$

where $\Pi_{E_i} = |E_i\rangle\langle E_i|$ are rank-one projectors onto the energy eigenstates $|E_i\rangle$ and E_i are the corresponding eigenvalues.

The WM is initially in thermal equilibrium with a heat bath at temperature T , and its initial state is given by the Gibbs state

$$\rho_0 = \frac{e^{-H/T}}{\text{Tr}[e^{-H/T}]}. \quad (2)$$

We now describe the three strokes of the cycle (see, e.g., Ref. [76]):

(1) *First stroke*: The WM, initially prepared in the state ρ_0 , is subjected to a quantum measurement. Since ρ_0 is a thermal equilibrium state, this stroke fuels the engine. Indeed, the Gibbs state (2) is passive [115,116], meaning that no work can be extracted from it via cyclic unitary operations.

(2) *Second stroke*: An appropriate unitary operation U is applied to the WM in order to extract work.

(3) *Third stroke*: The WM is recoupled to the heat bath and allowed to fully thermalize, thereby returning to its initial equilibrium state (2).

The action of a nonselective quantum projective measurement [117–119] on a state ρ is described by the quantum channel Φ_M , where

$$\Phi_M(\rho) = \sum_{i=1}^{d_b} \Pi_i \rho \Pi_i, \quad (3)$$

with $\Pi_i = |\Pi_i\rangle\langle\Pi_i|$ being projective operators satisfying the orthogonality condition $\Pi_i \Pi_j = \delta_{ij} \Pi_i$. Accordingly, during the first stroke the state changes from ρ_0 to

$$\rho_1 = \Phi_M(\rho_0). \quad (4)$$

The corresponding average energy change is

$$\langle Q_M \rangle := \text{Tr}[H(\rho_1 - \rho_0)]. \quad (5)$$

Since Φ_M generally increases the von Neumann entropy of the WM, the quantity $\langle Q_M \rangle$ is interpreted as (quantum) heatlike [3,4,120].

In Eq. (3), although the number of projectors is taken to be equal to the dimension of the WM, this assumption is not necessary for deriving our analytical results [121]. Allowing the number of projectors to be fewer than d_b enables measurements to be performed on only, e.g., a subsystem of the WM [122]. Consider a quantum system composed of N subsystems, where the n th subsystem has dimension d_n . If the measurement is performed on this subsystem, Eq. (3) becomes $\tilde{\Phi}_M(\rho) = \sum_{i=1}^{d_n} \tilde{\Pi}_i \rho \tilde{\Pi}_i$, where $\tilde{\Pi}_i = \mathbb{1}_{d_1} \otimes \cdots \otimes \mathbb{1}_{d_{n-1}} \otimes \tilde{\Pi}_i \otimes \mathbb{1}_{d_{n+1}} \otimes \cdots \otimes \mathbb{1}_{d_N}$. Here, the projectors $\tilde{\Pi}_i$ satisfy $\tilde{\Pi}_s \tilde{\Pi}_m = \delta_{sm} \tilde{\Pi}_s$. The total Hilbert space dimension is $d_b = \prod_{i=1}^N d_i$, and $\mathbb{1}_d$ denotes the identity operator on a d -dimensional space. For example, for a WM composed of two coupled qubits, i.e., $N = 2$, one may restrict the measurement to a single qubit [121,122], in which case $d_b = 4$ and $d_i = 2$ for $i = 1, 2$. In general, however, the measurement need not be restricted to a single subsystem and can act on any subset of the global system. We also note that the channel Φ_M in Eq. (3) is unital, since $\Phi_M(\mathbb{1}_{d_b}) = \mathbb{1}_{d_b}$.

In the second stroke, the state evolves unitarily from ρ_1 to $\rho_2 = U \rho_1 U^\dagger$, and the average work (extracted or invested, depending on its sign) is

$$\langle W \rangle = \text{Tr}[H(\rho_2 - \rho_1)]. \quad (6)$$

Finally, in the third stroke, the WM thermalizes back to the initial state ρ_0 , and the average heat released into the bath is

$$\langle Q_C \rangle = \text{Tr}[H(\rho_0 - \rho_2)].$$

When the cycle operates as a heat engine, the conditions $\langle Q_M \rangle > 0$, $\langle W \rangle < 0$, and $\langle Q_C \rangle < 0$ must be satisfied. The efficiency is then defined as

$$\eta = \langle W \rangle_{\text{ext}} / \langle Q_M \rangle = 1 + \langle Q_C \rangle / \langle Q_M \rangle, \quad (7)$$

where $\langle W \rangle_{\text{ext}} = -\langle W \rangle$. The second equality follows from the first law of thermodynamics, $\langle Q_M \rangle + \langle W \rangle + \langle Q_C \rangle = 0$. In Eq. (7), the efficiency η quantifies how much energy is recovered as useful work from the energy injected into the engine via quantum measurement.

B. Equivalence of the 3SQC to cyclic quantum battery

An important remark is in order.

Remark 1. Our 3SQC can be interpreted as a cyclic QB [122]. A (quantum) battery is a system that stores energy for later use. A central question in quantum thermodynamics is how much energy can be stored in and extracted from a quantum system. In this context, the WM functions as a battery: it starts in a thermal equilibrium state, is then charged via a quantum projective measurement (first stroke), and is partially or completely discharged through a unitary operation that extracts useful energy (second stroke). Finally, the battery is reset to its initial state by coupling it to a thermal bath (third stroke).

It is worth noting that additional strokes can be added between the charging and discharging processes to study, for example, the influence of quantum noise on the energy stored in the battery [123, 124].

This observation, namely, that measuring a WM can be interpreted as charging a QB via quantum measurement, can be useful to avoid using the terms heat or work and instead describe the process as either charging or discharging a QB. For example, when the measurement channel during the first stroke is considered to be a nonselective weak measurement [48, 119] rather than a projective one, we show in Appendix L, using a qubit example, that the change in the energy of the WM can sometimes occur with no entropy increase in its state and can therefore be considered worklike. In this case, one can recover all the energy given to the WM during the first stroke, and hence unit efficiency is achieved [76]. However, for nonselective quantum projective measurements, the change in energy during the first stroke is considered (quantum) heat [120].

C. Ergotropy, antiergotropy, and battery capacity

In the second stroke, if one maximizes $\langle W \rangle_{\text{ext}}$ over all possible unitaries $\in \mathcal{U}_{d_b}$, where the latter is the set of unitary matrices of dimension $d_b \times d_b$, the resulting quantity is known as *ergotropy* [114], which we denote by $\langle W \rangle_{\text{max}}$, while the unitary that attains it is denoted by U_{erg} . In this regime, the corresponding efficiency is $\eta_{\text{max}} := \langle W \rangle_{\text{max}} / \langle Q_M \rangle$, and the state ρ_2 becomes ρ_p . With regard to work maximization, one can consider either local or global unitaries; however, the analytical expressions for fluctuations derived in the following section apply to both cases, since we consider an arbitrary unitary U .

For QBs, ergotropy is not the only relevant quantity. Instead of maximizing $\langle W \rangle_{\text{ext}}$ over all unitaries, one can also minimize it. This corresponds to maximizing $\text{Tr}[U \rho_1 U^\dagger H]$ over all unitaries $\in \mathcal{U}_{d_b}$, and the resulting quantity is known as *antiergotropy* [58, 124, 125]. We denote it by $\langle W \rangle_{\text{min}}$, which is a negative quantity [113], and the unitary achieving it is labeled U_{aerg} . The action of U_{aerg} on the state ρ_1 produces the active state, which we denote by ρ_a .

From the difference between ergotropy and antiergotropy, one can define the quantum battery capacity (or, in short, the battery capacity) [58, 113, 124–132], given by $\mathcal{C}(\rho_1; H) := \langle W \rangle_{\text{max}} - \langle W \rangle_{\text{min}}$. It is straightforward to show that

$$\mathcal{C}(\rho_1; H) = \text{Tr}[(U_{\text{aerg}} \rho_1 U_{\text{aerg}}^\dagger - U_{\text{erg}} \rho_1 U_{\text{erg}}^\dagger) H].$$

Finally, an explicit example illustrating our 3SQC based on nonselective quantum projective measurement is provided in Appendix A.

III. RESULTS

In this section, we present the main analytical results. Detailed proofs, applications, and illustrative examples are provided in the Appendixes.

A. The importance of schemes beyond the two-point measurement scheme

In the previous section, we introduced the 3SQC and provided expressions for the average work, heat, and battery capacity. However, these thermodynamic quantities exhibit nonzero fluctuations, raising an important question: how can we assess fluctuations, e.g., in work and heat?

To estimate these fluctuations, the TPM scheme is typically employed [102,111]. This method involves measuring the system's energy at the beginning and end of its evolution, constructing the PD, and then deriving all cumulants from this distribution.

Since the initial state of the cycle commutes with the Hamiltonian, i.e., $[\rho_0, H] = 0$, where $[A, B] = AB - BA$ denotes the commutator, fluctuations in heat can be evaluated using the TPM without any issues—although we note that our paper also presents the expression for heat fluctuations even in the case of a noncommuting state and Hamiltonian.

However, the situation becomes more intricate when considering fluctuations in work during the second stroke. In general, $[\rho_1, H] \neq 0$, implying that the postmeasurement state ρ_1 contains quantum coherence in the energy eigenbasis. If the TPM approach were used, the required projective measurements would destroy this coherence, leading to work fluctuations that fail to capture its influence [133]. As a result, computing work fluctuations in the presence of coherence becomes both a challenging and an interesting problem.

Recently, it has been shown that the KDQ [78–100] is a promising tool—among many other proposed methods [87]—for analyzing work fluctuations in quantum systems where coherence plays a significant role. This quasiprobability satisfies all Kolmogorov axioms except for positivity; it can take on negative or complex values.

B. Derivation of cumulants using the Kirkwood-Dirac quasiprobability distribution

Remark 2. Below, when we consider the decomposition of H as in Eq. (1), we add a superscript to the eigenvalues and eigenstates of H to indicate the state of the QB. More precisely, when the QB is in the state ρ_0 , ρ_1 , or ρ_2 , we denote the eigenvalues by $E_i^{(0)}$, $E_j^{(1)}$, and $E_k^{(2)}$, while the corresponding eigenstates of H are denoted by $|E_i^{(0)}\rangle$, $|E_j^{(1)}\rangle$, and $|E_k^{(2)}\rangle$, respectively. This is purely a notational convention and does not affect the physics.

Let us now define Q_M and W as the heat and work at the (quantum) trajectory level—i.e., at the level of single realizations of the cycle [97,134]. In terms of Hamiltonian eigenvalues, they are given by

$$Q_M := E_j^{(1)} - E_i^{(0)} \quad \text{and} \quad W := E_k^{(2)} - E_j^{(1)}.$$

Following Refs. [92–95], the joint KDQ describing both heat and work at the trajectory level for the 3SQC is given by

$$q_{ijk} = \text{Tr} \left[|E_k^{(2)}\rangle \langle E_k^{(2)}| U |E_j^{(1)}\rangle \langle E_j^{(1)}| \Phi_M(|E_i^{(0)}\rangle \langle E_i^{(0)}| \rho_0) U^\dagger \right]. \quad (8)$$

In general, Eq. (8) can be negative and even complex. Its real part is known as the Margenau-Hill quasiprobability [18,80,93], as it may take negative values.

Using Eq. (8), the joint CF—i.e., the Fourier transform of the joint quasiprobability distribution of heat and work (see Appendix B)—is

$$\chi(\gamma_{Q_M}, \gamma_W) = \text{Tr} \left[e^{i\gamma_W H} U e^{i(\gamma_{Q_M} - \gamma_W) H} \Phi_M(e^{-i\gamma_{Q_M} H} \rho_0) U^\dagger \right]. \quad (9)$$

Here, γ_{Q_M} and γ_W are the Fourier conjugate variables corresponding to the stochastic variables Q_M and W , respectively. Note that in Eq. (9), if one considers an arbitrary generalized nonselective measurement, one only needs to replace the channel Φ_M in Eq. (3) by the appropriate one, and Eq. (9) still gives the cumulants of heat and work.

The averages of Q_M and W follow from the joint CF as

$$\langle Q_M \rangle = \left. \frac{\partial \ln \chi(\gamma_{Q_M}, \gamma_W)}{\partial (i\gamma_{Q_M})} \right|_{\gamma_{Q_M}=\gamma_W=0}, \quad (10)$$

$$\langle W \rangle = \left. \frac{\partial \ln \chi(\gamma_{Q_M}, \gamma_W)}{\partial (i\gamma_W)} \right|_{\gamma_{Q_M}=\gamma_W=0}, \quad (11)$$

respectively. Furthermore, Eqs. (10) and (11) reproduce the average quantities introduced earlier in Eqs. (5) and (6). In Appendix C, we prove analytically that $\langle Q_M \rangle \geq 0$ and $\langle Q_C \rangle \leq 0$, where the latter is in accordance with the second law of thermodynamics for our 3SQC, ensuring that heat flows on average only from the QB to the bath.

On the other hand, the cumulants of Q_M and W , beyond averages, are obtained from the joint CF via

$$\langle X^n \rangle_c = \left. \frac{\partial^n \ln \chi(\gamma_{Q_M}, \gamma_W)}{\partial (i\gamma_X)^n} \right|_{\gamma_{Q_M}=\gamma_W=0}, \quad (12)$$

where X can be Q_M or W , and $n \geq 2$.

C. A method to measure Eq. (9) using a qubit ancilla: Interferometric scheme

Before developing further results, an important question arises: is Eq. (9) a measurable quantity? If so, how can it be measured? Below, we answer this question affirmatively and propose a quantum circuit to achieve this task.

Consider a qubit ancilla [103–105] with states $|e\rangle$ and $|g\rangle$ as its excited and ground states, respectively. In the basis $\{|e\rangle, |g\rangle\}$, the Pauli operators are $\sigma_x = |g\rangle\langle e| + |e\rangle\langle g|$, $\sigma_y = i(|g\rangle\langle e| - |e\rangle\langle g|)$, and $\sigma_z = |e\rangle\langle e| - |g\rangle\langle g|$. Assume that the ancilla is prepared in the state $|e\rangle\langle e|$, and let $H_g = (\sigma_z + \sigma_x)/\sqrt{2}$ be the Hadamard gate. We then define the following operators, and the channel E_2 :

$$\begin{aligned} E_0 &:= H_g \otimes \mathbb{1}_{d_b}, \\ E_1 &:= |e\rangle\langle e| \otimes e^{-i\gamma_{Q_M}H} + |g\rangle\langle g| \otimes \mathbb{1}_{d_b}, \\ E_2(\cdot) &:= \sum_{i=1}^{d_b} \sigma_x \otimes \Pi_i(\cdot) \sigma_x \otimes \Pi_i, \\ E_3 &:= |e\rangle\langle e| \otimes \mathbb{1}_{d_b} + |g\rangle\langle g| \otimes e^{i\gamma_{Q_M}H}, \\ E_4 &:= |e\rangle\langle e| \otimes \mathbb{1}_{d_b} + |g\rangle\langle g| \otimes e^{-i\gamma_WH}, \\ E_5 &:= \sigma_x \otimes U, \\ E_6 &:= |e\rangle\langle e| \otimes e^{i\gamma_WH} + |g\rangle\langle g| \otimes \mathbb{1}_{d_b}. \end{aligned}$$

One can see that (1) E_0 acts only on the ancilla, preparing it in the state $(|e\rangle + |g\rangle)/\sqrt{2}$ while acting as the identity on the QB; (2) E_1 , E_3 , E_4 , and E_6 are controlled unitary operators acting on the QB, conditioned on the state of the ancilla; (3) the channel E_2 applies σ_x on the ancilla and the measurement channel Φ_M on the QB; and (4) E_5 is a unitary operator that applies σ_x on the ancilla and U on the QB. Furthermore, by the Stinespring dilation theorem [135], the measurement channel Φ_M in Eq. (3) can be expressed as

$$\Phi_M(\rho_0) = \text{Tr}_m[U_{bm}(\rho_0 \otimes \rho_m)U_{bm}^\dagger],$$

where $\rho_m = |0\rangle\langle 0|$ is the initial state of the meter, U_{bm} is the unitary coupling the meter and the QB, and Tr_m denotes the partial trace over the meter degrees of freedom. With an appropriate choice of U_{bm} , this dilation exactly reproduces the nonselective quantum projective measurement, i.e., $\text{Tr}_m[U_{bm}(\rho_0 \otimes \rho_m)U_{bm}^\dagger] = \sum_{i=1}^{d_b} \Pi_i \rho_0 \Pi_i$.

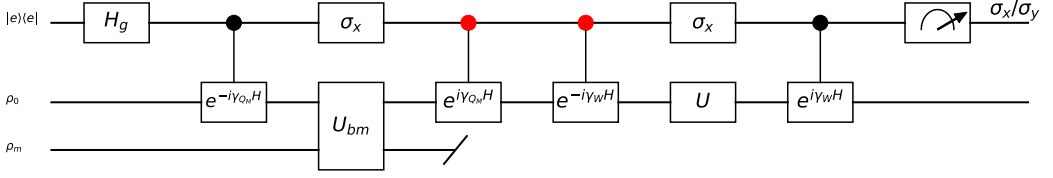


FIG. 2. Quantum circuit to infer the joint CF. From left to right: the first black dot indicates that, when the ancilla is in the excited state $|e\rangle\langle e|$, the QB is acted upon by $e^{-i\gamma_{Q_M}H}$; the first red dot indicates that, when the ancilla is in the ground state $|g\rangle\langle g|$, the QB is acted upon by $e^{i\gamma_{Q_M}H}$. The remaining dots are interpreted similarly. The final box represents a measurement on the ancilla. Measuring σ_x and σ_y of the ancilla yields the real and imaginary parts of Eq. (9), respectively.

It should be emphasized that the circuit in Fig. 2 can, with appropriate modifications, operate even when the ancilla is in the ground state; that is, the ancilla does not need to be initially prepared in the excited state. More generally, the notions of excited and ground states are not essential, in the sense that the Hamiltonian of the ancilla can be taken to be trivial [93].

At this stage, one can see (Fig. 2) that

$$\begin{aligned} \rho_f &:= \text{Tr}_m[E_6 E_5 E_4 E_3 \sigma_x U_{bm} ((E_1 E_0 (|e\rangle\langle e| \otimes \rho_0) E_0^\dagger E_1^\dagger) \otimes \rho_m) U_{bm}^\dagger \sigma_x E_3^\dagger E_4^\dagger E_5^\dagger E_6^\dagger] \\ &= E_6 E_5 E_4 E_3 E_2 (E_1 E_0 (|e\rangle\langle e| \otimes \rho_0) E_0^\dagger E_1^\dagger) E_3^\dagger E_4^\dagger E_5^\dagger E_6^\dagger \\ &= \frac{|e\rangle\langle e|}{2} \otimes e^{i\gamma_W H} U e^{-i\gamma_W H} e^{i\gamma_{Q_M} H} \Phi_M(e^{-i\gamma_{Q_M} H} \rho_0 e^{i\gamma_{Q_M} H}) e^{-i\gamma_{Q_M} H} e^{i\gamma_W H} U^\dagger e^{-i\gamma_W H} \\ &\quad + \frac{|g\rangle\langle g|}{2} \otimes U \Phi_M(\rho_0) U^\dagger + \frac{|e\rangle\langle g|}{2} \otimes e^{i\gamma_W H} U e^{i(\gamma_{Q_M} - \gamma_W) H} \Phi_M(e^{-i\gamma_{Q_M} H} \rho_0) U^\dagger \\ &\quad + \frac{|g\rangle\langle e|}{2} \otimes U \Phi_M(\rho_0 e^{i\gamma_{Q_M} H}) e^{-i(\gamma_{Q_M} - \gamma_W) H} U^\dagger e^{-i\gamma_W H}. \end{aligned}$$

Tracing out the QB degrees of freedom gives the reduced state of the ancilla:

$$\rho_a = \text{Tr}_b[\rho_f] = \frac{|e\rangle\langle e| + \chi(\gamma_{Q_M}, \gamma_W) |e\rangle\langle g| + \chi(\gamma_{Q_M}, \gamma_W)^* |g\rangle\langle e| + |g\rangle\langle g|}{2}, \quad (13)$$

where $\chi(\gamma_{Q_M}, \gamma_W)^*$ denotes the complex conjugate of $\chi(\gamma_{Q_M}, \gamma_W)$.

By measuring σ_x of the ancilla, one obtains the real part of the joint CF: $\text{Tr}[\sigma_x \rho_a] = \frac{\chi(\gamma_{Q_M}, \gamma_W) + \chi(\gamma_{Q_M}, \gamma_W)^*}{2}$, and by measuring σ_y , one obtains the imaginary part of the joint CF: $\text{Tr}[\sigma_y \rho_a] = \frac{i(\chi(\gamma_{Q_M}, \gamma_W) - \chi(\gamma_{Q_M}, \gamma_W)^*)}{2}$. Thus,

$$\chi(\gamma_{Q_M}, \gamma_W) = \text{Tr}[\sigma_x \rho_a] - i \text{Tr}[\sigma_y \rho_a].$$

Note that a measurement of σ_z reveals nothing about the joint CF, as the cumulants are encoded only in the off-diagonal elements of the ancilla state [Eq. (13)].

