

Spin-mixing-induced dynamics of spinor solitons in $F = 1$ Bose-Einstein condensates

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We explore soliton interactions in a homogeneous spinor $F = 1$ Bose-Einstein condensate (BEC) in the presence of a magnetic field, focusing on dark-bright-dark and bright-dark-bright configurations. We investigate how these interactions depend on the phase differences among bright solitons and their influence during the dynamics. Our findings align with prior nonspinor results, i.e., repulsion among in-phase bright solitons and attraction among out-of-phase pairs in self-repulsive atomic BECs. The potential bright soliton attraction, added to the short-range repulsion of dark-dark soliton interactions, can lead to bound states. However, we find that these bound states break in the presence of spinor interactions due to the particle-exchange dynamics between the hyperfine states of the components. Additionally, we develop an effective classical model to describe the soliton dynamics, using a Lagrangian approach. The accuracy of the model is tested by comparing it against numerical simulations. Our results suggest that the proposed model captures the essential features of soliton behavior in the presence of spin interactions, and provides congruent soliton trajectories and interspecies particle-exchange dynamics in most of the cases.

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I. INTRODUCTION

Among the physical systems that support the rise of nonlinear excitations, Bose-Einstein condensates (BECs) [1,2] occupy a prominent position due to their high controllability, multiple internal

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states, and long coherence times. In single-component BECs, excitations manifest as matter-wave dark [3] and bright [4] solitons, corresponding to repulsive and attractive interparticle interactions, respectively. Two-component BECs support more complex structures like dark-bright (DB) solitons [5–7], while spinor BECs can host a broader range of dark and bright soliton combinations [8–18]. These multisoliton structures emerge in repulsively interacting BECs [19], where bright solitons are supported by the effective potential created by dark solitons—a phenomenon originally noted in nonlinear optics [20–22] and analyzed in the context of integrable systems [23].

Solitons [24] exhibit particlelike behavior, allowing for effective descriptions of their motion in harmonic confinements [3, 19] and interactions between them [25, 26]. Experimental studies have extensively explored soliton dynamics, including collisions of single dark [27, 28] and bright solitons [29], as well as the dynamics of two-component [5], magnetic [30], and three-component solitons [10]. The subject continues to be at the forefront of experimental studies, most recently through the exploration of dense complexes (sometimes referred to also as gases) of dark-bright solitons [31].

Here, we investigate soliton interactions of a spinor $F = 1$ BEC in free space under the influence of a magnetic field, focusing on symmetric dark-bright-dark (DBD) and bright-dark-bright (BDB) configurations. We explore the dependence of the interactions on the phase differences among the bright solitons and examine the influence of spin interactions on the dynamics. In agreement with previous results [25, 26], we verify the repulsive (attractive) character of the effective intersoliton interaction between the bright solitary waves when the latter are in phase (out of phase) in our self-repulsive atomic condensates. In the attractive case, the formation of bound states is possible. Yet, spin interactions result in the breaking of the bound state, enabling the solitons to escape. This behavior is associated with significant changes in the individual particle numbers in each component. Furthermore, we propose an effective model based on the Lagrangian approach in order to capture the system’s dynamical behavior. Our results indicate that, despite its limitations, the proposed effective model successfully captures the dynamical behavior of the interacting solitons in the majority of cases.

The paper is organized as follows. In Sec. II, we introduce the physical model—a system of spinor Gross-Pitaevskii equations (SGPEs)—and describe how stationary single DBD and BDB soliton solutions are obtained. Subsequently, we use the single-soliton solutions as building blocks to consider the two-soliton system. Based on the symmetries of the system, we identify three different cases and discuss their properties and how these affect the dynamics. In Sec. III, we present the numerical results and develop an effective classical approach to describe the dynamics. Additionally, we provide a comparison between the soliton trajectories and the predictions from the model. The accuracy and limitations of this approach are assessed through fidelity measures comparing model predictions to actual soliton dynamics. Finally, in Sec. IV, we summarize our findings and discuss potential future challenges and directions for further research.

II. MODEL AND THEORETICAL SETUP

A. Spinor Gross-Pitaevskii equations

We consider a one-dimensional (1D) spin $F = 1$ BEC in free space. In the mean-field approximation, such a system can be described by the following dimensionless SGPEs, one for each of the three $m_F = 0, \pm 1$ hyperfine states [13, 32]:

$$i\partial_t \Psi_0 = -\frac{1}{2}\partial_x^2 \Psi_0 + g_n(|\Psi_+|^2 + |\Psi_0|^2 + |\Psi_-|^2)\Psi_0 + g_s(|\Psi_+|^2 + |\Psi_-|^2)\Psi_0 + 2g_s\Psi_+\Psi_0^*\Psi_-, \quad (1a)$$

$$i\partial_t \Psi_{\pm} = -\frac{1}{2}\partial_x^2 \Psi_{\pm} + g_n(|\Psi_+|^2 + |\Psi_0|^2 + |\Psi_-|^2)\Psi_{\pm} + g_s(|\Psi_{\pm}|^2 + |\Psi_0|^2 - |\Psi_{\mp}|^2)\Psi_{\pm} \\ + g_s\Psi_{\mp}^*\Psi_0^2 + q\Psi_{\pm}, \quad (1b)$$

where $\Psi(x, t) = (\Psi_+(x, t), \Psi_0(x, t), \Psi_-(x, t))^T$ is the three-component wavefunction, subject to the normalization condition $N = \int dx \sum_{m_F=-1}^1 |\Psi_{m_F}(x, t)|^2$, where N is the total number of particles. In the above equations, q is the quadratic Zeeman parameter, quadratically proportional to an

external uniform magnetic field applied along the spin- z direction, and $g_n = \frac{a_0+2a_2}{3}$ and $g_s = \frac{a_2-a_0}{3}$ are the spin-independent and spin-dependent interaction coefficients, respectively. Here, a_0 and a_2 are the s -wave scattering lengths, accounting for two atoms in the scattering channels with total spin $\mathcal{F} = 0$ and $\mathcal{F} = 2$, respectively. The spin-dependent coefficient is negative for ferromagnetic (F) and positive for antiferromagnetic (AF) spin interactions. Note that Eq. (1) was made dimensionless by measuring length, time, and energy in units of $a_\perp$, $\omega_\perp^{-1}$, and $\hbar\omega_\perp$, respectively. Here, $a_\perp = \sqrt{\frac{\hbar}{m\omega_\perp}}$ is the transverse harmonic oscillator length, corresponding to the experimentally relevant case [10], where a quasi-1D cigar-shaped system is realized utilizing a highly anisotropic trap with the longitudinal and transverse trapping frequencies obeying $\omega_x \ll \omega_\perp$. However, it must be emphasized that in our work, we strictly consider a homogeneous 1D system in free space, i.e., $\omega_x = 0$. Such possibilities have been explored recently in 1D settings in the realm of box potentials [33].

Equation (1) conserves, among other quantities, the energy, the total number of particles, and the total magnetization along the spin- z axis, $M_z = \frac{1}{N} \int dx (|\Psi_+|^2 - |\Psi_-|^2)$. However, due to the existence of spin interactions, and more specifically the spin-mixing terms $2g_s \Psi_+ \Psi_0^* \Psi_-$ and $g_s \Psi_\mp^* \Psi_0^2$, the individual particle numbers of each component, $N_{m_F} \equiv \int dx |\Psi_{m_F}(x)|^2$, are not conserved, unless $g_s = 0$. In the latter case, the spin degrees of freedom are said to be “frozen,” and thus throughout this work, we will refer to it as nonspinor (NS). Moreover, the absence of such a conservation will be shown to play a significant role in the full spinorial dynamics.

In case of the spinor systems, any static solution of Eq. (1) must fulfill the condition [34]

$$\Delta\phi \equiv 2\phi_0 - \phi_+ - \phi_- = 0 \text{ or } \pi, \quad (2)$$

where $\phi_{\pm,0}$ are the phases of the wavefunctions $\Psi_{\pm,0}$. More specifically, $\Delta\phi = 0$ corresponds to the F system, while $\Delta\phi = \pi$ corresponds to the AF system. This stems from the fact that a stationary state is the one that minimizes locally the energy functional. With respect to $\Delta\phi$, this is achieved by minimizing the phase-sensitive spin-mixing term

$$E_{\text{sm}} = 2g_s \int dx \text{Re} \{ \Psi_0^2 \Psi_-^* \Psi_+^* \} = 2g_s \int dx [(|\Psi_-||\Psi_+||\Psi_0|^2) \cos(\Delta\phi)]. \quad (3)$$

It is straightforward to see that E_{sm} becomes minimal when $\Delta\phi = 0$ ($\Delta\phi = \pi$) in case of an F (AF) system.