It must be emphasized that, unlike most of the literature, which focuses only on the statistics of work [104, 136, 137] or heat [138, 139], here we propose a method to measure the statistics of both heat and work. Furthermore, one could consider using two ancillas—one interacting with the QB during the first stroke to infer the marginal characteristic function of heat in Eq. (B4) and the other during the second stroke to infer the marginal characteristic function of work in Eq. (B5). However, we prefer using a single ancilla, as it also allows us to capture cumulants that account for correlations between stochastic heat Q_M and stochastic work W . Our circuit can be extended, e.g., to the quantum Otto cycle (four-stroke cycle) [4], and we believe it can be experimentally implemented using NMR techniques [140]. Other possible platforms [105] could also be suitable for experimental implementation. Finally, our circuit can be readily adapted to scenarios in which the measurement during the first stroke acts only on a subset of the QB or involves a weak, rather than projective, measurement.

D. Second cumulant expressions: Real part

Using Eq. (12), the second cumulants of heat and work are given by

$$\langle Q_M^2 \rangle_c = \text{Tr}[H^2 \rho_1] + \text{Tr}[H^2 \rho_0] - 2 \text{Tr}[H \Phi_M(H \rho_0)] - \langle Q_M \rangle^2 \quad (14)$$

and

$$\langle W^2 \rangle_c = \text{Tr}[H^2 \rho_2] + \text{Tr}[H^2 \rho_1] - 2 \text{Tr}[H U H \rho_1 U^\dagger] - \langle W \rangle^2.$$

It can be shown that the latter quantity generally has both real and imaginary parts (resulting from $\text{Tr}[H U H \rho_1 U^\dagger]$), while this is not the case for the second cumulant of Q_M . This is because $[\rho_0, H] = 0$, implying that $\langle Q_M^2 \rangle_c$ is real. In what follows, we use $\text{var}(Q_M) := \langle Q_M^2 \rangle_c$, where $\text{var}(X)$ denotes the variance of the stochastic variable X .

The real part of the second cumulant of work, denoted by $\text{Re}[\langle W^2 \rangle_c]$, measures work fluctuations [91,93–95]. Explicitly, we have

$$\text{var}(W) := \text{Re}[\langle W^2 \rangle_c] = \text{var}(U^\dagger H U - H)_{\rho_1} = \text{var}(H - U H U^\dagger)_{\rho_2}, \quad (15)$$

where $\text{var}(X')_\rho = \langle X'^2 \rangle_\rho - \langle X' \rangle_\rho^2$ and $\langle X'^p \rangle_\rho := \text{Tr}[X'^p \rho]$, with p a positive integer. In the literature, the operator $U^\dagger H U - H$ is known as the work operator [110,111]. More precisely, the average of this operator with respect to ρ_1 gives the average work, while the average of $(U^\dagger H U - H)^2$ gives the second moment of work.

E. Imaginary part of the second cumulant of work

In this section, we focus on the imaginary part of the second cumulant of work, denoted by $\text{Im}[\langle W^2 \rangle_c]$, which is given by

$$\text{Im}[\langle W^2 \rangle_c] = i \text{Tr}[[U^\dagger H U, H] \rho_1] = i \text{Tr}[[H, U H U^\dagger] \rho_2]. \quad (16)$$

It can also be expressed as $\text{Im}[\langle W^2 \rangle_c] = i \text{Tr}[[H, \rho_1] U^\dagger H U] = i \text{Tr}[[\rho_1, U^\dagger H U] H]$, and similar variants.

From, e.g., $\text{Tr}[[U^\dagger H U, H] \rho_1]$ in Eq. (16), one can see that the imaginary part is strongly influenced by the noncommutativity of H and $U^\dagger H U$ [91,93,94]. However, one may ask whether this expression admits alternative interpretations or can be related to thermodynamic quantities. In this paper, we answer both questions.

Defining $\tilde{H} = U^\dagger H U$, one can prove the following:

$$\text{Im}[\langle W^2 \rangle_c] = \frac{d}{d\epsilon} \text{Tr}[e^{-i\epsilon \tilde{H}} \rho_1 e^{i\epsilon \tilde{H}} H] |_{\epsilon=0}. \quad (17)$$

This can be interpreted as follows: (1) Prepare the QB in state ρ_1 , which is the postmeasurement state; (2) evolve ρ_1 under the unitary $e^{-i\epsilon \tilde{H}}$ generated by $\tilde{H}$; and (3) compute the average of the Hamiltonian H of the QB in the state $e^{-i\epsilon \tilde{H}} \rho_1 e^{i\epsilon \tilde{H}}$. Thus, $\text{Im}[\langle W^2 \rangle_c]$ can be interpreted as the linear response of the average energy $\text{Tr}[e^{-i\epsilon \tilde{H}} \rho_1 e^{i\epsilon \tilde{H}} H]$. Moreover, since the derivative of a constant is zero, we can equivalently write

$$\text{Im}[\langle W^2 \rangle_c] = \frac{d}{d\epsilon} \text{Tr}[(e^{-i\epsilon \tilde{H}} \rho_1 e^{i\epsilon \tilde{H}} - \rho_1) H] |_{\epsilon=0}, \quad (18)$$

interpreting $\text{Im}[\langle W^2 \rangle_c]$ as the linear response of the average work $\text{Tr}[e^{-i\epsilon \tilde{H}} \rho_1 e^{i\epsilon \tilde{H}} H] - \text{Tr}[\rho_1 H]$, since $e^{-i\epsilon \tilde{H}}$ is entropy preserving [3–5,114].

Now, one can show from Eq. (2) that $H = -T(\ln \rho_0 + \ln \text{Tr}[e^{-H/T}])$. Using this, one finds

$$\begin{aligned} \text{Im}[\langle W^2 \rangle_c] &= iT^2 \text{Tr}[\rho_1 [U^\dagger (\ln \rho_0) U, \ln \rho_0]] \\ &= iT^2 \text{Tr}[\rho_1 [\ln(U^\dagger \rho_0 U), \ln \rho_0]]. \end{aligned} \quad (19)$$

Here, we used $U^\dagger(\ln \rho_0)U = \ln(U^\dagger \rho_0 U)$. Furthermore, since $-\ln \rho$ is the quantum analog of the classical surprisal (i.e., the *quantum surprisal*), when $\rho = \rho_0$, $-\ln \rho_0$ can be interpreted as the thermal quantum surprisal. Therefore, $\text{Tr}[\rho_1[U^\dagger(\ln \rho_0)U, \ln \rho_0]]$ in Eq. (19) represents the average commutator between the thermal quantum surprisal and its rotated version in the state ρ_1 . In the theory of many-body quantum entanglement, $-\ln \rho$ is known as the modular Hamiltonian [141,142], whose average gives the von Neumann entropy. The variance of $-\ln \rho$ is also relevant in quantum information theory and is called the capacity of entanglement [141,143].

Finally, inspired by Eq. (18), one can show

$$\text{Im}[\langle W^2 \rangle_c] = \frac{d}{d\epsilon} \text{Tr}[-e^{i\epsilon T\bar{H}}(\ln \rho_0)e^{-i\epsilon T\bar{H}}\rho_1]|_{\epsilon=0}.$$

The quantity $\text{Tr}[-e^{i\epsilon T\bar{H}}(\ln \rho_0)e^{-i\epsilon T\bar{H}}\rho_1]$ represents the average of the rotated thermal quantum surprisal in ρ_1 , which coincides with the quantum cross entropy $S(\rho, \sigma) := -\text{Tr}[\rho \ln \sigma]$ [109]. To make the notation explicit, we define $\tilde{\rho}_0 := e^{i\epsilon T\bar{H}}\rho_0e^{-i\epsilon T\bar{H}}$. Using this, the imaginary part of the second cumulant of work can be written as the linear response of the quantum cross entropy, or of its change:

$$\text{Im}[\langle W^2 \rangle_c] = \frac{d}{d\epsilon} S(\rho_1, \tilde{\rho}_0)|_{\epsilon=0} = \frac{d}{d\epsilon} (S(\rho_1, \tilde{\rho}_0) - S(\rho_1, \rho_0))|_{\epsilon=0}. \quad (20)$$

To the best of our knowledge, this is the first study establishing a connection between the imaginary part of the second cumulant of work and quantum entropies, thereby providing an interpretation beyond the standard commutator-based viewpoint [108]. It should be emphasized that Eqs. (17), (18), and (20) demonstrate not only that the imaginary part is experimentally accessible, but also that they provide operational interpretations of the weak value [106–108]. Moreover, the imaginary part plays a central role in uncertainty relations and their extensions [144–146], since, as we see from Eq. (16), it is given (up to the imaginary unit i) by the average of the commutator of two Hermitian operators, underscoring the importance of explicitly accounting for it.

For more on the imaginary part, we refer the reader to Appendix D.

F. Second cumulant of the charging energy: Useful upper and lower bounds

Let us now return to the variance of Q_M . We can write

$$\begin{aligned} \text{var}(Q_M) &= \text{Tr}[(\Phi_M(H) - H)^2 \rho_0] - \text{Tr}[(\Phi_M(H) - H)\rho_0]^2 + \text{Tr}[H^2 \rho_1] - \text{Tr}[H\rho_1]^2 \\ &\quad - \text{Tr}[\Phi_M(H)^2 \rho_0] + \text{Tr}[\Phi_M(H)\rho_0]^2. \end{aligned} \quad (21)$$

To arrive at this result, we added and subtracted the term $\text{Tr}[\Phi_M(H)^2 \rho_0] + \text{Tr}[\Phi_M(H)\rho_0]^2$ in Eq. (14), and have used $\Phi_M(\cdot) = \Phi_M^\dagger(\cdot)$ [see Eq. (3)] since the measurement operators are Hermitian, and $\Phi_M^\dagger$ denotes the adjoint of Φ_M with respect to the Hilbert-Schmidt inner product, defined by $\text{Tr}[A \Phi_M(B)] = \text{Tr}[\Phi_M^\dagger(A)B]$ for all operators A, B on $\mathcal{H}_{db}$.

Equation (21) can be rewritten as

$$\text{var}(Q_M) = \text{var}(\Phi_M(H) - H)_{\rho_0} + \text{var}(H)_{\rho_1} - \text{var}(\Phi_M(H))_{\rho_0}. \quad (22)$$

This equation, written in terms of variances [or, more generally, Eq. (E3)], generalizes Eq. (15) to the case of nonunitary operations and shows that the first term alone is not sufficient; correction terms are needed, as captured by Eq. (21). More precisely, this means that if one defines a *heat operator* given by $\Phi_M^\dagger(H) - H$ (inspired by the *work operator*, i.e., $U^\dagger H U - H$) [110,111], the heat operator will only give the true average heat as in Eq. (5) and the first part of Eq. (21). However, the term

$$\text{Tr}[(\Phi_M(H^2) - \Phi_M(H)^2)\rho_0]$$

would not contribute. We will return to this quantity and show its usefulness. What is also important about Eq. (22) is that it holds even when the state ρ_0 does not commute with the Hamiltonian H ,

as explained in Appendix E. Also, it is valid, as we show in the same Appendix, for measurement channels that are characterized by Hermitian operators. When the operators are not Hermitian, one must use Eq. (E3).

It is easily seen that the terms in Eq. (22) are as follows: (1) $\text{var}(\Phi_M(H) - H)_{\rho_0}$ represents the variance of the heat operator $\Phi_M(H) - H$ with respect to the state ρ_0 . The operator $\Phi_M(H) - H$ is Hermitian, and its average in the state ρ_0 gives Eq. (5); (2) $\text{var}(H)_{\rho_1}$ represents the variance of H with respect to ρ_1 . More precisely, $\text{var}(H)_{\rho_1}$ measures fluctuations in the Hamiltonian of the battery in the postmeasurement state; (3) $\text{var}(\Phi_M(H))_{\rho_0}$ represents the variance of $\Phi_M(H)$ with respect to ρ_0 . Here, $\Phi_M(H)$ is the Heisenberg evolution of the operator H under the measurement channel Φ_M . Please note that while in $\text{var}(H)_{\rho_1}$ we are evolving the state of the QB, in $\text{var}(\Phi_M(H))_{\rho_0}$ we are evolving the Hamiltonian operator itself. To the best of our knowledge, the decomposition presented in Eq. (22), as well as its comparison with the *heat operator*, has not been discussed in the literature.

To further appreciate the importance of Eq. (22) (see Appendix E), note that, because of the decomposition given in Eq. (22), one can prove

$$0 \leq \text{LowB}_v \leq \text{var}(Q_M) \leq \text{UppB}_v \leq \text{UppB}_v', \quad (23)$$

where $\text{LowB}_v = \text{var}(\Phi_M(H) - H)_{\rho_0}$, $\text{UppB}_v = \text{var}(\Phi_M(H) - H)_{\rho_0} + \text{var}(H)_{\rho_1}$, and $\text{UppB}_v' = \text{var}(\Phi_M(H) - H)_{\rho_0} + \text{Tr}[\Phi_M(H^2)\rho_0]$. Thus, we see that the variance of the heat operator provides a lower bound on the fluctuations in heat or charging energy. Furthermore, since $\text{LowB}_v' = \text{var}(H)_{\rho_1} - \text{var}(\Phi_M(H))_{\rho_0} \geq 0$, as we prove in Appendix E, one can thus see that $\text{LowB}_v' \leq \text{var}(Q_M)$. However, we note that $\text{LowB}_v - \text{LowB}_v'$ can be negative or positive; see, for example, the upper panels of Figs. 4, 5, and 7, and Eq. (K3), where $\text{LowB}_v - \text{LowB}_v'$ is nonpositive, while in Eq. (L3) $\text{LowB}_v - \text{LowB}_v'$, i.e., for the case of weak measurement, can be positive or negative depending on the parameters. What we do not know is whether $\text{LowB}_v - \text{LowB}_v'$ is always nonpositive in the case of nonselective quantum projective measurements. Numerical plots suggest that it is, but we could not prove or disprove this analytically.

From the above discussion, it then follows naturally that the square of the heat reliability, defined as the ratio of the square of the heat average to the heat fluctuations, i.e., $\frac{\langle Q_M \rangle^2}{\text{var}(Q_M)}$ (also called the signal-to-noise ratio of the charging energy), can be bounded from above and below as

$$0 \leq \text{LowB}_r' \leq \text{LowB}_r \leq \frac{\langle Q_M \rangle^2}{\text{var}(Q_M)} \leq \text{UppB}_r, \quad (24)$$

where $\text{LowB}_r' = \frac{\langle Q_M \rangle^2}{\text{UppB}_v'}$, $\text{LowB}_r = \frac{\langle Q_M \rangle^2}{\text{UppB}_v}$, and $\text{UppB}_r = \frac{\langle Q_M \rangle^2}{\text{LowB}_v}$. This is valid independently of the (unital) quantum measurement channel and the Hamiltonian of the QB. Please note that one can also prove that $\frac{\langle Q_M \rangle^2}{\text{var}(Q_M)} \leq \text{UppB}_r' = \frac{\langle Q_M \rangle^2}{\text{LowB}_v'}$. Again, it remains unclear whether UppB_r' always provides an upper bound on heat reliability that is at most as large as that obtained from UppB_r for arbitrary nonselective quantum projective measurements. Numerical evidence supports this conclusion, although a formal proof or disproof is left for future work.

G. Ergotropy and antiergotropy regimes and the imaginary part of the second cumulant of work

The spectral decomposition of the state ρ_1 [i.e., Eq. (4)] is given by $\rho_1 = \sum_{k=1}^{d_b} p_k |\Pi_k\rangle\langle\Pi_k|$, with $p_k = \langle\Pi_k|\rho_0|\Pi_k\rangle = \langle\Pi_k|\rho_1|\Pi_k\rangle$. Assuming that the eigenvalues of ρ_1 are arranged in descending order, i.e., $p_k \geq p_{k+1}$, while those of H [Eq. (1)] are in increasing order, i.e., $E_s \leq E_{s+1}$, it was shown in Ref. [114] that there exists a unitary operator that extracts the ergotropy, given by

$$U_{\text{erg}} := \sum_{t=1}^{d_b} |E_t^{(2)}\rangle\langle\Pi_t|. \quad (25)$$

This unitary rearranges the populations of ρ_1 such that the final state is the passive state ρ_p , where $\rho_p = U_{\text{erg}}\rho_1 U_{\text{erg}}^\dagger = \sum_{s=1}^{d_b} p_s |E_s^{(2)}\rangle\langle E_s^{(2)}|$ and $p_s \geq p_{s+1}$.

For unitaries that maximize the average work—i.e., Eq. (25)—one can show that the imaginary part of the second cumulant of work vanishes. More precisely, since $[\rho_1, U_{\text{erg}}^\dagger H U_{\text{erg}}] = 0$, we find

$$\text{Im}[\langle W^2 \rangle_c] = 0.$$

Thus, this implies that the unitary that extracts the ergotropy also nullifies the imaginary part of the second cumulant of work.

Now, consider the converse question: does a vanishing imaginary part necessarily imply that maximum work has been extracted? The answer is no—a system as simple as a single qubit is sufficient to demonstrate this.

Example 1. Consider a qubit with Hamiltonian $H_l := \sigma_z$ prepared in the state $\rho_l := (\mathbb{1}_2 + r_x \sigma_x)/2$. The state is evolved by the unitary $U_l := e^{-i\theta_l \sigma_y}$, i.e., a rotation about the y axis. For this example, the imaginary part of the second cumulant of work satisfies

$$i \text{Tr}[[U_l^\dagger H_l U_l, H_l] \rho_l] = 0.$$

One can show that ergotropy is achieved by applying a Hadamard gate followed by a bit-flip gate on ρ_l . For our example, this corresponds to $\theta_l = \pi/4 + 2\pi n$, where $n \in \mathbb{Z}$.

More precisely, given that $r_x > 0$ and $\theta_l \in (0, \pi/2)$, the average work extracted is given by $\text{Tr}[H_l(\rho_l - U_l \rho_l U_l^\dagger)] = r_x \sin(2\theta_l)$, which equals the ergotropy, i.e., r_x , when $\theta_l = \pi/4$. This proves that even when the work extracted is less than the ergotropy, the imaginary part of the second cumulant of work can be zero.

Let us delve deeper to understand why the imaginary part of the second cumulant of work vanishes in the ergotropy regime. Marginalizing over i in Eq. (8), it follows that the quasiprobability describing work at the trajectory level (see Refs. [78–91, 93–100]) is given by

$$q_{jk} := \sum_i q_{ijk} = \text{Tr}[|E_k^{(2)}\rangle\langle E_k^{(2)}| U |E_j^{(1)}\rangle\langle E_j^{(1)}| \rho_l U^\dagger]. \quad (26)$$

This equation, when written as $\text{Tr}[U^\dagger |E_k^{(2)}\rangle\langle E_k^{(2)}| U |E_j^{(1)}\rangle\langle E_j^{(1)}| \rho_l]$, shows that q_{jk} can be interpreted as the Kirkwood-Dirac distribution of the state ρ_l with respect to the eigenbases $\{|E_j^{(1)}\rangle\}$ and $\{|E_k^{(2)}\rangle\}$. In the literature, see e.g. Ref. [83], people focus on understanding when q_{jk} is positive or negative (or even complex), and in this case they call the state ρ_l , when q_{jk} is non-negative for all $j, k \in [1, d_b]$, a Kirkwood-Dirac classical state [83]. Though some studies suggest the name Kirkwood-Dirac positive state is better and more logical, see e.g. Ref. [147].

Now, plugging Eq. (25) into Eq. (26), one can show that

$$q_{jk} = p_k |\langle \Pi_k | E_j^{(1)} \rangle|^2 \geq 0. \quad (27)$$

Thus, we find that in the regime of maximum work extraction, not only is the imaginary part of the second cumulant of work zero, but the marginal KDQ of work also becomes a proper probability. This shows that the marginal quasiprobability distribution of work given in Eq. (B3) becomes a proper PD in the ergotropy regime.

Equation (27) can be rewritten as

$$q_{jk} = \text{Tr}[|E_j^{(1)}\rangle\langle E_j^{(1)}| U_{\text{erg}}^\dagger |E_k^{(2)}\rangle\langle E_k^{(2)}| U_{\text{erg}}] \text{Tr}[|E_k^{(2)}\rangle\langle E_k^{(2)}| \rho_p]. \quad (28)$$

The interpretation of this equation is as follows: first, project the state ρ_p onto the eigenstate $|E_k^{(2)}\rangle\langle E_k^{(2)}|$ of H ; then, evolve the resulting postmeasurement state $|E_k^{(2)}\rangle\langle E_k^{(2)}|$ under $U_{\text{erg}}^\dagger$; and finally, project onto $|E_j^{(1)}\rangle\langle E_j^{(1)}|$. This procedure corresponds exactly to the TPM scheme. This shows that even though (1) $[\rho_1, H] \neq 0$, (2) $[\rho_1, U_{\text{erg}}] \neq 0$, and (3) $[U_{\text{erg}}, H] \neq 0$, one can still use the TPM scheme in the ergotropy regime; *however*, one must do it from the passive state ρ_p toward the state ρ_l .