B. Soliton solutions

Equation (1) with $g_s = 0$ (NS system) reduces to the so-called Manakov model [35,36], in particular, in its three-component form, which has also been explored experimentally in Ref. [37]. This system of equations is integrable and admits exact vector soliton solutions [25,26,38,39]. In what follows, we set $g_n = 1$. We consider solutions of the dark type,

$$\Psi_d = [\nu \tanh(D\tilde{x}) + i\lambda] e^{-i\mu_d t}, \quad (4)$$

and bright type,

$$\Psi_b = \eta \text{sech}(D\tilde{x}) e^{i\nu x - i\mu_b t + i\phi(t)}. \quad (5)$$

Here, ν, η are the amplitudes of the dark and bright components, respectively. D is a parameter characterizing the common inverse width. $\tilde{x}(t) = x - x_0(t)$ reflects the spatial dependence, with $x_0(t)$ representing the instantaneous position of the soliton center and $v = \dot{x}_0(t)$ is the soliton velocity. μ_d and μ_b are the corresponding chemical potentials, and ϕ is the bright soliton phase. λ is related to the dark soliton velocity and $v^2 + \lambda^2 = n_0$, where n_0 is the constant background density of the dark component.

In what follows, we will consider vector solitons of the form DBD and BDB, with the letters denoting the $(+1, 0, -1)$ hyperfine components notation ordering. The aforementioned soliton

parameters are connected through

$$D^2 = v_+^2 + v_-^2 - \eta_0^2, \quad v = D \frac{\lambda_+}{v_+} = D \frac{\lambda_-}{v_-}, \quad (6)$$

in the DBD case, and

$$D^2 = v_0^2 - \eta_+^2 - \eta_-^2, \quad v = D \frac{\lambda_0}{v_0}, \quad (7)$$

in the BDB case, while the phase of the bright soliton evolves as

$$\phi(t) = \frac{1}{2}(D^2 - v^2)t + (\mu_b - \mu_d)t. \quad (8)$$

Furthermore, we will consider symmetric configurations, namely, $v_+ = v_- = v$ and $\eta_+ = \eta_- = \eta$, respectively. Under this assumption, the analytical static solutions of the integrable NS system for $v = 0$ read

$$\Psi = \begin{pmatrix} \sqrt{\frac{\mu-q}{2}} \tanh(D\tilde{x}) \\ \sqrt{\mu-q-D^2} \operatorname{sech}(D\tilde{x}) \\ \sqrt{\frac{\mu-q}{2}} \tanh(D\tilde{x}) \end{pmatrix}, \quad (9)$$

in the DBD case, and

$$\Psi = \begin{pmatrix} \sqrt{\frac{1}{2}(\mu-D^2)} \operatorname{sech}(D\tilde{x}) \\ \sqrt{\mu} \tanh(D\tilde{x}) \\ \sqrt{\frac{1}{2}(\mu-D^2)} \operatorname{sech}(D\tilde{x}) \end{pmatrix}, \quad (10)$$

in the BDB case. Note here that we have set $\mu_{0,\pm} = \mu$. The latter choice has been made with an eye toward the F and AF systems, where N_{m_F} is not conserved. Therefore, only an overall chemical potential can be defined. Equations (9) and (10) are used as an ansatz. Note that in some cases we will include a phase term, $e^{i\phi_{m_F}}$, in order to fulfill Eq. (2). For the symmetric stationary AF configuration considered here, for which the component wavefunctions can be chosen real up to constant component phases, the relative phase condition $\Delta\phi = \pi$ causes the spin-dependent contributions to cancel pointwise in the stationary equations. Consequently, the AF stationary solutions have the same density profiles as the corresponding NS (Manakov) solutions, differing only in their relative component phases. On the numerical side, we used a fixed-point iteration scheme, based on Newton's method [40], to obtain the stationary solutions for the F, NS, and AF systems.

C. Two-soliton system

In order to investigate the interactions between two DBD as well as two BDB solitons, we initialize static single solitons of these types, centered at $x_0^L = -\frac{d_0}{2}$ and $x_0^R = \frac{d_0}{2}$, where d_0 is the initial distance between the soliton centers. Then, we construct the soliton pair by adding up the bright components and multiplying and normalizing the dark components.

At this point, it should be noted that the aforementioned recipe can be considered a good approximation only for marginally overlapping solitons, i.e., $d_0 \gg D^{-1}$. Therefore, in this work, we restrict ourselves to studying only those systems that initially fulfill that condition, and particularly we set the condition $d_0 > 4D^{-1}$, so as to ensure that the separation of the solitons is much larger than their individual widths.

In what follows, we investigate the cases where the bright solitons are in phase (IP) or out of phase (OP). In order to realize the aforementioned cases, the phases of the single-soliton states are modified by acting with operators of the form $\hat{T}_\theta = \operatorname{diag}(e^{i\theta_+}, e^{i\theta_0}, e^{i\theta_-})$ on them, under the

condition that the resulting states fulfill again condition (2) and ensure they are stationary solutions. They are then combined to form the initial two-soliton wavefunctions, which read

$$\Psi(t=0) = \begin{pmatrix} \sqrt{\frac{2}{\mu-q}} \Psi_+^L \Psi_+^R \\ \Psi_0^L + \Psi_0^R \\ \sqrt{\frac{2}{\mu-q}} \Psi_-^L \Psi_-^R \end{pmatrix}, \quad (11)$$

in case of the two-DBD system, and

$$\Psi(t=0) = \begin{pmatrix} \Psi_+^L + \Psi_+^R \\ \frac{1}{\sqrt{\mu}} \Psi_0^L \Psi_0^R \\ \Psi_-^L + \Psi_-^R \end{pmatrix}, \quad (12)$$

in case of the two-BDB system. In the above states, L and R stand for a left-centered ($x_0 = -\frac{d_0}{2}$) and right-centered ($x_0 = \frac{d_0}{2}$) soliton, respectively.

Starting from the two-DBD system, two different cases can be identified: one IP configuration, where $\Delta\phi_0^{[LR]} = \phi_0^L - \phi_0^R = 0$, and one OP configuration, where $\Delta\phi_0^{[LR]} = \pi$. Here, $\phi_{m_F}^L$ ($\phi_{m_F}^R$) denotes the phase of the left- (right-) centered soliton occupying the m_F component. The two-BDB system consists of two pairs of bright solitons, one for each of the $m_F = \pm 1$ components. Combining single BDB solutions, two-soliton states with all possible phase difference combinations can be created. However, due to the symmetries of the system, some of these states are physically equivalent. More specifically, they are connected through an $m_F = \pm 1$ component exchange transformation, a trivial global phase transformation or a transformation of the form $\Psi' = \hat{T}\Psi$, where $\hat{T} = \text{diag}(e^{i\theta_+}, e^{i\theta_0}, e^{i\theta_-})$. Recall that the condition $2\theta_0 - (\theta_- + \theta_+) = 2k\pi$ ensures that two states solve the same SGPE, and thus are physically equivalent. After recognizing these symmetries, the system can be reduced to three cases: (IP) $\Delta\phi_-^{[LR]} = \Delta\phi_+^{[LR]} = 0$, (OP) $\Delta\phi_-^{[LR]} = \Delta\phi_+^{[LR]} = \pi$, and (IP/OP) $\Delta\phi_{\{-,+\}}^{[LR]} = \{0, \pi\}$, or $\Delta\phi_{\{-,+\}}^{[LR]} = \{\pi, 0\}$, where $\Delta\phi_{\pm}^{[LR]} = \phi_{\pm}^L - \phi_{\pm}^R$.