Remark 3. This result shows that, although the KDQ distribution is required to obtain the cumulants of work in the ergotropy regime in the presence of initial quantum coherence, the KDQ itself is real and also positive. Thus, Eq. (27) aligns with the results proven in Ref. [83], which show that noncommutation alone is not sufficient to guarantee nonclassicality (i.e., negative or even complex values of the KDQ).

Furthermore, if we consider $\rho_1 \rightarrow \rho_p$ (work extraction) as the forward process, then we can define $\rho_p \rightarrow \rho_1$ (energy injection) as the backward process. Our result shows that the statistics of the work extracted in the forward process (derived from the KDQ distribution) are equivalent to the statistics of the energy injected in the backward process (derived from the TPM scheme). The only difference is that the random variables describing these quantities have opposite signs.

Appendixes G–I are recommended for more details.

H. (Physical interpretation of) fluctuations in the battery capacity operator

Since all thermodynamic quantities generally exhibit nonzero fluctuations due to thermal and quantum effects, an important question is how to estimate the fluctuations in battery capacity. First, one can show that the capacity of the QB can be expressed as $\text{Tr}[\rho_1(U_{\text{aerg}}^\dagger H U_{\text{aerg}} - U_{\text{erg}}^\dagger H U_{\text{erg}})]$. This is simply the expectation value of the operator

$$O := U_{\text{aerg}}^\dagger H U_{\text{aerg}} - U_{\text{erg}}^\dagger H U_{\text{erg}}, \quad (29)$$

in the state ρ_1 . Inspired by the notion of the *work operator* [110, 111] [see, e.g., Eq. (15)], we define the variance of the battery capacity as the variance of the operator O (which we term the battery capacity operator) in the state ρ_1 , i.e.,

$$\text{var}(\mathcal{C}_{(\rho_1; H)}) := \text{var}(O)_{\rho_1}.$$

Similarly to the property $\mathcal{C}(\rho_1; H) = \mathcal{C}(V \rho_1 V^\dagger; H)$ [113], one can show that $\text{var}(\mathcal{C}_{(\rho_1; H)}) = \text{var}(\mathcal{C}_{(V \rho_1 V^\dagger; H)})$, for any unitary operator V . The reason is that when we evolve ρ_1 to $V \rho_1 V^\dagger$, the unitaries U_{erg} and U_{aerg} become, respectively, $U_{\text{erg}} V^\dagger$ and $U_{\text{aerg}} V^\dagger$; hence, the first and second moments of O are not affected by V .

Defining the ergotropy and antiergotropy operators (whose averages yield, respectively, the ergotropy and antiergotropy) as

$$O_{\text{erg}} := H - U_{\text{erg}}^\dagger H U_{\text{erg}}, \quad O_{\text{aerg}} := H - U_{\text{aerg}}^\dagger H U_{\text{aerg}},$$

so that $O = O_{\text{erg}} - O_{\text{aerg}}$, the variance of the battery capacity operator can be expressed as

$$\text{var}(\mathcal{C}_{(\rho_1; H)}) = \text{var}(O_{\text{erg}})_{\rho_1} + \text{var}(O_{\text{aerg}})_{\rho_1} - 2\text{cov}(O_{\text{erg}}, O_{\text{aerg}})_{\rho_1}, \quad (30)$$

where $\text{cov}(O_{\text{erg}}, O_{\text{aerg}})_{\rho_1}$ denotes the quantum covariance where $\text{cov}(O_{\text{erg}}, O_{\text{aerg}})_{\rho_1} = \frac{1}{2} \text{Tr}[\{O_{\text{erg}} - \langle W \rangle_{\text{max}}, O_{\text{aerg}} - \langle W \rangle_{\text{min}}\} \rho_1]$, with $\{A, B\} = AB + BA$ the anticommutator [93]. From Eq. (30), we see that the fluctuations of the battery capacity operator are given by the sum of the variances of the ergotropy and antiergotropy operators, corrected by $-2 \text{cov}(O_{\text{erg}}, O_{\text{aerg}})_{\rho_1}$, where $\text{cov}(O_{\text{erg}}, O_{\text{aerg}})_{\rho_1}$ accounts for correlations between energy extraction and energy injection.

From Eq. (29), when the QB is in the active (passive) state, the battery capacity reduces to the ergotropy (minus the antiergotropy); i.e., one can only extract (inject) energy from (into) the QB. For states intermediate between active and passive, both ergotropy and antiergotropy contribute, with the dominant contribution determined by the state of the QB. Specifically, if the QB is energetic enough to deliver more energy to the outside than it can absorb, the capacity is primarily influenced by the ergotropy, since extraction dominates over injection.

We now turn to the variance of the battery capacity operator. First, one can prove that

$$\text{var}(\mathcal{C}_{(\rho_1; H)}) = \text{var}(O_1)_{\rho_p} = \text{var}(O_2)_{\rho_a}, \quad (31)$$

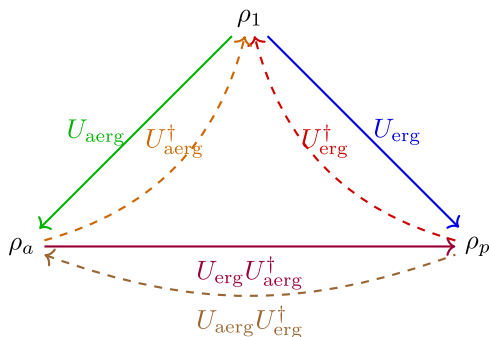


FIG. 3. Triangle diagram showing that charging from ρ_1 to ρ_a occurs via U_{aerg} , discharging from ρ_1 to ρ_p occurs via U_{erg} , and discharging from ρ_a to ρ_p occurs via $U_{\text{erg}} U_{\text{aerg}}^\dagger$. Dashed arrows indicate reverse evolution.

where

$$O_1 = U_{\text{erg}} U_{\text{aerg}}^\dagger H U_{\text{aerg}} U_{\text{erg}}^\dagger - H, \quad O_2 = H - U_{\text{aerg}} U_{\text{erg}}^\dagger H U_{\text{erg}} U_{\text{aerg}}^\dagger,$$

and $\mathcal{C}(\rho_1; H) = \text{Tr}[O_1 \rho_p] = \text{Tr}[O_2 \rho_a]$. Thus, from Eq. (31) we see that if the QB is prepared in the passive state ρ_p and evolved via $U_{\text{aerg}} U_{\text{erg}}^\dagger$ (see Fig. 3), the resulting variance of O_1 in state ρ_p captures the fluctuations of work during the charging process. Similarly, evolving the active state ρ_a to the passive state via $U_{\text{erg}} U_{\text{aerg}}^\dagger$ (see Fig. 3) allows the variance of O_2 in state ρ_a to quantify the fluctuations of work extracted during discharging (see Appendix J).

IV. CONCLUSIONS AND OUTLOOKS

In this paper, we focused on fluctuations in a three-stroke quantum cycle [148–151] based on quantum measurement [76,77,122]. This cycle, as we have shown and discussed, is beneficial not only for understanding the thermodynamics of quantum engines but also for QBs.

First, we presented how to compute cumulants of work and heat based on the KDQ distribution, a tool that has recently gained increasing interest in the quantum thermodynamics community due to its remarkable properties. The cumulants derived from the KDQ distribution can be experimentally inferred using a qubit ancilla, as illustrated in Fig. 2.

Furthermore, we proposed two interpretations of the imaginary part of the second cumulant of work: as the linear response of (i) an average work defined in the main text and (ii) the quantum cross entropy. This question regarding the interpretation of the imaginary part of the second cumulant of work had already been pointed out in the literature [87], which we address in this paper.

By manipulating the expression in Eq. (14), we were able to prove two upper and two lower bounds on fluctuations in the charging energy, from which bounds on the heat reliability follow directly. In the literature, the negativity and complex parts of the KDQ are usually associated with advantages in work extraction [91,93] and metrology [152,153]. In this work, we showed that in the ergotropy regime the marginal KDQ distribution of work becomes a proper probability distribution. This means that the exotic properties of the KDQ (i.e., taking negative and complex values) do not appear in the regime of maximum work extraction.

Furthermore, we proposed an operator to compute fluctuations in the battery capacity, which we call the battery capacity operator. We proved that the variance of this operator can be interpreted as a measure of work fluctuations. For further results, i.e., applications of the results presented in the main text and their interpretations, see Appendixes K–M.

Overall, we believe that our work will be useful not only for quantum batteries and quantum engines based on quantum measurements [45,48,122,154–160], but also for quantum refrigerators [161]. From an experimental perspective, our results are relevant in this direction, as they describe QBs under both unitary and nonunitary operations.

However, many open questions remain. First, it is important to connect these results with thermodynamic uncertainty relations [25], which would help us understand the trade-off between reliability and dissipation. Our results actually highlight that many of the uncertainty-relation bounds [162–166] developed in the literature can be applied to the variances considered in this work. Second, it would be important to consider how quantum correlations [121,167,168] could influence these results. Third, one could consider adding strokes between charging and discharging to study the battery’s capacity to store energy over longer times. For example, one may consider the QB to be subjected to quantum noise between charging and discharging strokes, such as a damping channel [123,125]. Fourth, it would be interesting to apply our results to relativistic QBs [169], where fluctuations are still largely neglected. Fifth, in the case of nonselective quantum projective measurements, it is also worth considering whether the average of the Kadison variance in the premeasurement state, i.e., the average of $\Phi_M(H^2) - \Phi_M(H)^2$ in the state ρ_0 , always provides a lower bound on heat fluctuations that is at least as tight as the variance of the heat operator $\Phi_M(H) - H$ evaluated in the same state. Sixth, although we proved that Eq. (26) is non-negative in the ergotropy regime independently of the considered measurement, it would be interesting to consider the case of local unitaries [72,131] and see whether Eq. (26) is still non-negative. Finally, we believe that our study may also be useful for understanding quantum energy teleportation, which is also based on quantum measurements [170], from the perspective of fluctuations.

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

APPENDIX A: ILLUSTRATION OF THE 3SQC WITH AN EXAMPLE

First, note that our 3SQC, particularly when quantum noise is introduced between the charging and discharging strokes, strongly resembles a quantum communication protocol [123]. One can view the setup as involving two agents, Alice and Bob. First, Alice charges the QB using a quantum projective measurement, an operation analogous to encoding information. She then transmits the QB through a channel that may, in principle, introduce environmental noise. Upon receiving the QB, Bob applies suitable unitary operations to extract useful energy, which corresponds to the decoding process. Finally, Bob may send the QB back through a thermalizing channel, allowing Alice to begin a new cycle.

Second, if one considers applying a unitary in the second stroke depending on the measurement outcome during the first stroke, i.e., a selective measurement, our 3SQC closely resembles quantum energy teleportation [170], highlighting the richness of our cycle.

Now, consider a QB with an equally spaced Hamiltonian, i.e., $H = \sum_{i=1}^{d_B} (i-1)\omega|i-1\rangle\langle i-1|$, where $|0\rangle\langle 0|$ and $|d_B-1\rangle\langle d_B-1|$ are, respectively, the ground state and the highest excited state of H . Assume that the QB is prepared in the ground state, which corresponds to a thermal state in the zero-temperature limit. For the quantum projective measurement, we consider a nontrivial action only on the ground state and the highest excited state. More precisely, the projectors are chosen as $\Pi_0 = \frac{(|0\rangle+|d_B-1\rangle)(\langle 0|+\langle d_B-1|)}{2}$, $\Pi_s = |s\rangle\langle s|$ for $s = 1, \dots, d_B-2$, and $\Pi_{d_B-1} = \frac{(|0\rangle-|d_B-1\rangle)(\langle 0|-\langle d_B-1|)}{2}$. A direct calculation shows that the charging energy, the ergotropy, and the efficiency in the ergotropy regime are given by $\langle Q_M \rangle = \omega(d_B-1)/2$, $\langle W \rangle_{\max} = \omega(d_B-2)/2$, and $\eta_{\max} = 1 - \frac{1}{d_B-1}$. The unitary that achieves the ergotropy is $U_{\text{erg}} = (|1\rangle\langle d_B-1| + \text{H.c.}) + \mathbb{1}_{\text{rest}}$, where $\mathbb{1}_{\text{rest}}$ denotes

the identity operator acting on the orthogonal complement of the subspace involved in the first term and H.c. stands for Hermitian conjugate. This example shows that the efficiency vanishes for $d_B = 2$, while it approaches unity when $d_B \gg 1$. The former case is expected, since for $d_B = 2$ the postmeasurement state of the qubit is maximally mixed, and therefore no work can be extracted using unitary operations.

On the other hand, if one considers the energy injection during the second stroke, the antiertropy, i.e., $\langle W \rangle_{\min}$, is achieved by the unitary $U_{\text{aerg}} = (|d_B - 2\rangle\langle 0| + \text{H.c.}) + \mathbb{1}_{\text{rest}}$, and is equal to $\omega(2 - d_B)/2$. Consequently, the battery capacity $\mathcal{C}(\rho_1; H)$ is given by $\omega(d_B - 2)$.

If one were to consider replacing the quantum measurement channel in Eq. (3) with a unitary operation given by $V = (|d_B - 1\rangle\langle 0| + \text{H.c.}) + \mathbb{1}_{\text{rest}}$, one can easily show that, in this case, the efficiency equals unity when choosing $U = (|0\rangle\langle d_B - 1| + \text{H.c.}) + \mathbb{1}_{\text{rest}}$, since $V^\dagger = U$. Furthermore, in this case, the battery capacity is equal to $\omega(d_B - 1)$, which corresponds to the energy difference between the highest excited state and the ground state of the QB. This situation of charging through a unitary operation is not particularly interesting, since one can achieve unit efficiency; the use of quantum measurement, on the other hand, allows us to explore the fundamental limits of energy injection and extraction beyond unitary operations, and thus makes the cycle more realistic.

APPENDIX B: PROOF OF EQ. (9) AND THE MARGINAL (QUASI)PROBABILITY DISTRIBUTIONS OF THE JOINT KDQ DISTRIBUTION $P(Q_M, W)$

Let us now briefly prove Eq. (9).

Proof. Inspired by Refs. [24,84,92–95,101,102], we define the joint PD of Q_M and W with respect to the joint KDQ, also known as the joint quasiprobability distribution:

$$P(Q_M, W) = \sum_{ijk} q_{ijk} \delta(Q_M - (E_j^{(1)} - E_i^{(0)})) \delta(W - (E_k^{(2)} - E_j^{(1)})). \quad (\text{B1})$$

Taking the Fourier transform, i.e.,

$$\chi(\gamma_{Q_M}, \gamma_W) = \int P(Q_M, W) e^{i\gamma_{Q_M} Q_M} e^{i\gamma_W W} dQ_M dW,$$

we obtain

$$\chi(\gamma_{Q_M}, \gamma_W) = \sum_{ijk} q_{ijk} e^{i\gamma_{Q_M} (E_j^{(1)} - E_i^{(0)})} e^{i\gamma_W (E_k^{(2)} - E_j^{(1)})}.$$

Using Eq. (8), one can then derive Eq. (9). ■

Remark 4. The marginal (quasi)probability distributions of the joint KDQ distribution $P(Q_M, W)$ are given by

$$P(Q_M) = \sum_{ij} q_{ij} \delta(Q_M - (E_j^{(1)} - E_i^{(0)})) \quad (\text{B2})$$

and

$$P(W) = \sum_{jk} q_{jk} \delta(W - (E_k^{(2)} - E_j^{(1)})), \quad (\text{B3})$$

where $q_{ij} = \sum_k q_{ijk}$ and $q_{jk} = \sum_i q_{ijk}$. Moreover, one can prove that while $P(Q_M)$ is a proper PD since $q_{ij} \geq 0$, $P(W)$ is, in general, a quasiprobability distribution, as q_{jk} can be negative or even complex. Now it is clear that the marginal CFs of heat and work are given, respectively, by

$$\chi(\gamma_{Q_M}) = \text{Tr}[e^{i\gamma_{Q_M} H} \Phi_M(e^{-i\gamma_{Q_M} H} \rho_0)] \quad (\text{B4})$$

and

$$\chi(\gamma_W) = \text{Tr}[e^{i\gamma_W H} U e^{-i\gamma_W H} \rho_1 U^\dagger]. \quad (\text{B5})$$

APPENDIX C: SECOND LAW OF THERMODYNAMICS

In the 3SQC, we focus on initial states in thermal equilibrium for two main reasons: first, these states are considered *free states* within resource theories [18,20]; second, starting, for example, from the ground state (thermal state at zero temperature) can be useful to find better ways to charge quantum systems.

For example, in Refs. [121,122], the authors considered the initial state of the QB, made of two qubits, to be out of equilibrium. To prepare this state, they included the coupling between the qubits, with each qubit weakly interacting with its own thermal bath. They showed that, in this case, the steady state of the QB would deviate from a thermal state. When the coupling between the qubits is turned off, each qubit is individually in a thermal state. However, the joint state of the two qubits can still be out of equilibrium, in the sense that work can be extracted from it, which is not possible for a global thermal state.

As shown in the plots in Refs. [121,122], if the initial state is out of equilibrium—though not necessarily noncommuting with the Hamiltonian—the average work extracted can exceed the heat; that is, the ergotropy can exceed the charging energy. This scenario does not occur in our setting, since the cycle always starts from a thermal state, enabling a meaningful definition and analysis of the engine's efficiency. Consequently, the efficiency will always be bounded by 1.

To understand this, we must mention that, since $-\langle W \rangle = \langle Q_M \rangle + \langle Q_C \rangle$ from the first law of thermodynamics, the condition for $-\langle W \rangle$ to exceed $\langle Q_M \rangle$ is that $\langle Q_C \rangle$ must be positive [122]. When considering an initial equilibrium state, i.e., Eq. (2), the measurement channel in Eq. (3) (or, more generally, any unital channel), one can prove the following result:

$$\langle Q_C \rangle \leq 0.$$

Proof. First, let us define $\Sigma_{ik} := (E_k^{(2)} - E_i^{(0)})/T$ and $p_{ik} := \text{Tr}[|E_k^{(2)}\rangle\langle E_k^{(2)}|U\Phi_M(|E_i^{(0)}\rangle\langle E_i^{(0)}|\rho_0)U^\dagger]$. We can now easily prove that $\sum_{ik} p_{ik} e^{-\Sigma_{ik}} = 1$, where we use the fact that the channel in Eq. (3) is unital, i.e., $\Phi_M(\mathbb{1}_{d_b}) = \mathbb{1}_{d_b}$. Using Jensen's inequality, i.e.,

$$\sum_{ik} p_{ik} e^{-\Sigma_{ik}} = 1 \geq e^{-\sum_{ik} p_{ik} \Sigma_{ik}},$$

one obtains $\sum_{ik} p_{ik} \Sigma_{ik} = -\langle Q_C \rangle/T \geq 0$; thus, $\langle Q_C \rangle \leq 0$ [95]. This result is a manifestation of the second law of thermodynamics: heat can only flow from the QB to the bath, and $-\langle Q_C \rangle/T$ is nothing but the entropy generated in the bath. Thus, $-\langle W \rangle \leq \langle Q_M \rangle$, and the efficiency η is therefore bounded by unity. ■

Actually, there is another simpler proof. First, note that the average heat released to the bath can also be written in terms of entropies [95]. More precisely, one can prove that

$$\langle Q_C \rangle = -T(S(\rho_2) - S(\rho_0) + S(\rho_2||\rho_0)), \quad (\text{C1})$$

where $S(\rho) = -\text{Tr}[\rho \ln \rho]$ and $S(\rho||\sigma) = \text{Tr}[\rho(\ln \rho - \ln \sigma)]$ are, respectively, the von Neumann entropy and the Kullback-Leibler divergence, and $S(\rho_2) - S(\rho_0) = S(\rho_1) - S(\rho_0) \geq 0$ measures the entropy increase in the QB state caused by the nonselective quantum projective measurement, while $S(\rho_2||\rho_0)$ measures the deviation of the state ρ_2 from the equilibrium state ρ_0 . The entropic decomposition in Eq. (C1) proves that $\langle Q_C \rangle \leq 0$.

Analogously, note that one can easily prove that

$$\langle Q_M \rangle = T(S(\rho_1) - S(\rho_0) + S(\rho_1||\rho_0)) \geq 0.$$

This means that a nonselective quantum projective measurement, or more generally any unital measurement channel, cannot decrease the energy of the quantum battery; it can only increase it or keep it unchanged. Overall, since $\rho_2 = U\rho_1 U^\dagger$, it follows that $S(\rho_2) = S(\rho_1)$; therefore, one can see that

$$\langle W \rangle = T(S(\rho_2||\rho_0) - S(\rho_1||\rho_0)),$$

where the latter can be either positive or negative. Thus, our 3SQC, under unital measurement channels, e.g., the one in Eq. (3), can only function as a heat engine—where work extraction is possible in the second stroke (2)—or as an accelerator, where during the second stroke the battery energy is increased rather than decreased. The latter can be understood as follows: energy is supplied to the QB in both the first and second strokes, and all of this energy is ultimately dissipated to the bath, and thus wasted.

Finally, we must mention that

$$\tilde{P}(Q_M, W) = e^{(W+Q_M)/T} P(-Q_M, -W), \quad (\text{C2})$$

where $(W + Q_M)/T = \Sigma_{ik}$, $\tilde{P}(Q_M, W) = \sum_{kji} \tilde{q}_{kji} \delta(Q_M + (E_j^{(1)} - E_i^{(0)})) \delta(W + (E_k^{(2)} - E_j^{(1)}))$, and $\tilde{q}_{kji} = \text{Tr}[\Phi_M^\dagger (U^\dagger(\rho_0|E_k^{(2)}\rangle\langle E_k^{(2)}|)U|E_j^{(1)}\rangle\langle E_j^{(1)}|)|E_i^{(0)}\rangle\langle E_i^{(0)}|]$. Equation (C2) can be interpreted as a fluctuation theorem [23,24,31,44,95] for joint quasiprobability distributions, where the backward process is governed by $U^\dagger$ followed by $\Phi_M^\dagger$, and the initial state of the backward cycle is the thermal state ρ_0 [44]. From Eq. (C2), it follows that the joint CFs satisfy $\tilde{\chi}(\gamma_{Q_M}, \gamma_W) = \chi(-\gamma_{Q_M} + i/T, -\gamma_W + i/T)$, where $\tilde{\chi}(\gamma_{Q_M}, \gamma_W) = \int \tilde{P}(Q_M, W) e^{i\gamma_{Q_M} Q_M} e^{i\gamma_W W} dQ_M dW$. However, it remains an open question whether Ref. [31] can be used to derive a thermodynamic uncertainty relation. This question is motivated by the fact that, for our 3SQC, we have

$$\tilde{P}(\phi, \Sigma_{ik}) = e^{\Sigma_{ik}} P(-\phi, -\Sigma_{ik}),$$

where $\phi = W, Q_M$.

APPENDIX D: LOWER BOUNDS ON WORK VARIANCE IN TERMS OF THE IMAGINARY PART OF THE SECOND CUMULANT OF WORK

First, note that the imaginary part of the second cumulant of work satisfies

$$\text{Im}[\langle W^2 \rangle_c] = i \text{Tr}[[U^\dagger H U - H, H] \rho_1] = \frac{i}{2} \text{Tr}[[U^\dagger H U - H, H + U^\dagger H U] \rho_1].$$

Using the Robertson-Schrödinger uncertainty relation [144,145], which states that for two Hermitian operators A and B and an arbitrary state ρ' ,

$$\frac{|\text{Tr}[\rho' [A, B]]|^2}{4} \leq \text{var}(A)_{\rho'} \text{var}(B)_{\rho'},$$

one can show that

$$|\text{Im}[\langle W^2 \rangle_c]|^2 \leq \text{var}(W) \text{var}(U^\dagger H U + H)_{\rho_1}.$$

This provides a lower bound on work fluctuations:

$$\text{Re}[\langle W^2 \rangle_c] \geq \frac{|\text{Im}[\langle W^2 \rangle_c]|^2}{\text{var}(U^\dagger H U + H)_{\rho_1}}. \quad (\text{D1})$$

Thus, the imaginary part can be used to develop a lower bound on fluctuations in work. Equation (D1) has not appeared previously in the literature.

One can go further and prove that [145]

$$\text{Re}[\langle W^2 \rangle_c] \geq \frac{|\text{Im}[\langle W^2 \rangle_c]|^2}{\text{var}(U^\dagger H U + H)_{\rho_1}} + \frac{|\text{Tr}[\rho_1 \{A - \text{Tr}[A \rho_1], B - \text{Tr}[B \rho_1]\}/2]|^2}{\text{var}(U^\dagger H U + H)_{\rho_1}}, \quad (\text{D2})$$

where $A = U^\dagger H U - H$ and $B = U^\dagger H U + H$, with $\{A, B\} = AB + BA$ the anticommutator. However, testing the bound in Eqs. (D1) and (D2) is left for future work.

Finally, note that the imaginary part of the second cumulant of work can also be written as

$$\text{Im}[\langle W^2 \rangle_c] = \frac{d}{d\epsilon} S(\rho_1 \| \tilde{\rho}_0) \Big|_{\epsilon=0} = \frac{d}{d\epsilon} (S(\rho_1 \| \tilde{\rho}_0) - S(\rho_1 \| \rho_0)) \Big|_{\epsilon=0}, \quad (\text{D3})$$

where (see Appendix C) $S(\rho_1 \| \rho_0)$ is the Kullback-Leibler divergence.

APPENDIX E: PROOF OF EQ. (23) AND GENERALIZATION OF EQ. (22) TO ARBITRARY COMPLETELY POSITIVE AND TRACE-PRESERVING MAPS

1. Proof of Eq. (23)

Proof. First, note that, because of our decomposition in Eq. (22), and since $\text{var}(\Phi_M(H) - H)_{\rho_0}$, $\text{var}(H)_{\rho_1}$, and $\text{var}(\Phi_M(H))_{\rho_0}$ are all non-negative, it follows that

$$\text{var}(Q_M) \leq \text{var}(\Phi_M(H) - H)_{\rho_0} + \text{var}(H)_{\rho_1}.$$

Furthermore, note that since H is Hermitian and Φ_M is a completely positive and unital map—i.e., $\text{id}_n \otimes \Phi_M$ maps positive operators to positive operators on any extended Hilbert space and preserves the identity operator, where id_n stands for the identity map—the Kadison-Schwarz inequality [112] implies that

$$\Phi_M(H^2) \geq \Phi_M(H)^2.$$

The difference $\Phi_M(H^2) - \Phi_M(H)^2$ can be referred to as the *Kadison variance* [112,171]. Furthermore, since ρ_0 is positive semidefinite (where the latter means that $\langle \psi | \rho_0 | \psi \rangle \geq 0$ for all $|\psi\rangle \in \mathcal{H}_d$), it follows that

$$\text{Tr}[(\Phi_M(H^2) - \Phi_M(H)^2)\rho_0] \geq 0, \quad (\text{E1})$$

thus obtaining the lower bound in Eq. (23). Therefore, we have shown that the variance of the charging energy can be lower bounded by the variance of the heat operator $\Phi_M(H) - H$. Please note that, as a direct consequence of Eq. (E1), one can easily see that $\text{var}(H)_{\rho_1} \geq \text{var}(\Phi_M(H))_{\rho_0}$, and thus $\text{var}(Q_M) \geq \text{var}(H)_{\rho_1} - \text{var}(\Phi_M(H))_{\rho_0}$.

Now, from the fact that both $\Phi_M(H^2)$ and $\Phi_M(H)^2$ are positive semidefinite, i.e., they are ≥ 0 , one can easily prove that

$$\text{var}(Q_M) \leq \text{UppB}_v \leq \text{var}(\Phi_M(H) - H)_{\rho_0} + \text{Tr}[\Phi_M(H^2)\rho_0]. \quad (\text{E2})$$

To stay self-contained, let us prove that $\Phi_M(H^2)$ and $\Phi_M(H)^2$ are positive semidefinite. First, note that from Eq. (1) we have $H^2 = \sum_{i=1}^{d_b} E_i^2 \Pi_{E_i} \geq 0$, and due to the fact that Φ_M maps positive operators to positive operators, one can clearly see that $\Phi_M(H^2) \geq 0$. Furthermore, one can prove that $\Phi_M(H)^2 = \sum_{l=1}^{d_b} \Pi_l H \Pi_l \sum_{s=1}^{d_b} \Pi_s H \Pi_s = \sum_{l=1}^{d_b} \Pi_l H \Pi_l H \Pi_l = \sum_{l=1}^{d_b} (\langle \Pi_l | H | \Pi_l \rangle)^2 \Pi_l$, since $\Pi_i \Pi_j = \delta_{ij} \Pi_i$; hence, it is clear that now $\Phi_M(H)^2 \geq 0$. Note that this proof does not require that $H \geq 0$.

Finally, note that now Eq. (24) follows straightforwardly. ■

2. Two examples validating our bounds

Example 2. Consider a two-qubit QB with the Hamiltonian $H = (\epsilon_A \sigma_z \otimes \mathbb{1}_2 + \epsilon_B \mathbb{1}_2 \otimes \sigma_z)/2$, and prepared in thermal equilibrium with respect to H . Consider that $|\Pi_1\rangle = \cos(\frac{\alpha}{2})|ee\rangle + \sin(\frac{\alpha}{2})|gg\rangle$, $|\Pi_2\rangle = \cos(\frac{\beta}{2})|eg\rangle + \sin(\frac{\beta}{2})|ge\rangle$, $|\Pi_3\rangle = -\sin(\frac{\beta}{2})|eg\rangle + \cos(\frac{\beta}{2})|ge\rangle$, and $|\Pi_4\rangle = -\sin(\frac{\alpha}{2})|ee\rangle + \cos(\frac{\alpha}{2})|gg\rangle$, with $\alpha, \beta \in [0, \pi]$. From Fig. 4, we see that our bounds are respected.

Example 3. Now consider a three-qubit QB with Hamiltonian $H = (\epsilon_A \sigma_z \otimes \mathbb{1}_2 \otimes \mathbb{1}_2 + \epsilon_B \mathbb{1}_2 \otimes \sigma_z \otimes \mathbb{1}_2 + \epsilon_C \mathbb{1}_2 \otimes \mathbb{1}_2 \otimes \sigma_z)/2$ prepared in thermal equilibrium with respect to H . Consider $|\Pi_1\rangle = \cos(\frac{\alpha}{2})|eee\rangle + \sin(\frac{\alpha}{2})|ggg\rangle$, $|\Pi_2\rangle = \sin(\frac{\alpha}{2})|eee\rangle - \cos(\frac{\alpha}{2})|ggg\rangle$, $|\Pi_3\rangle = \cos(\frac{\beta}{2})|eeg\rangle + \sin(\frac{\beta}{2})|gge\rangle$, $|\Pi_4\rangle = \sin(\frac{\beta}{2})|eeg\rangle - \cos(\frac{\beta}{2})|gge\rangle$, $|\Pi_5\rangle = \cos(\frac{\gamma}{2})|ege\rangle + \sin(\frac{\gamma}{2})|geg\rangle$, $|\Pi_6\rangle = \sin(\frac{\gamma}{2})|ege\rangle - \cos(\frac{\gamma}{2})|geg\rangle$, $|\Pi_7\rangle = \cos(\frac{\delta}{2})|egg\rangle + \sin(\frac{\delta}{2})|gee\rangle$, and $|\Pi_8\rangle = \sin(\frac{\delta}{2})|egg\rangle - \cos(\frac{\delta}{2})|gee\rangle$, with $\alpha, \beta, \gamma, \delta \in [0, \pi]$. From Fig. 5, we see that our bounds are again respected as demonstrated analytically.

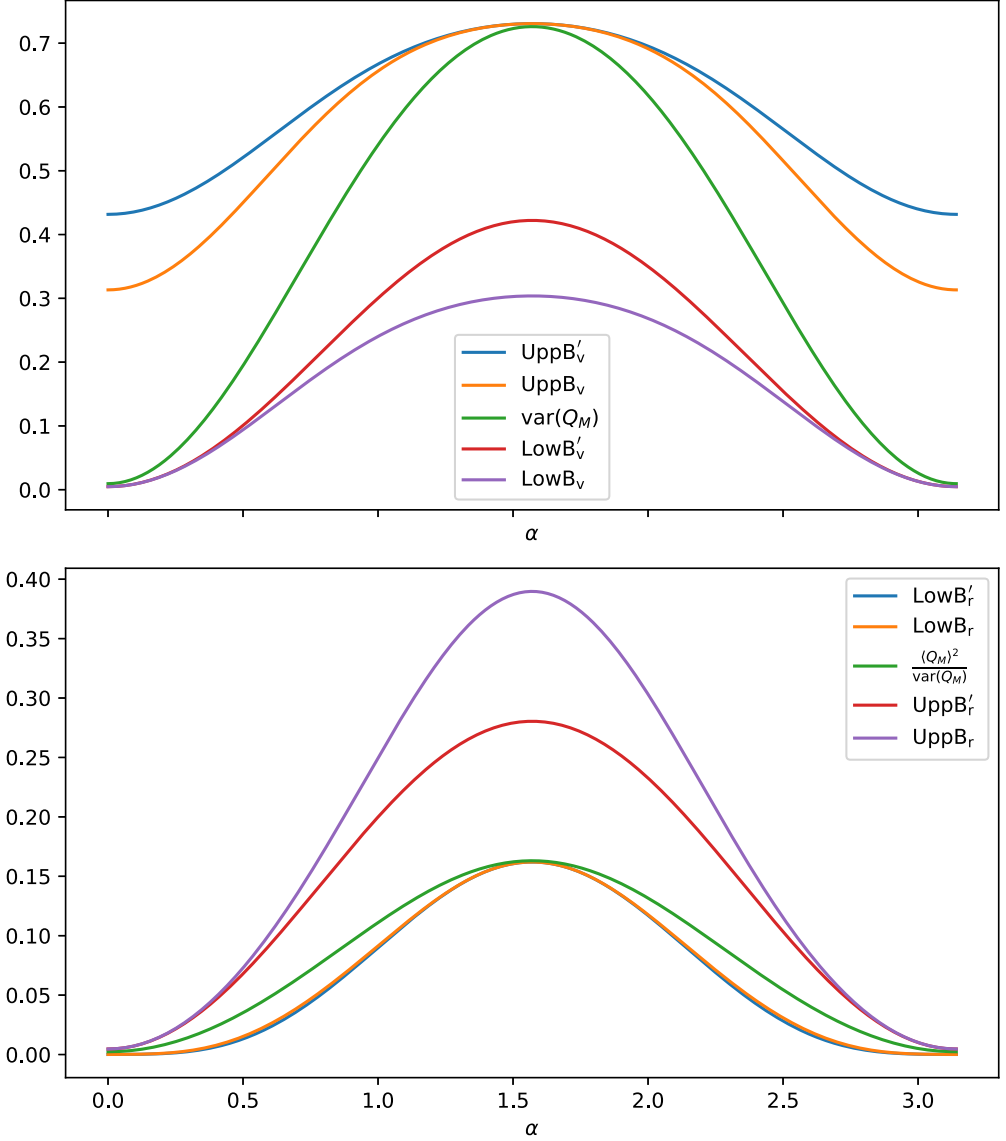


FIG. 4. The parameters are $\epsilon_A = 1$, $\epsilon_B = 0.7$, $\beta = \pi/4$, and $T = 1$. The upper panel shows that $\text{var}(H)_{\rho_1} - \text{var}(\Phi_M(H))_{\rho_0}$ gives a lower bound that is tighter than, or in the worst case equal to, the variance of the heat operator, $\text{var}(\Phi_M(H) - H)_{\rho_0}$.

3. Generalization of Eq. (22) to arbitrary completely positive and trace-preserving maps

We believe that Eq. (22) will be valuable for assessing the reliability of the charging process, specifically the quality of energy transferred to the QB. Furthermore, this equation can also be relevant for discharging protocols implemented via completely positive and trace-preserving (CPTP) quantum operations. For example, the authors in Refs. [46,47] recently considered energy extraction, i.e., discharging, through CPTP and even non-CPTP (but still physically realizable) operations.

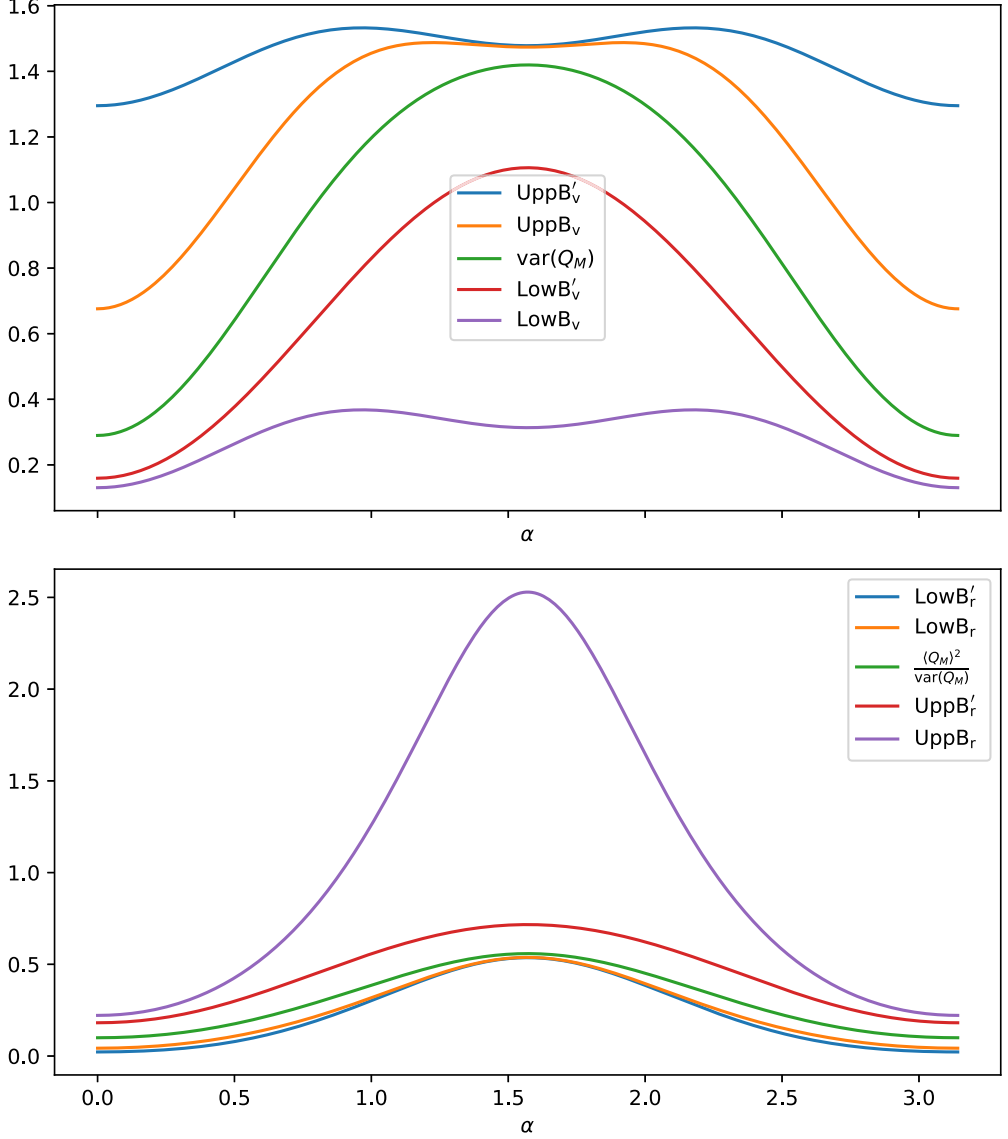


FIG. 5. The parameters are $\epsilon_A = 1$, $\epsilon_B = 0.4$, $\epsilon_C = 1.2$, $\beta = \pi/2$, $\gamma = \pi/3$, $\delta = \pi/6$, and $T = 0.5$. The upper panel shows that $\text{var}(H)_{\rho_1} - \text{var}(\Phi_M(H))_{\rho_0}$ provides a tighter lower bound than the variance of the heat operator, $\text{var}(\Phi_M(H) - H)_{\rho_0}$. The heat operator variance still gives a valid lower bound, but it is less informative. The second panel shows that its associated signal-to-noise ratio can sometimes significantly overestimate the square of the heat reliability.

To support our statement, let us consider replacing Φ_M in Eq. (9) by an arbitrary CPTP map, denoted by Φ . In this case, the expression in Eq. (21) becomes

$$\text{Re}[\langle X^2 \rangle_c] = \text{Tr}[H^2 \Phi(\rho_0)] + \text{Tr}[H^2 \rho_0] - 2\text{Re}[\text{Tr}[H \Phi(H \rho_0)]] - (\text{Tr}[(\Phi(\rho_0) - \rho_0)H])^2.$$

Here, we replace Q_M by X , since the interpretation of this variable depends on the specific CPTP map Φ , and thus it cannot always be interpreted as heat. For example, when Φ is a unitary operation, $\text{Re}[\langle X^2 \rangle_c]$ quantifies work fluctuations. On the other hand, when Φ is a measurement channel, e.g., Φ_M in Eq. (3), $\text{Re}[\langle X^2 \rangle_c]$ is interpreted as heat fluctuations, and Eq. (22) is recovered.

Now, let us define $\Phi^\dagger$ as the adjoint map of Φ , where $\Phi^\dagger$ is defined by

$$\text{Tr}[X \Phi(Y)] = \text{Tr}[\Phi^\dagger(X)Y]$$

for any operators X and Y . This implies that if the action of Φ is given by $\Phi(X) = \sum_j K_j X K_j^\dagger$, then $\Phi^\dagger(X) = \sum_j K_j^\dagger X K_j$. One can then prove that

$$\begin{aligned} \text{Re}[\langle X^2 \rangle_c] &= \text{Tr}[(\Phi^\dagger(H) - H)^2 \rho_0] - \text{Tr}[(\Phi^\dagger(H) - H) \rho_0]^2 + \text{Tr}[H^2 \Phi(\rho_0)] - \text{Tr}[H \Phi(\rho_0)]^2 \\ &\quad - \text{Tr}[\Phi^\dagger(H)^2 \rho_0] + \text{Tr}[\Phi^\dagger(H) \rho_0]^2 \\ &= \text{var}(\Phi^\dagger(H) - H)_{\rho_0} + \text{var}(H)_{\Phi(\rho_0)} - \text{var}(\Phi^\dagger(H))_{\rho_0}. \end{aligned} \quad (\text{E3})$$

This expression shows that our results can be used to study fluctuations in quantum batteries under both projective and weak measurements. Note, however, that if one considers a measurement channel characterized by, e.g., non-Hermitian operators, Eq. (E3) must be used instead of Eq. (22). An example of a weak measurement channel with non-Hermitian operators can be found in Ref. [76].

For further generalizations and additional bounds beyond those in Eq. (23), we refer the reader to Ref. [172]. More precisely, in Ref. [172] one of the authors provides additional lower and upper bounds on fluctuations for arbitrary observables beyond Hamiltonian operators.

Finally, note that Eq. (22) remains valid even if $[\rho_0, H] \neq 0$. More precisely, in the derivation of Eq. (22) we do not assume that H and ρ_0 commute (although in our setup they do). This can be easily verified, since we used $\text{Re}[-2 \text{Tr}[H \Phi_M(H \rho_0)]] = -(\text{Tr}[H \Phi_M(H \rho_0)] + \text{Tr}[H \Phi_M(\rho_0 H)])$, which correctly accounts for $[\rho_0, H] \neq 0$, in contrast to $\text{Re}[-2 \text{Tr}[H \Phi_M(H \rho_0)]] = -2 \text{Tr}[H \Phi_M(H \rho_0)]$, valid when $[\rho_0, H] = 0$. For information on the imaginary part of the second cumulant of heat, see Ref. [172].

In Appendix F, we consider fluctuations in the energy extracted from or injected into a qubit used as a QB through nonselective quantum projective measurement, with the qubit prepared in a state that does not commute with the Hamiltonian. We also test the validity of the bounds in Eqs. (23) and (24).

APPENDIX F: FLUCTUATIONS IN THE ENERGY EXTRACTED FROM OR INJECTED INTO A QUBIT AS A QB THROUGH QUANTUM MEASUREMENT: STATE AND HAMILTONIAN OF THE QB DO NOT COMMUTE

Let us now consider an example of a qubit acting as a quantum battery, with Hamiltonian $H_a = \sigma_z/2$. The QB is initially prepared in the pure state

$$|\psi\rangle = \cos\left(\frac{\alpha_1}{2}\right)|e\rangle + e^{i\chi_1} \sin\left(\frac{\alpha_1}{2}\right)|g\rangle,$$

and is subjected to a nonselective quantum projective measurement described by the operation

$$\Phi_{Ma}(\cdot) = \sum_{i=1}^2 |\Pi_i\rangle\langle\Pi_i|(\cdot)|\Pi_i\rangle\langle\Pi_i|,$$

where the measurement basis vectors are defined as

$$\begin{aligned} |\Pi_1\rangle &= \cos\left(\frac{\alpha_2}{2}\right)|e\rangle + e^{i\chi_2} \sin\left(\frac{\alpha_2}{2}\right)|g\rangle, \\ |\Pi_2\rangle &= \sin\left(\frac{\alpha_2}{2}\right)|e\rangle - e^{i\chi_2} \cos\left(\frac{\alpha_2}{2}\right)|g\rangle, \end{aligned}$$

where $\alpha_1, \alpha_2 \in [0, \pi]$ and $\chi_1, \chi_2 \in [0, 2\pi)$.

Consider $\rho_i = |\psi\rangle\langle\psi|$. Then, using Eqs. (5) and (22), the energy extracted from or injected into the battery, as well as its fluctuations, is given, respectively, by

$$\langle Q_M \rangle = \text{Tr}[H_a(\Phi_{Ma}(\rho_i) - \rho_i)] \quad (\text{F1})$$

and

$$\text{var}(Q_M) = \text{var}(\Phi_{Ma}(H_a) - H_a)_{\rho_i} + \text{var}(H_a)_{\Phi_{Ma}(\rho_i)} - \text{var}(\Phi_{Ma}(H_a))_{\rho_i}. \quad (\text{F2})$$

In Fig. 6, we plot these quantities, as well as $\langle Q_M \rangle^2 / \text{var}(Q_M)$. By varying the measurement angle α_2 , one can either inject into or extract energy from the QB. However, energy extraction tends to dominate. This can be explained by the fact that the QB has a higher probability of being in the excited state before the measurement, since $\cos^2(\alpha_1/2) \approx 0.854$ for $\alpha_1 = \pi/4$.

Interestingly, as shown in Fig. 6, the maximum magnitude of the extracted energy is larger than that of the injected energy. The fluctuations are also higher during energy extraction (i.e., when the battery loses energy). Nevertheless, this does not significantly affect the ratio $\langle Q_M \rangle^2 / \text{var}(Q_M)$; in fact, $\langle Q_M \rangle^2 / \text{var}(Q_M)$ attains its largest values in the energy extraction regime.

Note that the initial state does not commute with the Hamiltonian, i.e., $[\rho_i, H_a] \neq 0$, thus supporting our claim regarding the validity of Eq. (22) for initial states with nonzero coherence. Finally, in Fig. 7 we plot the lower and upper bounds given in Eqs. (23) and (24), and observe that they are respected.

APPENDIX G: THE IMAGINARY PART OF EQ. (8)

Let us now consider the (joint) quasiprobability describing both heat and work at the trajectory level for the 3SQC, i.e., Eq. (8). One can easily show that in the ergotropy regime, i.e., by plugging the unitary in Eq. (25) into Eq. (8), one obtains

$$q_{ijk} = \langle E_i^{(0)} | \rho_0 | E_i^{(0)} \rangle |\langle \Pi_k | E_i^{(0)} \rangle|^2 |\langle \Pi_k | E_j^{(1)} \rangle|^2 \geq 0.$$

Thus, the joint KDQ becomes a proper probability in the ergotropy regime, i.e., under the unitary in Eq. (25).

Equation (8), when marginalized over k , yields information only on the cumulants of heat [or, equivalently, the cumulants of the charging energy; see Eq. (B4)]. It is easy to show that this expression is always non-negative. This non-negativity holds because the initial state commutes with the Hamiltonian. However, we emphasize that Eq. (8) remains valid even when the initial state does not commute with the Hamiltonian.

Inspired by Eq. (18), one can prove that the imaginary part of Eq. (8) can be written as

$$\begin{aligned} \text{Im}[q_{ijk}] &= \frac{d}{d\epsilon} \text{Tr} \left[|E_j^{(1)}\rangle\langle E_j^{(1)}| \hat{V} \Phi_M(|E_i^{(0)}\rangle\langle E_i^{(0)}|) \hat{V}^\dagger \right] \Big|_{\epsilon=0} \\ &= \frac{d}{d\epsilon} \langle E_j^{(1)} | \hat{V} \Phi_M(|E_i^{(0)}\rangle\langle E_i^{(0)}|) \hat{V}^\dagger | E_j^{(1)} \rangle \Big|_{\epsilon=0}, \end{aligned} \quad (\text{G1})$$

where $\hat{V} = e^{i\epsilon \text{Tr}[|E_i^{(0)}\rangle\langle E_i^{(0)}| \rho_0] U^\dagger |E_k^{(2)}\rangle\langle E_k^{(2)}| U/2}$ is a unitary evolution generated by $\text{Tr}[|E_i^{(0)}\rangle\langle E_i^{(0)}| \rho_0] U^\dagger |E_k^{(2)}\rangle\langle E_k^{(2)}| U/2$, and note that we use $[|E_i^{(0)}\rangle\langle E_i^{(0)}|, \rho_0] = 0$ for the derivation of Eq. (G1).

In Eq. (G1), the first equality corresponds to the linear response of the average of the energy projector $|E_j^{(1)}\rangle\langle E_j^{(1)}|$ in the state $\hat{V} \Phi_M(|E_i^{(0)}\rangle\langle E_i^{(0)}|) \hat{V}^\dagger$. The second equality gives the linear response of the transition probability $\langle E_j^{(1)} | \hat{V} \Phi_M(|E_i^{(0)}\rangle\langle E_i^{(0)}|) \hat{V}^\dagger | E_j^{(1)} \rangle$ [107]. Finally, we note that $\langle E_j^{(1)} | \hat{V} \Phi_M(|E_i^{(0)}\rangle\langle E_i^{(0)}|) \hat{V}^\dagger | E_j^{(1)} \rangle$ also corresponds to a quantity known in the literature as the fidelity.

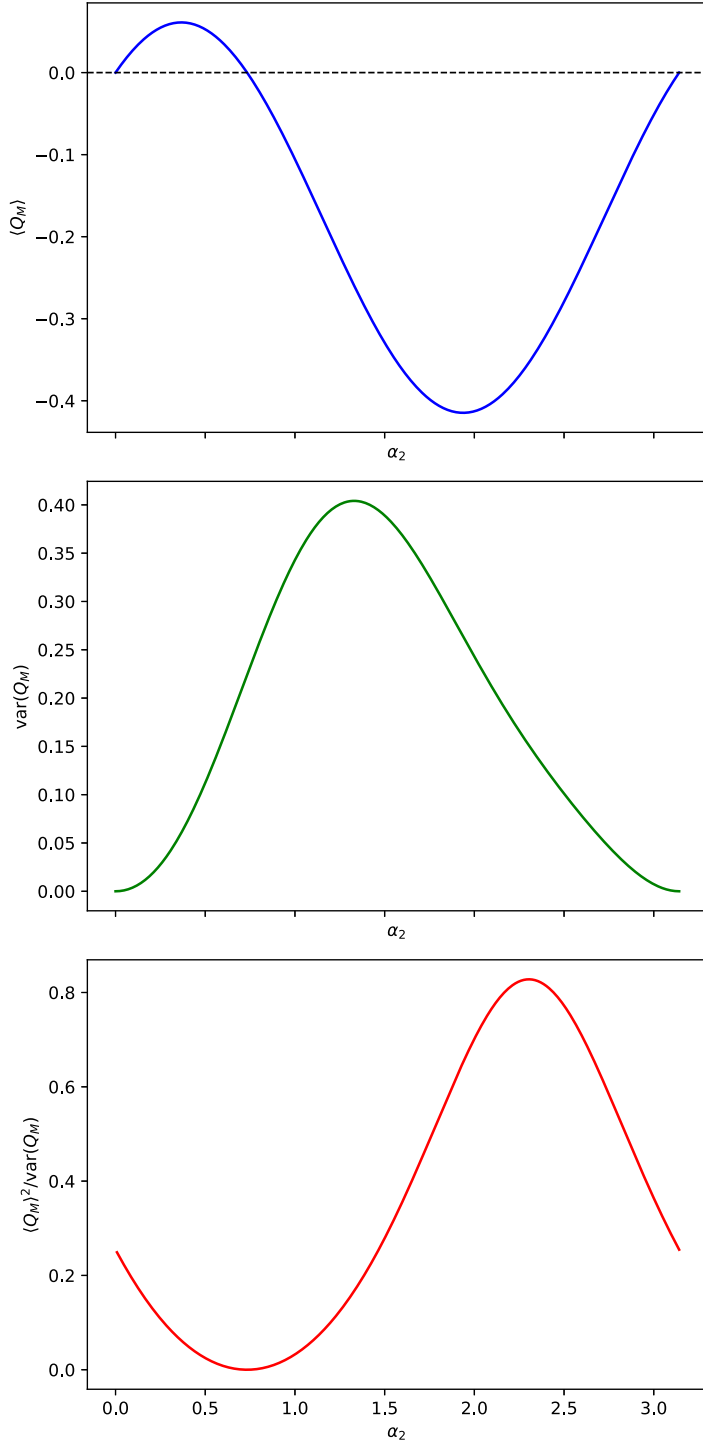


FIG. 6. The parameters used are $\chi_2 = 0$, $\alpha_1 = \pi/4$, and $\chi_1 = \pi/7$.

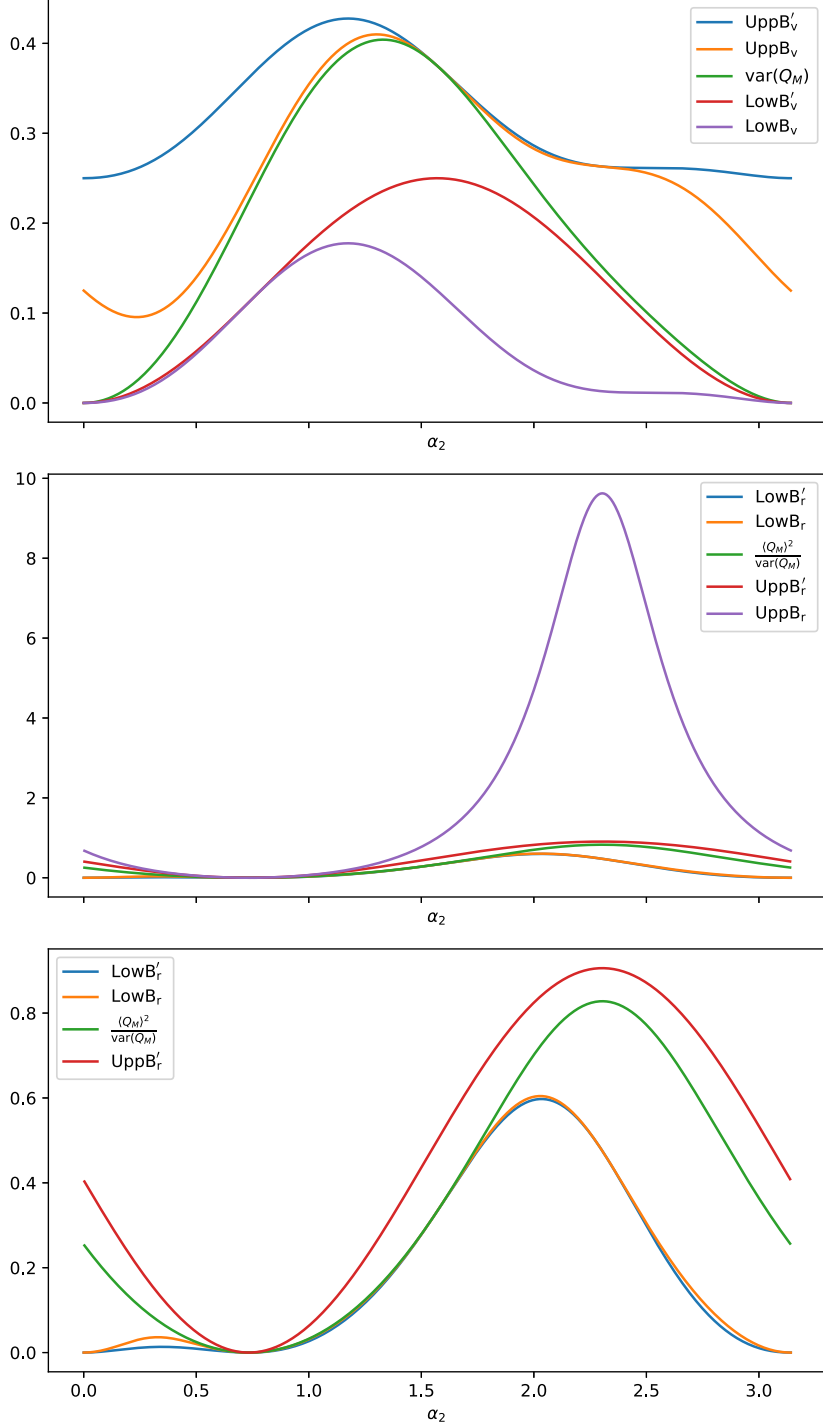


FIG. 7. The parameters are the same as in Fig. 6. Please note that, again, the signal-to-noise ratio associated with the heat operator can sometimes significantly overestimate the square of the charging energy reliability. The last panel is the same as the second one, except that we eliminate UppB_r , since the latter overestimates the square of the heat reliability. We observe that the remaining bounds still provide meaningful estimates of the square of the heat reliability.

APPENDIX H: TPM SCHEME AND QUANTUM STATES WITH NONZERO COHERENCE: AN EXPLICIT EXAMPLE

Consider a qubit system prepared in the pure state $\sigma_i = |\sigma_i\rangle\langle\sigma_i|$, $|\sigma_i\rangle = \frac{|e\rangle+|g\rangle}{\sqrt{2}}$, with Hamiltonian $H = |e\rangle\langle e|$. One can easily show that the unitary operator (a Hadamard gate followed by a bit-flip gate) $U_{\text{erg}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ extracts the ergotropy by driving the qubit to the ground state.

Furthermore, one can verify that

- (1) $[\sigma_i, H] = -i\sigma_y/2$,
- (2) $[\sigma_i, U_{\text{erg}}] = \sigma_z/\sqrt{2}$,
- (3) $[U_{\text{erg}}, H] = \sigma_x/\sqrt{2}$.

Despite these noncommutation relations, the KDQ distribution can be shown to reduce to a proper PD.

Moreover, while the application of U_{erg} to σ_i corresponds to a discharging process, the application of $U_{\text{erg}}^\dagger$ to the ground state $|g\rangle\langle g|$ corresponds to a charging process. These two processes satisfy

- (1) $\text{Tr}[(U_{\text{erg}}^\dagger H U_{\text{erg}} - H)\sigma_i] = -\text{Tr}[(U_{\text{erg}} H U_{\text{erg}}^\dagger - H)|g\rangle\langle g|]$,
- (2) $\text{var}(U_{\text{erg}}^\dagger H U_{\text{erg}} - H)_{\sigma_i} = \text{var}(U_{\text{erg}} H U_{\text{erg}}^\dagger - H)_{|g\rangle\langle g|}$ [see Eq. (15)], where $\text{var}(U_{\text{erg}} H U_{\text{erg}}^\dagger - H)_{|g\rangle\langle g|}$ follows from the TPM scheme since $[H, |g\rangle\langle g|] = 0$.

This is in agreement with the statement made in Remark 3.

APPENDIX I: PROOF OF THE POSITIVITY OF EQ. (26) IN THE ERGOTROPY AND ANTIERGOTROPY REGIMES EVEN WHEN THE MEASUREMENT CHANNEL HAS FEWER PROJECTORS THAN d_b

Consider the case when the measurement channel Φ_M has a number of projectors fewer than the Hilbert space dimension d_b . In this case, by considering a measurement only on the n th subsystem, where the working medium is composed of N subsystems, we have

$$\rho_1 = \tilde{\Phi}_M(\rho_0) = \sum_{i=1}^{d_n} \tilde{\Pi}_i \rho_0 \tilde{\Pi}_i, \quad (\text{I1})$$

where $\tilde{\Pi}_i = \mathbb{1}_{d_1} \otimes \cdots \otimes \mathbb{1}_{d_{n-1}} \otimes \tilde{\Pi}_i \otimes \mathbb{1}_{d_{n+1}} \otimes \cdots \otimes \mathbb{1}_{d_N}$. In general, the measurement need not be restricted to a single subsystem and can be applied to any part of the working medium.

Diagonalizing the state in Eq. (I1) gives $\rho_1 = \sum_{s=1}^{d_b} l_s |\rho_s\rangle\langle\rho_s|$. Assuming that the eigenvalues of ρ_1 are arranged in descending order, i.e., $l_k \geq l_{k+1}$, while those of H [Eq. (1)] are in increasing order, i.e., $E_s \leq E_{s+1}$, one can prove that in the ergotropy regime

$$\begin{aligned} q_{jk} &= \text{Tr}[(E_k^{(2)}\rangle\langle E_k^{(2)}| U_{\text{erg}} |E_j^{(1)}\rangle\langle E_j^{(1)}| \rho_1 U_{\text{erg}}^\dagger] \\ &= l_k |\langle \rho_k | E_j^{(1)} \rangle|^2 (\geq 0) \\ &= \text{Tr}[|E_j^{(1)}\rangle\langle E_j^{(1)}| U_{\text{erg}}^\dagger |E_k^{(2)}\rangle\langle E_k^{(2)}| U_{\text{erg}}] \text{Tr}[|E_k^{(2)}\rangle\langle E_k^{(2)}| \rho_p], \end{aligned} \quad (\text{I2})$$

where $U_{\text{erg}} := \sum_{v=1}^{d_b} |E_v^{(2)}\rangle\langle\rho_v|$. Please note that q_{jk} remains non-negative not only when the state ρ_2 becomes passive, but for any unitary acting on ρ_1 that produces a state with the same eigenstates as H .

Finally, using the fact that $U_{\text{aerg}} := \sum_{t=1}^{d_b} |E_{d_b+1-t}^{(2)}\rangle\langle\Pi_t|$ is the unitary that maximizes the magnitude of the energy injected into the QB, the quantity q_{jk} satisfies the following equalities:

$$\begin{aligned} q_{jk} &= \text{Tr}[|E_k^{(2)}\rangle\langle E_k^{(2)}| U_{\text{aerg}} |E_j^{(1)}\rangle\langle E_j^{(1)}| \rho_1 U_{\text{aerg}}^\dagger] \\ &= p_{d_b+1-k} |\langle \Pi_{d_b+1-k} | E_j^{(1)} \rangle|^2 \\ &= \text{Tr}[|E_j^{(1)}\rangle\langle E_j^{(1)}| U_{\text{aerg}}^\dagger |E_k^{(2)}\rangle\langle E_k^{(2)}| U_{\text{aerg}}] \text{Tr}[|E_k^{(2)}\rangle\langle E_k^{(2)}| \rho_a]. \end{aligned}$$

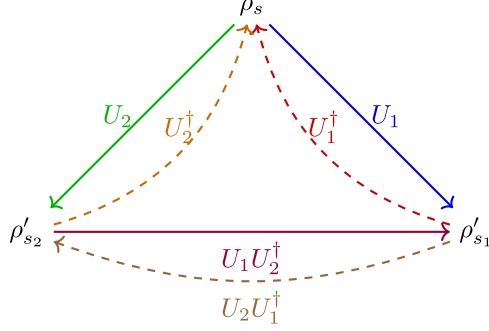


FIG. 8. Triangle diagram of quantum states connected by unitaries and their daggers.

This quantity is also non-negative, i.e., $q_{jk} \geq 0$. Again, this holds even when the measurement channel has fewer projectors than d_b . The same proof as in Eq. (12) applies.

APPENDIX J: VARIANCE OF THE DIFFERENCE OF TWO WORK OPERATORS CAN BE INTERPRETED AS A VARIANCE OF A WORK OPERATOR

Consider an arbitrary QB with Hamiltonian H_s and prepared in the state ρ_s . Suppose ρ_s evolves under either U_1 or U_2 . In this case, we have two work operators $O_{w_i} = U_i^\dagger H_s U_i - H_s$ for $i = 1$ or 2 , where we do not care about the sign of the average work for our purposes. One can show that $\text{var}(O_{w_1} - O_{w_2})_{\rho_s} = \text{var}(O_{w_2} - O_{w_1})_{\rho_s}$ satisfies the following equation:

$$\text{var}(O_{w_1} - O_{w_2})_{\rho_s} = \text{var}(O'_{w_1})_{\rho'_{s_1}} = \text{var}(O'_{w_2})_{\rho'_{s_2}}, \quad (\text{J1})$$

where (see Fig. 8) $\rho'_{s_1} = U_1 \rho_s U_1^\dagger$ and $\rho'_{s_2} = U_2 \rho_s U_2^\dagger$ are states; on the other hand, $O'_{w_1} = U_1 U_2^\dagger H_s U_2 U_1^\dagger - H_s$ and $O'_{w_2} = U_2 U_1^\dagger H_s U_1 U_2^\dagger - H_s$ are work operators [110]. Equation (J1) shows that the variance of the difference of the work operators O_{w_1} and O_{w_2} in state ρ_s is equivalent to the (1) variance of the work operator O'_{w_1} in state ρ'_{s_1} or (2) variance of the work operator O'_{w_2} in state ρ'_{s_2} . Please note that this statement can be easily generalized to the case in which the Hamiltonian H_s of the QB is not fixed.

APPENDIX K: SINGLE QUBIT AS A QUANTUM BATTERY

First, one may wonder why we are interested in work and heat fluctuations. In the literature, it is common to search for unitaries that maximize the average extracted work—leading to the notion of ergotropy. A natural question then arises: what are the fluctuations in work under optimal extraction? In this case, we can simply insert the optimal unitary that maximizes $\langle W \rangle_{\text{ext}}$ into Eq. (15) and compute the work fluctuations. We denote this quantity by $\text{var}(W)_{\text{max}}$, which we can refer to as *ergotropy operator fluctuations* (see Sec. IIIH) or simply *ergotropy fluctuations* in this section.

1. Expressions for the cumulants of heat and work in terms of transition probabilities

Let us consider an arbitrary qubit Hamiltonian and a nonselective quantum projective measurement, given, respectively, by

$$H = \frac{\nu}{2}(|+\rangle\langle+| - |-\rangle\langle-|) \quad (\text{K1})$$

and

$$\Phi_M(\cdot) = \sum_{i=1}^2 \Pi_i(\cdot) \Pi_i.$$

Here, ν is the energy gap of the qubit, and $|+\rangle$ and $|-\rangle$ denote the excited and ground states of the Hamiltonian, respectively. The operators Π_i are projectors onto the eigenstates of the measured qubit observable.

Let us now define the following transition probabilities:

$$\begin{aligned}\theta_1 &:= \langle -|\Phi_M(|+\rangle\langle +|)|-\rangle = \sum_{i=1}^2 |\langle -|\Pi_i|+\rangle|^2, \\ \theta_2 &:= \langle -|U\Phi_M(|+\rangle\langle +|)U^\dagger|-\rangle = \sum_{i=1}^2 |\langle -|U\Pi_i|+\rangle|^2, \\ \theta_3 &:= |\langle -|U|+\rangle|^2.\end{aligned}$$

Using these, we can express the averages and variances as

$$\begin{aligned}\langle Q_M \rangle &= \nu\theta_1 \tanh\left(\frac{\nu}{2T}\right), \\ \langle W \rangle &= \nu(\theta_2 - \theta_1) \tanh\left(\frac{\nu}{2T}\right), \\ \langle Q_C \rangle &= -\nu\theta_2 \tanh\left(\frac{\nu}{2T}\right), \\ \text{var}(Q_M) &= \theta_1\nu^2 - \langle Q_M \rangle^2 = \theta_1\nu^2 \left(1 - \theta_1 \tanh^2\left(\frac{\nu}{2T}\right)\right), \\ \text{var}(W) &= \theta_3\nu^2 - \langle W \rangle^2 = \nu^2 \left(\theta_3 - (\theta_1 - \theta_2)^2 \tanh^2\left(\frac{\nu}{2T}\right)\right).\end{aligned}\tag{K2}$$

From the average quantities, we can identify only two modes of operation:

(1) *Heat engine*: $\langle W \rangle < 0$, $\langle Q_M \rangle > 0$, and $\langle Q_C \rangle < 0$. This occurs when $\theta_1 > \theta_2$. That is, the QB receives energy during the first stroke through quantum measurement, during the second stroke part of this energy is extracted as useful work, and finally the remaining energy is dumped into the bath.

(2) *Accelerator*: $\langle W \rangle > 0$, $\langle Q_M \rangle > 0$, and $\langle Q_C \rangle < 0$. This occurs when $\theta_1 < \theta_2$. That is, the QB receives energy during the first stroke through quantum measurement, during the second stroke useful energy is added to the battery, and finally all of this energy is released to the bath.

The refrigerator and heater modes are not possible. This is because $\langle Q_M \rangle \geq 0$, as we have already proved in Appendix C. This can be understood via the notion of passivity [114]: since the battery starts in thermal equilibrium, its energy cannot be lowered by measurement—only increased. Moreover, for the heater mode to be possible, both $\langle Q_M \rangle$ and $\langle Q_C \rangle$ must have the same sign, which cannot happen here [see Eq. (K2)].

2. Explicit verification of the bounds in Eqs. (23) and (24)

First, one can easily prove that

$$\text{var}(H)_{\rho_1} = \frac{\nu^2}{4} \left[1 - (1 - 2\theta_1)^2 \tanh^2\left(\frac{\nu}{2T}\right) \right].$$

On the other hand, by using the fact that

$$\Phi_M(H)^2 = \frac{\nu^2}{4} (2\alpha_{1+} - 1)^2 \mathbb{1}_2, \quad \text{with} \quad \alpha_{1+} := |\langle \Pi_1 | + \rangle|^2,$$

one can show that

$$\text{var}(\Phi_M(H))_{\rho_0} = \left(\frac{v}{2}(2\alpha_{1+} - 1) \right)^2 - \left(\frac{v}{2}(1 - 2\theta_1) \tanh\left(\frac{v}{2T}\right) \right)^2,$$

and after a few lines of algebra, one arrives at

$$\text{var}(\Phi_M(H) - H)_{\rho_0} = \frac{\theta_1 v^2}{2} - \langle Q_M \rangle^2.$$

Now using all these equations [and the fact that $0 \leq \alpha_{1+} \leq 1$ and $\theta_1 = 2\alpha_{1+}(1 - \alpha_{1+})$], one can prove that

$$\begin{aligned} \text{var}(\Phi_M(H) - H)_{\rho_0} &= \text{var}(Q_M) - v^2 \alpha_{1+}(1 - \alpha_{1+}) \\ &= \text{var}(Q_M) - \frac{\theta_1 v^2}{2} \leq \text{var}(Q_M) \leq \text{var}(\Phi_M(H) - H)_{\rho_0} \\ &\quad + \text{var}(H)_{\rho_1} \leq \text{var}(\Phi_M(H) - H)_{\rho_0} + \frac{v^2}{4}. \end{aligned}$$

Please note that $\text{var}(H)_{\rho_1} - \text{var}(\Phi_M(H))_{\rho_0} - \text{var}(\Phi_M(H) - H)_{\rho_0} = \langle Q_M \rangle^2 \geq 0$; thus,

$$0 \leq \text{LowB}_v \leq \text{LowB}_v' \leq \text{var}(Q_M) \leq \text{UppB}_v \leq \text{UppB}_v'. \quad (\text{K3})$$

This result shows that LowB_v' provides a lower bound on heat fluctuations that is at least as tight as LowB_v .

3. Maximizing charging energy, work extracted, and efficiency

It can be shown that θ_1 is bounded above by $1/2$, with the maximum value (i.e., $1/2$) attained when

$$|\Pi_1\rangle = |\Pi_1\rangle^{(c)} := \frac{1}{\sqrt{2}}(|+\rangle + e^{i\phi}|-\rangle), \quad (\text{K4})$$

where $\phi \in [0, 2\pi)$. Equation (K4) defines the bases on the Bloch sphere where the projection of the eigenstates of the qubit Hamiltonian (K1) yields a probability of $\frac{1}{2}$. These bases maximize the heat, thereby enhancing the charging process of the qubit. The maximum charging energy is given by $\frac{v}{2} \tanh(\frac{v}{2T})$. In this case, the state ρ_1 becomes the maximally mixed state. Please note that Eq. (K4) is an eigenstate of the operator $\sigma_c(\phi) = e^{-i\phi}|+\rangle\langle-| + e^{i\phi}|-\rangle\langle+|$. The latter is the operator that must be measured during the first stroke to achieve the maximum charging energy.

Furthermore, from Eq. (K2), it can be seen that when the qubit starts in the ground state, i.e., $T \rightarrow 0$, the charging process becomes even more efficient. Thus, initializing the QB in states closer to the ground state is desirable for better charging. Additionally, Eq. (K2) shows that lowering the temperature also reduces the fluctuations of work and heat.

Since, for the charging energy to be maximal, the measurement must drive the battery toward the maximally mixed state—a state that cannot be changed unitarily—work extraction or injection are impossible; hence, the efficiency is also zero.

Now consider work extraction. One can show that if $\langle \Pi_1 | \rho_1 | \Pi_1 \rangle \geq \langle \Pi_2 | \rho_1 | \Pi_2 \rangle$, then the unitary that maximizes the average work is

$$U_{rg1} := |+\rangle\langle\Pi_2| + |-\rangle\langle\Pi_1|. \quad (\text{K5})$$

Otherwise, the optimal unitary is

$$U_{rg2} := |+\rangle\langle\Pi_1| + |-\rangle\langle\Pi_2| = U_{rg1}(|\Pi_2\rangle\langle\Pi_1| + |\Pi_1\rangle\langle\Pi_2|) = (|+\rangle\langle-| + |-\rangle\langle+|)U_{rg1}. \quad (\text{K6})$$

Actually, one can show that when U_{rg1} maximizes the work extracted, U_{rg2} maximizes the magnitude of the energy injected into the quantum battery (i.e., the magnitude of the antiergotropy), and vice versa.

Using Eq. (K5), one can prove that the maximum value of ergotropy occurs when $|\Pi_1\rangle = |\Pi_1\rangle^{(0)} := \frac{1}{2}|+\rangle + e^{i\phi}\frac{\sqrt{3}}{2}|-\rangle = \cos(\frac{\pi}{3})|+\rangle + e^{i\phi}\sin(\frac{\pi}{3})|-\rangle$, and thus $|\Pi_2\rangle = |\Pi_2\rangle^{(1)} := -e^{-i\phi}\frac{\sqrt{3}}{2}|+\rangle + \frac{1}{2}|-\rangle = -e^{-i\phi}\sin(\frac{\pi}{3})|+\rangle + \cos(\frac{\pi}{3})|-\rangle$. On the other hand, for the unitary in Eq. (K6), one can prove that the maximum value of ergotropy occurs when $|\Pi_1\rangle = |\Pi_2\rangle^{(1)}$ and $|\Pi_2\rangle = |\Pi_1\rangle^{(0)}$. Overall, one can show that the maximum value of ergotropy is given by $\frac{\nu}{8}\tanh(\frac{\nu}{2T})$.

One can show that the operator that must be measured during the first stroke for the discharging process during the second stroke to reach highest possible value of ergotropy [i.e., $\frac{\nu}{8}\tanh(\frac{\nu}{2T})$] is $\sigma_d(\phi) = \sqrt{3}(e^{-i\phi}|+\rangle\langle-| + e^{i\phi}|-\rangle\langle+|)/2 - (|+\rangle\langle+| - |-\rangle\langle-|)/2$. What we mean is that we maximize the ergotropy over all possible qubit nonselective projective measurements during the first stroke.

The value of the charging energy when the discharging process (i.e., ergotropy) reaches its highest value is $\frac{3\nu}{8}\tanh(\frac{\nu}{2T})$. In this case, the efficiency is $\frac{1}{3}$.

Using Eq. (K5) or (K6), one can prove that the maximum value of efficiency $\eta_{\max}$ is $\frac{1}{2}$. That is, the best we can recover from the charging energy is $\frac{1}{2}$. The proof is as follows: in the ergotropy regime, one can show that $\theta_1 = 2\theta_2(1 - \theta_2)$. Using this and substituting it into the efficiency expression, we obtain

$$\eta_{\max} = \frac{1}{2} - \frac{\theta_2}{2(1 - \theta_2)}.$$

This expression reaches its maximum when $\theta_2 = 0$. Furthermore, one can see from Eq. (K2) that in this case, both charging energy and ergotropy go to zero. This is reminiscent of the Carnot efficiency being attained when all average quantities vanish. Put differently, when the engine is more efficient, the amount of work extracted is zero.

When θ_2 goes to 0, what is happening is that the projectors Π_1 and Π_2 become eigenstates of the Hamiltonian; hence, both the charging energy and the work extracted go to zero. This means that measuring an observable that is diagonal in the energy eigenbasis is the case where the 3SQC can be most efficient. Unfortunately, this comes with vanishing work.

The results of this section show that maximizing ergotropy, then computing the charging energy and the efficiency for the measured observable that maximizes the ergotropy, is the more interesting regime.

Finally, and briefly, one can show that $\text{var}(Q_M) - \langle Q_M \rangle^2 \geq 0$, where equality is achieved when $\theta_1 = 1/2$ and $T \rightarrow 0$. This means that, under the condition $T \rightarrow 0$, not only measuring $\sigma_c(\phi)$ is optimal in terms of charging energy, but also this charging happens with the highest possible reliability, which is 1.

4. Comparison between a unitary that achieves ergotropy and one that does not, in terms of enhancing charging, discharging, and their reliabilities

Let us now choose a specific basis for H and Φ_M . Consider a Hamiltonian defined in the z basis, as

$$H = \frac{\nu}{2}(|e\rangle\langle e| - |g\rangle\langle g|).$$

The measurement projectors are given by

$$\begin{aligned} |\Pi_1\rangle &:= \cos\left(\frac{\alpha}{2}\right)|e\rangle + e^{i\chi}\sin\left(\frac{\alpha}{2}\right)|g\rangle, \\ |\Pi_2\rangle &:= \sin\left(\frac{\alpha}{2}\right)|e\rangle - e^{i\chi}\cos\left(\frac{\alpha}{2}\right)|g\rangle, \end{aligned}$$

where $\alpha \in [0, \pi]$ and $\chi \in [0, 2\pi)$.

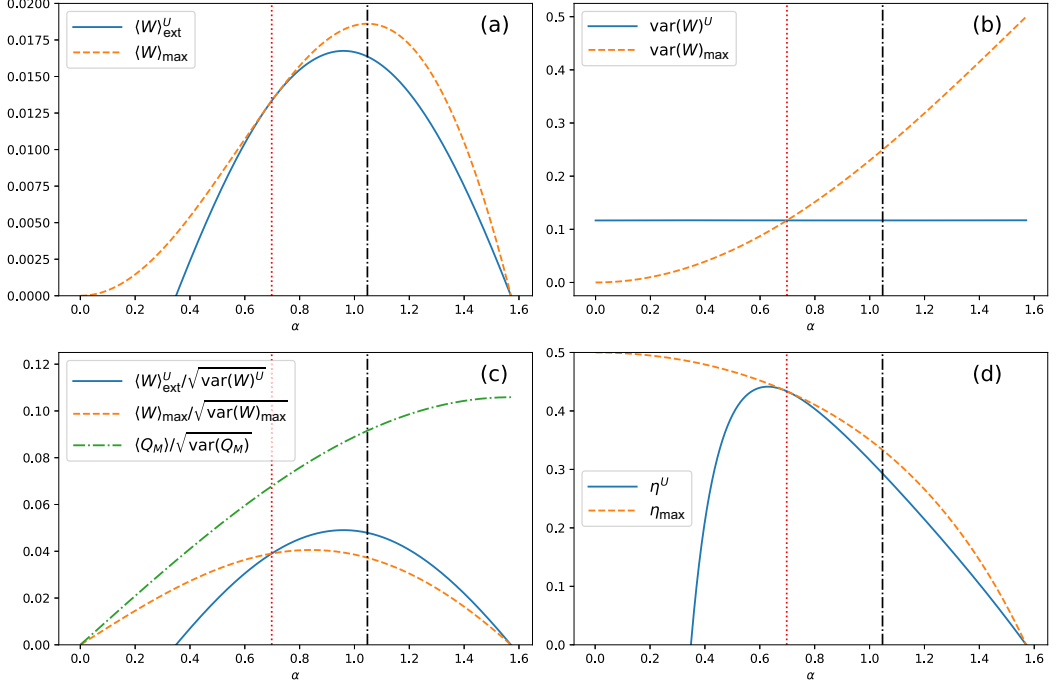


FIG. 9. The parameters used are $T = 10/3$, $v = 1$, $\theta = \pi/9$, and $\chi = \pi/2$. Panel (a) shows the ergotropy (orange dashed line) and the average work extracted using the unitary in Eq. (K7) (blue solid line), while panel (b) presents their corresponding fluctuations. Panel (c) displays the reliability of three quantities: the charging energy (green dash-dot line), the ergotropy, and the work extracted using the unitary in Eq. (K7). Panel (d) shows the efficiency η_{max} and the efficiency η^U corresponding to the unitary in Eq. (K7). In all panels, the vertical red dashed line (vertical gray dash-dotted line) corresponds to $\alpha = \frac{2\pi}{9}$ ($\alpha = \frac{\pi}{3}$); at this angle, panels (b) and (c) show the intersection of the orange dashed and blue solid lines [panel (a) shows that the maximum value of ergotropy is attained].

From what we explained above, any measurement in the xy plane of the Bloch sphere—i.e., when $\alpha = \pi/2$ and for any value of χ —suffices to maximize the charging energy. The optimal operator is given by $\sigma_c(\chi) = \cos(\chi)\sigma_x + \sin(\chi)\sigma_y$. Further, note that when $\alpha = \pi/2$ and $\chi = 0$, Eq. (K6) corresponds to a Hadamard gate, while Eq. (K5) corresponds to a Hadamard gate followed by a bit-flip gate (or equivalently, a phase-flip gate followed by a Hadamard gate). Considering the optimal operator for discharging, one can show that it is given by $\sigma_d(\chi) = \sqrt{3}(\cos(\chi)\sigma_x + \sin(\chi)\sigma_y)/2 - \sigma_z/2$.

Now, consider work extraction via the unitary operator U defined as

$$U = e^{-i\theta\sigma_x}, \quad (\text{K7})$$

where $\theta \in [0, 2\pi]$. More precisely, Eq. (K7) realizes a unitary rotation about the x axis.

When applying U during work extraction, we denote the average work extracted as $\langle W \rangle_{\text{ext}}^U$, the variance as $\text{var}(W)^U$, the reliability as $\langle W \rangle_{\text{ext}}^U / \sqrt{\text{var}(W)^U}$, and the efficiency as η^U .

Our aim is to compare the usefulness of the unitaries given in Eqs. (K7) and (K5) [or (K6)] in terms of the work extracted, its fluctuations, and its reliability. Of course, Eqs. (K5) and (K6) are, by definition, superior in terms of the work extracted (and thus the efficiency).

In Fig. 9, we present four panels. Here are our key observations.

From panel (a), we observe that by choosing the right measurement angle α , one can achieve the ergotropy using the unitary in Eq. (K7). However, the work extracted via this unitary does not reach the ergotropy across the entire domain of α .

The expression for the ergotropy [using Eq. (K6)] is given by

$$\langle W \rangle_{\max} = \nu \cos(\alpha) \sin^2 \left(\frac{\alpha}{2} \right) \tanh \left(\frac{\nu}{2T} \right). \quad (\text{K8})$$

One can show that the maxima of this expression occur at

$$\alpha = \pm \frac{\pi}{3} + 2\pi n, \quad n \in \mathbb{Z}. \quad (\text{K9})$$

For example, the vertical gray dash-dotted line in Fig. 9 corresponds to $\alpha = \pi/3$. Furthermore, note that Eq. (K8) corresponds to ergotropy only when it is positive, which occurs for $\alpha \in (2\pi n, \frac{\pi}{2} + 2\pi n) \cup (\frac{3\pi}{2} + 2\pi n, 2\pi(n+1))$; otherwise, it becomes negative and corresponds to antiergotropy, in which case it is denoted by $\langle W \rangle_{\min}$. On the other hand, when using Eq. (K5), we have

$$\langle W \rangle_{\max} = -\nu \cos(\alpha) \cos^2 \left(\frac{\alpha}{2} \right) \tanh \left(\frac{\nu}{2T} \right). \quad (\text{K10})$$

The maxima occur at

$$\alpha = \pm \frac{2\pi}{3} + 2\pi n, \quad n \in \mathbb{Z}. \quad (\text{K11})$$

Again, note that Eq. (K10) corresponds to ergotropy only for $\alpha \in (\frac{\pi}{2} + 2\pi n, \frac{3\pi}{2} + 2\pi n)$ excluding $\alpha = \pi + 2\pi n$, where the extracted work vanishes; otherwise, it becomes negative and corresponds to antiergotropy, in which case it is denoted by $\langle W \rangle_{\min}$.

Regarding the fluctuations shown in panel (b), we see that as α increases, the fluctuations in ergotropy increase, whereas the fluctuations in the work extracted using the unitary in Eq. (K7) remain (approximately) constant. However, note that for $\alpha \leq 2\pi/9$ we have $\text{var}(W)^U \geq \text{var}(W)_{\max}$, while for $\alpha > 2\pi/9$ we have $\text{var}(W)^U < \text{var}(W)_{\max}$.

From panel (b), one can see that unitaries [i.e., Eqs. (K5) and (K6)] maximizing work extracted may also increase fluctuations, which can reduce the reliability of the extracted work, as demonstrated in panel (c).

More precisely, from panel (c), we observe that within the regime $[0, 2\pi/9)$, the ergotropy reliability is higher than $\langle W \rangle_{\text{ext}}^U / \sqrt{\text{var}(W)^U}$. Beyond $\alpha = 2\pi/9$, however, the trend is reversed. The problem is that when $\alpha > 2\pi/9$, ergotropy achieves higher values. Thus, we see that regions where ergotropy attains higher values may not correspond to higher reliabilities, suggesting a trade-off between ergotropy and its reliability.

In panel (c), we also plot the reliability of the charging energy (green dash-dot line). We observe that the reliability of the charging energy increases monotonically and is greater than the other two reliabilities. This indicates that the charging process (even though it is done through *quantum measurement*) is more reliable than the discharging process (done via unitary operations).

In panel (d), we show the efficiencies. As expected, the efficiency reaches high values in the ergotropy regime. Furthermore, note that the upper bound of the efficiency, i.e., $\frac{1}{2}$, is reached in the limit where both work and heat approach zero. In this limit, $\theta_1, \theta_2 \rightarrow 0$.

Overall, from Fig. 9, we observe that although maximizing the average work over all unitaries can be advantageous in terms of the average work extracted [panel (a)] and efficiency [panel (d)], it can significantly increase fluctuations in the work [panel (b)], which may reduce its reliability [panel (c)]. In particular, the extracted work (or ergotropy) can exhibit larger fluctuations and decreased reliability.

5. Fluctuations in the battery capacity operator: Purity and quantum coherence as resources

Let us now look at the expression for the battery capacity and its fluctuations. When $p_2 = \langle \Pi_2 | \rho_1 | \Pi_2 \rangle \geq p_1 = \langle \Pi_1 | \rho_1 | \Pi_1 \rangle$, the battery capacity is given by

$$\mathcal{C}(\rho_1; H) = v(p_2 - p_1) = v \cos(\alpha) \tanh\left(\frac{v}{2T}\right). \quad (\text{K12})$$

From this equation, we see that the battery capacity is maximal when the difference $p_2 - p_1$ is maximal. Furthermore, given the constraint $p_1 + p_2 = 1$, one can show that the capacity reaches its maximum when p_2 is maximal (and equivalently, p_1 is minimal). On the other hand, for fluctuations in the battery capacity, we have

$$\text{var}(\mathcal{C}_{(\rho_1; H)}) = v^2(1 - (p_2 - p_1)^2) = \frac{v^2(1 + 6e^{v/T} + e^{2v/T} - (-1 + e^{v/T})^2 \cos(2\alpha))}{2(1 + e^{v/T})^2}. \quad (\text{K13})$$

The first equality in the latter equation, i.e., Eq. (K13), shows that the variance of the battery capacity is maximal when $p_1 = p_2$, i.e., when the state ρ_1 is maximally mixed. Thus, we see that as p_2 increases, not only do we increase the battery capacity, but we also decrease fluctuations in the battery capacity. This highlights the relationship between the purity of the state ρ_1 and the battery capacity. Furthermore, please note that when $p_1 = \langle \Pi_1 | \rho_1 | \Pi_1 \rangle \geq p_2 = \langle \Pi_2 | \rho_1 | \Pi_2 \rangle$, we only need to change p_1 with p_2 and p_2 with p_1 in Eqs. (K12) and (K13).

From Fig. 10, we see that increasing the gap of the qubit has a positive influence on the battery capacity and its reliability, while it also increases its fluctuations. In these plots, one can see that the highest values of battery capacity and the lowest values of its fluctuations occur at $\alpha = 0$. This corresponds to the z basis in the Bloch sphere, i.e., the basis where the charging energy is zero. From this figure, it is also clear that the purity, as well as the reliability, exhibits the same behavior as the battery capacity.

In Refs. [113,173], the authors have shown that purity and quantum coherence can be resources to enhance the battery capacity. This naturally raises the question: what is the influence of purity and quantum coherence on fluctuations in the battery capacity?

Example 4. Consider an arbitrary qubit state given by

$$\rho := \frac{1}{2}(\mathbb{1}_2 + r_x \sigma_x + r_y \sigma_y + r_z \sigma_z), \quad (\text{K14})$$

with $r_x^2 + r_y^2 + r_z^2 \leq 1$, and a Hamiltonian given by

$$H_q := \frac{1}{2}(a_0 \mathbb{1}_2 + a_x \sigma_x + a_y \sigma_y + a_z \sigma_z),$$

where a_i are arbitrary real coefficients.

The eigenvalues of ρ are $\lambda_{\pm} = \frac{1}{2}(1 \pm \sqrt{r_x^2 + r_y^2 + r_z^2})$. On the other hand, the eigenvalues of H_q are given by $E_{\pm} = \frac{1}{2}(a_0 \pm \sqrt{a_x^2 + a_y^2 + a_z^2})$.

One can show that the corresponding active and passive states of ρ in Eq. (K14) are given, respectively, by

$$\begin{aligned} \rho_a &= \lambda_+ | +q \rangle \langle +q | + \lambda_- | -q \rangle \langle -q |, \\ \rho_p &= \lambda_- | +q \rangle \langle +q | + \lambda_+ | -q \rangle \langle -q |, \end{aligned}$$

where $| +q \rangle$ and $| -q \rangle$ are the eigenstates of H_q corresponding to E_+ and E_- , respectively.

The expressions for ergotropy, antiergotropy, and battery capacity are, respectively, given by

$$\begin{aligned} \langle W \rangle_{\max} &= \frac{a_0 + a_x r_x + a_y r_y + a_z r_z}{2} - E_+ \lambda_- - E_- \lambda_+, \\ \langle W \rangle_{\min} &= \frac{a_0 + a_x r_x + a_y r_y + a_z r_z}{2} - E_+ \lambda_+ - E_- \lambda_-, \end{aligned}$$

and

$$\mathcal{C}(\rho; H_q) = \sqrt{a_x^2 + a_y^2 + a_z^2} \times \sqrt{r_x^2 + r_y^2 + r_z^2} = \sqrt{a_x^2 + a_y^2 + a_z^2} \times \sqrt{2\mathcal{P} - 1}, \quad (\text{K15})$$

where $\mathcal{P} = \text{Tr}[\rho^2]$ is the purity of the state ρ .

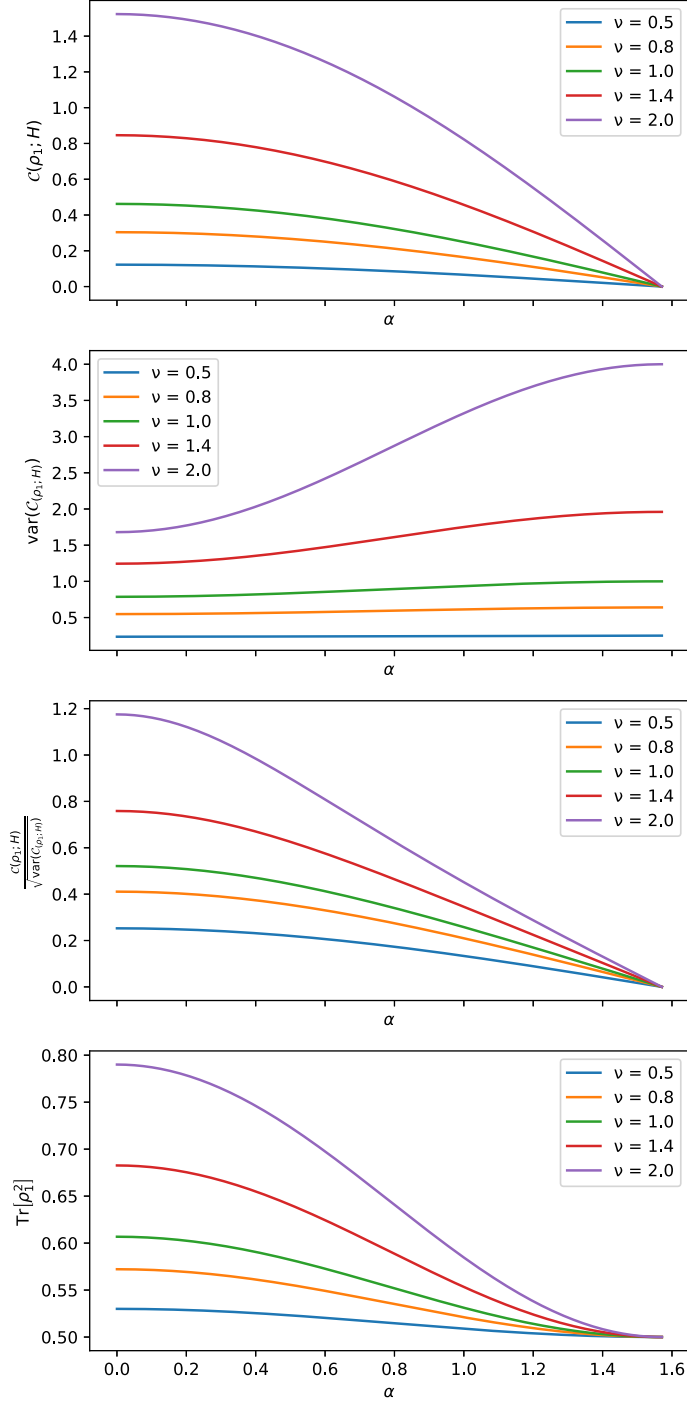


FIG. 10. Here, $T = 1$. From top to bottom: the *first panel* shows the battery capacity [Eq. (K12)], the *second panel* shows the fluctuations in the battery capacity [Eq. (K13)], the *third panel* shows the battery capacity reliability defined by $\frac{\mathcal{C}(\rho_1; H)}{\sqrt{\text{var}(\mathcal{C}_{(\rho_1; H)})}}$, and the *fourth panel* shows the purity of the state ρ_1 , i.e., $\text{Tr}[\rho_1^2] = p_1^2 + p_2^2$, which is equal to $\frac{3+2e^{\nu/T}+3e^{2\nu/T}+(-1+e^{\nu/T})^2 \cos(2\alpha)}{4(1+e^{\nu/T})^2}$.

Using the battery capacity operator, one can show that the fluctuations are given by

$$\text{var}(\mathcal{C}_{(\rho; H_q)}) = (a_x^2 + a_y^2 + a_z^2)(1 - r_x^2 - r_y^2 - r_z^2) = 2(a_x^2 + a_y^2 + a_z^2)(1 - \mathcal{P}). \quad (\text{K16})$$

From the expressions for the battery capacity [Eq. (K15)] and its fluctuations [Eq. (K16)], one can see that the more mixed the state ρ is (i.e., the lower $\mathcal{P}$), the lower the capacity, while the fluctuations increase. This shows that the purity of the initial state is a valuable resource for achieving higher battery capacity and lower fluctuations (see Fig. 10). Furthermore, one can see easily that when the battery capacity is maximal, then in this case the fluctuations vanish, and the state ρ becomes a pure state, where ρ can be any state on the surface of the Bloch sphere. On the other hand, when the state is maximally mixed, then the battery capacity vanishes while the variance is maximal.

Now consider (without loss of generality) $a_x = a_y = 0$. In this case, the Hamiltonian is diagonal in the z basis. Equations (K15) and (K16) simplify to

$$\mathcal{C}(\rho; H_q) = a_z \sqrt{r_x^2 + r_y^2 + r_z^2} \quad (\text{K17})$$

and

$$\text{var}(\mathcal{C}_{(\rho; H_q)}) = a_z^2 (1 - r_x^2 - r_y^2 - r_z^2). \quad (\text{K18})$$

The latter can be decomposed as

$$\text{var}(\mathcal{C}_{(\rho; H_q)}) = \text{var}(\mathcal{C}_{(\rho; H_q)})^{(\text{inco})} - (a_z C_{l_1}(\rho))^2,$$

where

$$\text{var}(\mathcal{C}_{(\rho; H_q)})^{(\text{inco})} := a_z^2 (1 - r_z^2)$$

and

$$C_{l_1}(\rho) := \sqrt{r_x^2 + r_y^2} = \sqrt{2\mathcal{P} - 1 - r_z^2}$$

are the incoherent variance and the amount of quantum coherence in the state ρ with respect to the z basis quantified by the l_1 norm [174], respectively.

This decomposition shows that quantum coherence with respect to the z basis can serve as a resource for both enhancing the battery capacity and reducing fluctuations. Please note that this result can be extended to arbitrary bases (e.g., x and y bases); that is, we can choose any qubit Hamiltonian and isolate the coherent and incoherent parts in the battery capacity and its fluctuations.

Finally, in Appendix L, we show that replacing projective measurement with weak measurement can improve the performance of the 3SQC. In Ref. [175], the authors considered the influence of weak measurements on three-, four-, and five-stroke cycles, and demonstrated that such measurements can enhance the performance of the engine. However, the question of fluctuations is also important and must be addressed. To this end, we apply our previously developed framework to the case of weak measurement as well.

APPENDIX L: FLUCTUATIONS IN THE 3SQC BASED ON WEAK MEASUREMENT OF THE OPERATOR σ_x

Consider a qubit used as a working medium, with Hamiltonian $H = \sigma_z/2$, initially prepared in the thermal state $\rho_0 = \frac{e^{-H/T}}{\text{Tr}[e^{-H/T}]}$. Suppose that during the charging stroke we perform a weak measurement of the operator σ_x on the qubit. The measurement operators describing the weak measurement of σ_x are given by [175]

$$K_{\pm} = m_0 \mathbb{1}_2 \pm m_1 \sigma_x, \quad (\text{L1})$$

where $m_0 \in [0, 1/\sqrt{2}]$, $2(m_0^2 + m_1^2) = 1$, $\sum_{l \in \{+, -\}} K_l^\dagger K_l = \mathbb{1}_2$, and $K_+^\dagger K_- = K_-^\dagger K_+ = (2m_0^2 - 1/2)\mathbb{1}_2$. The postmeasurement state is then given by

$$\rho_1 = \Phi'_M(\rho_0) := \sum_{i \in \{+, -\}} K_i \rho_0 K_i.$$

Furthermore, one can show that ρ_1 remains diagonal in the eigenbasis of σ_z . This fact means that the unitary that achieves ergotropy is nothing but σ_x .

For this example of a battery charged via weak measurement, one can compute the following average quantities and variances:

$$\begin{aligned} \langle Q_M \rangle &= (1 - 2m_0^2) \tanh\left(\frac{1}{2T}\right), \\ \langle W \rangle_{\max} &= (1 - 4m_0^2) \tanh\left(\frac{1}{2T}\right), \quad \text{only for } m_0 \in [0, 1/2], \\ \text{var}(Q_M) &= (1 - 2m_0^2) \left(1 - m_0^2 + m_0^2 \cosh\left(\frac{1}{T}\right)\right) \text{sech}^2\left(\frac{1}{2T}\right), \\ \text{var}(W)_{\max} &= \left(1 - 4m_0^2 + 8m_0^4 + (4m_0^2 - 8m_0^4) \cosh\left(\frac{1}{T}\right)\right) \text{sech}^2\left(\frac{1}{2T}\right). \end{aligned}$$

These results can be obtained straightforwardly using the tools developed in Sec. III and Appendix E. Note that, since the operators K_+ and K_- are Hermitian, one can still use Eq. (22) to compute fluctuations in heat.

From $\langle W \rangle_{\max}/\langle Q_M \rangle$, the efficiency can be written as

$$\eta_{\max} = \frac{1}{2} + \frac{1 - 6m_0^2}{2(1 - 2m_0^2)}, \quad (\text{L2})$$

so that the efficiency exceeds one half when $m_0 < \sqrt{1/6} \approx 0.41$.

From Fig. 11, we observe that weak measurement can be an important resource to enhance both the charging and discharging processes, as well as their reliabilities. More precisely, first from the first panel of Fig. 11 we see that by tuning the parameter m_0 , one can increase both the charging energy and the ergotropy beyond the maximum values attainable under projective measurement [see Appendix K]. Note that as $m_0 \rightarrow \frac{1}{2}$, the measurement operators K_j , where $j \in \{+, -\}$, become projectors. More precisely, K_+ and K_- become eigenstates of σ_x , with respective eigenvalues $+1$ and -1 ; hence, for $m_0 \neq \frac{1}{2}$, the measurement corresponds to a weak measurement of σ_x .

In the second panel, the efficiency of the cycle is plotted. It is evident from the figure that the efficiency can surpass $1/2$; in fact, it can even become unity: the efficiency expression is given by $\eta_{\max} = 2 - \frac{1}{1 - 2m_0^2}$. In this case, what is happening is that when m_0 goes to zero, the post-weak-measurement state becomes equivalent to applying a σ_x operator to the equilibrium initial state ρ_0 . This explains why the efficiency is equal to 1, since applying another σ_x allows one to extract all the energy that was just injected into the QB. If we keep increasing m_0 beyond $1/2$, then one can only inject energy into the qubit through σ_x . In this case, the expression $(1 - 4m_0^2) \tanh(\frac{1}{2T})$ plays the role of anti-ergotropy. Furthermore, note that the battery capacity is equal to ergotropy for $m_0 \in [0, 1/2]$, while beyond that it is equal to minus anti-ergotropy. At $m_0 = 1/2$, of course the battery capacity is zero, since the battery state is maximally mixed.

From the third panel, we observe that fluctuations in both the charging energy and the work are maximal in the projective limit $m_0 \rightarrow 1/2$. Conversely, as $m_0 \rightarrow 0$, the fluctuations vanish (thanks to $T = 1/10$), indicating that the reliability of these quantities diverges in this limit. Overall, we conclude that weak measurement can be utilized to enhance the charging and discharging performances of the cycle beyond what is achievable with projective measurement.

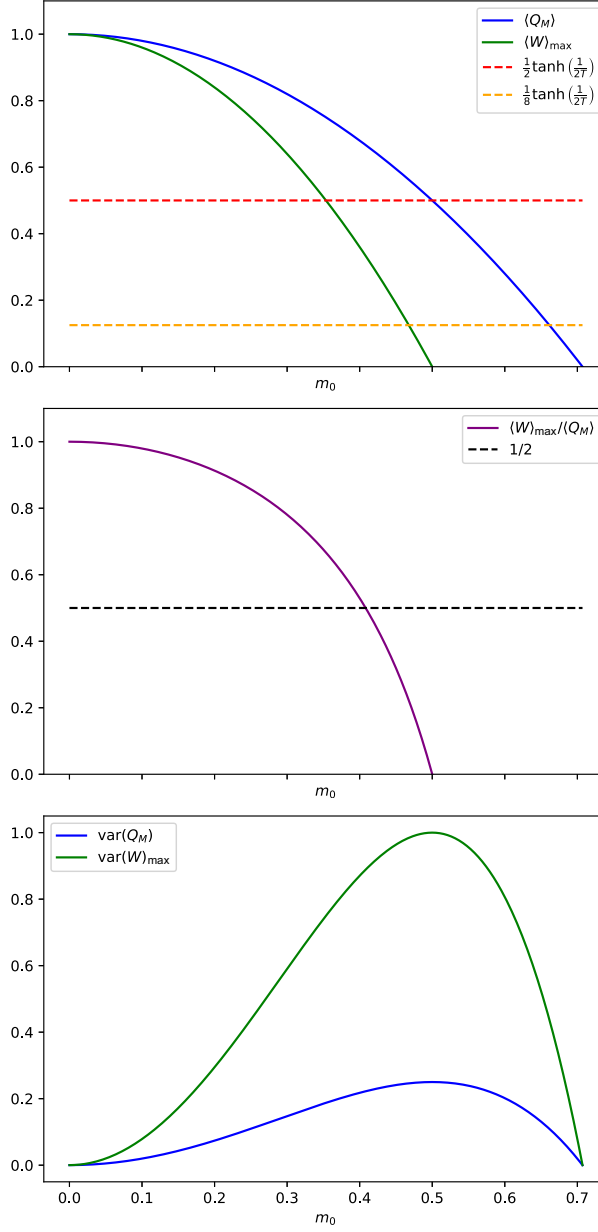


FIG. 11. Here, $T = 1/10$. In the last panel (numerical plot not shown), the fluctuations in the charging energy come almost entirely from the second term in Eq. (22), i.e., $\text{var}(H)_{\rho_1}$, while the first and last terms are negligible, i.e., $\text{var}(\Phi'_M(H) - H)_{\rho_0} \approx 0$ and $\text{var}(\Phi'_M(H))_{\rho_0} \approx 0$. This means that, thanks to $T = 1/10$, fluctuations in the charging energy are equal to fluctuations in the Hamiltonian of the battery in the post-measurement state. The reason is that while the operators $\Phi'_M(H) - H$ and $\Phi'_M(H)$ are proportional to σ_z , the probability of finding the qubit in the excited state before the measurement is almost zero for $T = 1/10$, suppressing the variance of these operators in ρ_0 . In fact, whenever the initial state ρ_0 is an eigenprojector of the Hamiltonian H , the fluctuations are equal to $\text{var}(H)_{\rho_1}$ even for an arbitrary finite-dimensional quantum system, i.e., $\text{var}(Q_M) = \text{var}(H)_{\rho_1}$. Furthermore, at $m_0 = 1/2$, the fluctuations reach their maximum (see the third panel), with $\text{var}(W)_{\max} = 1$ and $\text{var}(Q_M) = 0.250\,045$. The third panel also shows that, in fact, work fluctuations are larger than those of heat.

Finally, since the operators in Eq. (L1) are Hermitian, one can easily prove the validity of Eqs. (23) and (24). Strictly speaking, one can show that $\text{var}(\Phi'_M(H) - H)_{\rho_0} + \text{Tr}[\Phi'_M(H^2)\rho_0] - \text{var}(Q_M) = \frac{1}{4}(1 - 4m_0^2)^2 \geq 0$, $\text{var}(\Phi'_M(H) - H)_{\rho_0} + \text{var}(H)_{\rho_1} - \text{var}(Q_M) = \frac{1}{4}(1 - 4m_0^2)^2 \text{sech}^2(\frac{1}{2T}) \geq 0$, $\text{var}(Q_M) - (\text{var}(H)_{\rho_1} - \text{var}(\Phi'_M(H))_{\rho_0}) = (1 - 2m_0^2)^2 \text{sech}^2(\frac{1}{2T}) \geq 0$, and $\text{var}(Q_M) - \text{var}(\Phi'_M(H) - H)_{\rho_0} = 2m_0^2(1 - 2m_0^2) \geq 0$.

Please note that

$$\begin{aligned} & \text{var}(H)_{\rho_1} - \text{var}(\Phi'_M(H))_{\rho_0} - \text{var}(\Phi'_M(H) - H)_{\rho_0} \\ &= (1 - 2m_0^2) \left(-1 + 3m_0^2 + m_0^2 \cosh\left(\frac{1}{T}\right) \right) \text{sech}^2\left(\frac{1}{2T}\right), \end{aligned} \quad (\text{L3})$$

which can be positive or negative depending on the parameters m_0 and T . This shows that $\text{LowB}_{V'} - \text{LowB}_V$ can be either positive or negative. More precisely, Eq. (L3) is ≥ 0 whenever $m_0^2 \geq 1/(3 + \cosh(1/T))$, and is negative otherwise. For example, for $T = 1$, when $m_0 < 0.47$, we have $\text{LowB}_{V'} < \text{LowB}_V$, while for $0.47 < m_0 < \sqrt{1/2}$, we have $\text{LowB}_{V'} > \text{LowB}_V$.

APPENDIX M: FLUCTUATIONS IN THE GLOBAL AND LOCAL BATTERY CAPACITY OPERATORS: TWO-QUBIT BELL-DIAGONAL STATES

Consider a two-qubit Hamiltonian H_{AB} given by

$$H_{AB} = H_A + H_B,$$

with $H_A = E_A \sigma_z \otimes \mathbb{1}_2$ and $H_B = E_B \mathbb{1}_2 \otimes \sigma_z$, and a state ρ_{AB} given by

$$\rho_{AB} = \frac{1}{4}(\mathbb{1}_2 \otimes \mathbb{1}_2 + c_1 \sigma_x \otimes \sigma_x + c_2 \sigma_y \otimes \sigma_y + c_3 \sigma_z \otimes \sigma_z),$$

where we assume $c_1 \geq c_2 \geq c_3 \geq 0$ and $E_A \geq E_B \geq 0$.

The eigenvalues of H_{AB} and ρ_{AB} , arranged in descending order, are given, respectively, as follows [128,154]:

$$\text{eig}(H_{AB})^\downarrow = \{E_A + E_B, E_A - E_B, E_B - E_A, -(E_A + E_B)\}$$

and

$$\text{eig}(\rho_{AB})^\downarrow = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}.$$

Here, $\lambda_1 = \frac{1+c_1+c_2+c_3}{4}$, $\lambda_2 = \frac{1+c_1-c_2+c_3}{4}$, $\lambda_3 = \frac{1-c_1+c_2+c_3}{4}$, and $\lambda_4 = \frac{1-c_1-c_2-c_3}{4}$. The coefficients c_i are chosen such that $\lambda_j \in [0, 1]$ for $j = 1, 2, 3$, and 4. The eigenstates of ρ_{AB} are, as one can verify, the four maximally entangled states, i.e., the Bell states.

For these two qubits with Hamiltonian H_{AB} and in state ρ_{AB} , it has been shown that the global and local battery capacities are [128,154], respectively, given by $C_g(\rho_{AB}; H_{AB}) = 2c_1E_A + 2c_2E_B$ and $C_l(\rho_A; H_A) = C_l(\rho_B; H_B) = 0$. Here, $\rho_{A(B)}$ is obtained from ρ_{AB} by tracing out qubit B (A). Furthermore, one can easily see that the global battery capacity is zero when $c_1 = c_2 = 0$. In this case, the state ρ_{AB} becomes diagonal in the eigenbasis of H_{AB} and, more precisely, the state ρ_{AB} becomes maximally mixed. This means that quantum coherence with respect to the energy eigenbasis acts as a resource for nonzero battery capacity, and the higher the coherence, the higher the battery capacity.

In Refs. [128,132,154,155], the authors considered projectively measuring qubit B in the eigenbasis of H_B and then the influence of this local measurement on the global and local capacities of the battery. For example, in Ref. [154], after the measurement, the authors considered two scenarios: (1) for the first scenario, the global state of the two qubits is computed from the equally distributed conditional postmeasurement states, while (2) for the second scenario, they considered nonequally distributed conditional postmeasurement states.

For the first scenario, the eigenvalues of the final state ρ'_{AB} were given by [128,154]

$$\text{eig}(\rho'_{AB}) = \frac{1}{4}\{1 + c_3, 1 - c_3, 1 - c_3, 1 + c_3\},$$

while for the second scenario we have

$$\text{eig}(\rho'_{AB}) = \frac{1}{2}\{\mu_0(1+c_3), \mu_1(1-c_3), \mu_0(1-c_3), \mu_1(1+c_3)\},$$

where $\mu_0 + \mu_1 = 1$ and $\mu_0 \geq \mu_1$.

For the first scenario, it was shown that the global and local capacities are given [128,154], respectively, as follows: $\mathcal{C}_g(\rho'_{AB}; H_{AB}) = 2c_3E_A$ and $\mathcal{C}_l(\rho'_A; H_A) = \mathcal{C}_l(\rho'_B; H_B) = 0$. Here, ρ'_A and ρ'_B are the states obtained from ρ'_{AB} by partial trace. Thus, we see that this first scenario worsens the global battery capacity since $\mathcal{C}_g(\rho_{AB}; H_{AB}) \geq \mathcal{C}_g(\rho'_{AB}; H_{AB})$, while noting that the local capacities are unchanged (see Ref. [154]). Furthermore, note that one can easily see that the global battery capacity is zero when $c_3 = 0$.

Before measuring qubit B , the global battery capacity was not a function of c_3 , but depended only on c_1 and c_2 . After measuring qubit B , the parameter c_3 becomes important and is the main source of nonzero battery capacity. The reason c_1 and c_2 no longer contribute is that the authors in Ref. [154] considered the local measurement to be in the eigenbasis of H_B , which erases quantum coherence.

For the second scenario, it was shown that $\mathcal{C}_g(\rho'_{AB}; H_{AB}) = 2(\mu_0 - \mu_1)E_B + 2c_3E_A$, $\mathcal{C}_l(\rho'_A; H_A) = 2(\mu_0 - \mu_1)c_3E_A$, and $\mathcal{C}_l(\rho'_B; H_B) = 2(\mu_0 - \mu_1)E_B$. In this case, it was shown in Ref. [154] that while the global battery capacity can increase or decrease, the local battery capacity is enhanced, and that while $\mathcal{C}_l(\rho'_A; H_A)$ can be zero when $c_3 = 0$, $\mathcal{C}_l(\rho'_B; H_B)$ is always positive as long as $\mu_0 > \mu_1$. Furthermore, one can prove that [132,155]

$$\mathcal{C}_g(\rho'_{AB}; H_{AB}) \geq \mathcal{C}_l(\rho'_A; H_A) + \mathcal{C}_l(\rho'_B; H_B). \quad (\text{M1})$$

Equality is satisfied when either $c_3 = 0$ or $\mu_0 = 1$, or when both conditions are satisfied.

Note that in the derivation of the global battery capacity for the second scenario, we assumed that $\mu_0 \leq \frac{1+c_3}{2}$. When $\mu_0 > \frac{1+c_3}{2}$, we just need to apply the exchange $E_A \leftrightarrow E_B$ in the expression for the global battery capacity.

Remark 5. Please note that to compute the battery capacities, we only need the eigenvalues of ρ_{AB} , ρ'_{AB} , and H_{AB} . The same holds true when considering fluctuations. Now, let us see whether these two scenarios can reduce fluctuations in the (global and/or local) battery capacity.

First, the fluctuations in the battery capacity operator when the two qubits are in the state ρ_{AB} are given, respectively, as follows: $\text{var}(\mathcal{C}_{g(\rho_{AB}; H_{AB})}) = 4(E_A^2 + E_B^2) - 8c_3E_AE_B - (2c_1E_A + 2c_2E_B)^2$, $\text{var}(\mathcal{C}_{l(\rho_A; H_A)}) = 4E_A^2$, and $\text{var}(\mathcal{C}_{l(\rho_B; H_B)}) = 4E_B^2$.

We see that, unlike the global battery capacity, i.e., $\mathcal{C}_g(\rho_{AB}; H_{AB})$, the variance $\text{var}(\mathcal{C}_{g(\rho_{AB}; H_{AB})})$ depends on c_3 , and that increasing this parameter can reduce fluctuations in the battery capacity operator, while the global battery capacity remains unchanged. Furthermore, one can show that

$$\text{var}(\mathcal{C}_{g(\rho_{AB}; H_{AB})}) \leq \text{var}(\mathcal{C}_{l(\rho_A; H_A)}) + \text{var}(\mathcal{C}_{l(\rho_B; H_B)}). \quad (\text{M2})$$

This equation shows that fluctuations in the global battery capacity operator are less than or equal to the sum of the fluctuations in the local battery capacity operators. Equality is attained when $c_1 = c_2 = c_3 = 0$. In this case, the state ρ_{AB} is the maximally mixed state.

Now consider fluctuations in the global and local battery capacities for the first scenario. One can show that $\text{var}(\mathcal{C}_{g(\rho'_{AB}; H_{AB})}) = 4(E_A^2 + E_B^2) - (2c_3E_A)^2$, $\text{var}(\mathcal{C}_{l(\rho_A; H_A)}) = \text{var}(\mathcal{C}_{l(\rho'_A; H_A)}) = 4E_A^2$, and $\text{var}(\mathcal{C}_{l(\rho_B; H_B)}) = \text{var}(\mathcal{C}_{l(\rho'_B; H_B)}) = 4E_B^2$. From these equations, we see that not only do the local battery capacities remain unchanged, but so do the fluctuations of the battery capacity operators at the local level. For the global battery capacity, we observe that while the battery capacity is reduced, the fluctuations in the global battery capacity operator increase. Furthermore, one can again show that

$$\text{var}(\mathcal{C}_{g(\rho'_{AB}; H_{AB})}) \leq \text{var}(\mathcal{C}_{l(\rho'_A; H_A)}) + \text{var}(\mathcal{C}_{l(\rho'_B; H_B)}), \quad (\text{M3})$$

and equality is attained when $c_3 = 0$.

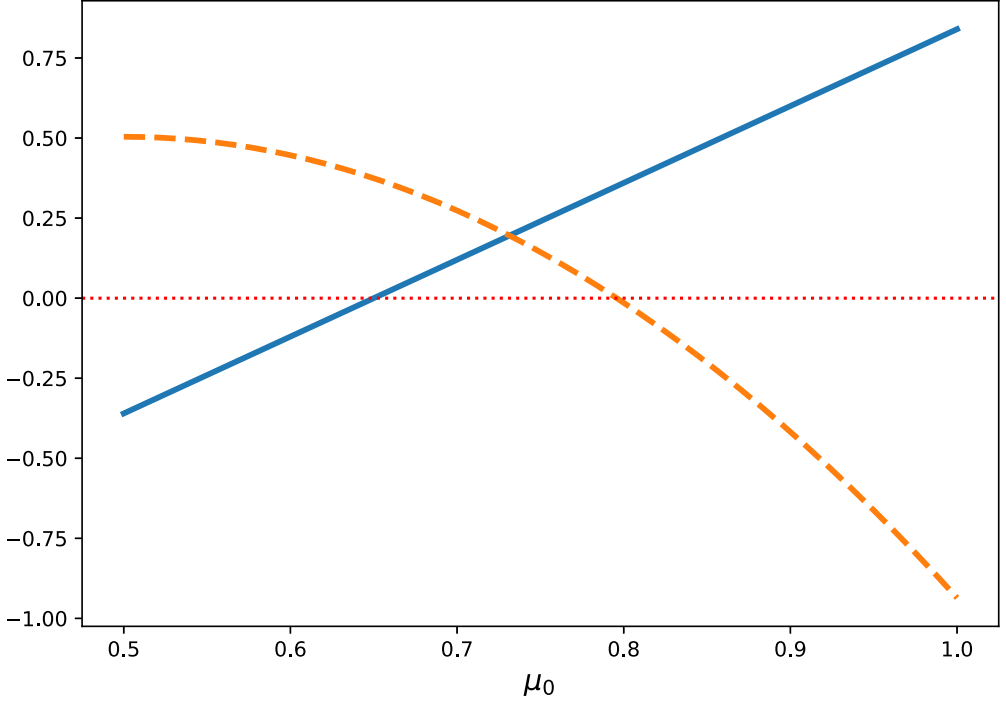


FIG. 12. The blue solid line shows the difference between the battery capacity in the second scenario [i.e., $2(\mu_0 - \mu_1)E_B + 2c_3E_A$] and the battery capacity before the measurement [i.e., $2c_1E_A + 2c_2E_B$], while the orange dashed line shows the difference in their corresponding fluctuations presented in the main text, as a function of $\mu_0 \in [0.5, 1]$ for the fixed parameters $E_A = E_B = 0.6$, $c_1 = 0.3$, and $c_2 = c_3 = 0.1$. As can be seen, choosing higher values of μ_0 can make the difference in battery capacities positive and the corresponding difference in fluctuations negative. The battery capacities are equal at $\mu_0 = 0.65$, while the fluctuations are equal at $\mu_0 \approx 0.8$. For example, in the domain $[0.5, 0.65)$, increasing μ_0 toward 0.65 reduces the plotted differences, but this is still not sufficient to enhance the battery capacity or reduce fluctuations. In the domain $(0.65, 0.8)$, only the battery capacity is higher than before the measurement, while the difference in fluctuations remains positive. Finally, in the domain $(0.8, 1]$, both the battery capacity is enhanced and the difference in fluctuations is negative, indicating a reduction in the fluctuations of the battery capacity for the second scenario compared to before the measurement.

For the second scenario, one can show that $\text{var}(\mathcal{C}_{g(\rho'_{AB}; H_{AB})}) = 4(E_A^2 + E_B^2) + 8E_AE_B(\mu_0 - \mu_1)c_3 - (2(\mu_0 - \mu_1)E_B + 2c_3E_A)^2$, $\text{var}(\mathcal{C}_{l(\rho'_A; H_A)}) = 4E_A^2(1 - ((\mu_0 - \mu_1)c_3)^2) \leq 4E_A^2$, and $\text{var}(\mathcal{C}_{l(\rho'_B; H_B)}) = 4E_B^2(1 - (\mu_0 - \mu_1)^2) \leq 4E_B^2$. The difference between fluctuations in the battery capacity for the first and second scenarios is given by $4E_B^2(1 - 2\mu_0)^2$, which shows that the first scenario actually increases the fluctuations.

At the local level, we see that not only are the local battery capacities enhanced, but also the fluctuations in the local battery capacity operators can be reduced. For fluctuations in the global capacity, as with the global capacity itself, they can either decrease or increase, as we see from Fig. 12. Furthermore, one can again show that

$$\text{var}(\mathcal{C}_{g(\rho'_{AB}; H_{AB})}) \leq \text{var}(\mathcal{C}_{l(\rho'_A; H_A)}) + \text{var}(\mathcal{C}_{l(\rho'_B; H_B)}), \quad (\text{M4})$$

where equality is attained when $c_3 = 0$ and/or $\mu_1 = 0$.

To sum up, our results show that, in addition to the fact that the global battery capacity is greater than or equal to the sum of the local battery capacities [128,132,154,155], we have proved that

fluctuations in the battery capacity operator at the global level are less than or equal to the sum of fluctuations in the local battery capacity operators [see Eqs. (M2)–(M4)]. This means that the square of the global battery capacity reliability, i.e., $\frac{C_g(\rho'_{AB}; H_{AB})^2}{\text{var}(C_g(\rho'_{AB}; H_{AB}))}$, can be lower bounded from Eqs. (M1) and (M4) as follows [assuming that $\text{var}(C_g(\rho'_{AB}; H_{AB})) > 0$]:

$$\frac{C_g(\rho'_{AB}; H_{AB})^2}{\text{var}(C_g(\rho'_{AB}; H_{AB}))} \geq \frac{(C_l(\rho'_A; H_A) + C_l(\rho'_B; H_B))^2}{\text{var}(C_l(\rho'_A; H_A)) + \text{var}(C_l(\rho'_B; H_B))}.$$

Please note that this advantage at the global level requires implementing complex unitaries, which can be difficult and significantly more costly than in the local case. Overall, it would be interesting to consider the fluctuations in the battery capacity operator proposed in Sec. III H for more complex systems.

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