At this point, it is important to note that, since the left and right dark solitons occupying the same component share a common background, the phases of the individually created single solitons are added up over the entirety of space. In some cases, this can lead to the violation of condition (2) from the resulting two-soliton system. In the IP and OP cases, this problem can be solved by multiplying the affected component of the two-soliton system with a proper overall phase. However, in the IP/OP case of the two-BDB system, this is not possible. More specifically, in case of the phase-sensitive F and AF systems, the violation of the above conditions at the right side ($x > 0$) and at the left side ($x < 0$) of the system, respectively, is unavoidable. This violation excites the affected part of the system. Therefore, one cannot directly speak of an interaction between two solitons; instead, it pertains to the interaction between a soliton and an excited state. The distinction is physical rather than numerical. The phase condition (2) ensures that the spin-mixing contribution to the energy is locally minimized. Consequently, violating this condition means that the initial state is not a stationary spinor-soliton configuration, but already represents an excited state of the spinor system. As a result, intrinsic spin dynamics is triggered through the spin-mixing interactions, independently of the subsequent collision process. Therefore, the ensuing evolution no longer reflects the interaction between two stationary solitons, but rather the interaction of a stationary soliton with an excited spinor state. Although such a configuration could in principle be prepared experimentally through independent phase imprinting of the individual hyperfine components, provided sufficient state-selective control is available, the relevant initialization would constitute a deliberately excited nonequilibrium state rather than a stationary spinor-soliton pair. Since the purpose of the present work is to study genuine soliton-soliton interactions, we do not consider the IP/OP scenario further.

Our goal is to study how the interaction depends on the phase difference between the bright solitons. Thus, it is reasonable to anticipate that the initially created phase differences will be

conserved during the dynamics. We will show that, in both IP and OP cases, this assumption is fulfilled. First, it is important to note that in these cases, the initial states are symmetric with respect to the $m_F = \pm 1$ components; specifically, $\Psi_+(t=0) = \Psi_-(t=0)$. Given that Eq. (1) is symmetric under the exchange of the $m_F = \pm 1$ components, this symmetry will be conserved over time, ensuring that $\Psi_+(t) = \Psi_-(t)$. Therefore, Eq. (1) reduces to

$$i\partial_t \Psi_0 = -\frac{1}{2}\partial_x^2 \Psi_0 + g_n(|\Psi_1|^2 + |\Psi_0|^2)\Psi_0 + g_s|\Psi_1|^2\Psi_0 + g_s\Psi_1^2\Psi_0^*, \quad (13a)$$

$$i\partial_t \Psi_1 = -\frac{1}{2}\partial_x^2 \Psi_1 + g_n(|\Psi_1|^2 + |\Psi_0|^2)\Psi_1 + g_s|\Psi_0|^2\Psi_1 + g_s\Psi_1^*\Psi_0^2 + q\Psi_1, \quad (13b)$$

where $\Psi_1 = \sqrt{2}\Psi_\pm$. A key feature of the above equations is that they conserve the parities of each of the Ψ_0 and Ψ_1 components individually, if the latter have initially definite (but not necessarily equal) parities. The initial two-soliton states of the cases fulfill by construction this condition. More specifically, the dark components have even parity, while the bright ones in the IP (OP) configuration have even (odd) parity. The connection between the phase differences $\Delta\phi_0^{[LR]} = 0, \pi$ and the parity of the corresponding wavefunctions becomes clear if one considers that the phase of every complex function $f(x) = |f(x)|e^{i\phi(x)}$ with even parity fulfills $\phi(x) - \phi(-x) = 2k\pi$ ($k = 0, 1, 2, \dots$), while in case of a function with odd parity the phase fulfills $\phi(x) - \phi(-x) = (2k+1)\pi$ ($k = 0, 1, 2, \dots$). Consequently, the parity conservation leads to phase difference conservation.

III. NUMERICAL RESULTS AND DISCUSSION

In this section, we present the dynamics of the aforementioned states comparing the F, NS, and AF systems. For this purpose, we set the chemical potential to $\mu = 2$ (without loss of generality), the spin-independent interaction coefficient $g_n = 1$, and the spin-dependent one $g_s = -5 \times 10^{-3}$, $g_s = 0$, and $g_s = 5 \times 10^{-3}$ in case of F, NS, and AF systems, respectively. The latter are representative values in the neighborhood of realistic ones for ^{87}Rb and ^{23}Na [41]. Since our goal is to investigate the interactions between different-sized solitons, we vary the quadratic Zeeman term (q) within the intervals $[-1, -0.1]$ and $[0.1, 1]$ in case of a DBD and BDB system, respectively. The motivation to use these values stems from Ref. [42], where, among others, the q intervals of stability of trapped single BDB and DBD solitons have been investigated. Our chosen q values lie within the stability intervals. Moreover, we confirmed that they lead to stable single-soliton solutions in free space as well, which is a key ingredient for our considerations.

For the time evolution of the initial states, a fourth-order Runge-Kutta integrator is utilized, while a second-order finite differences method is used for the spatial derivatives. The spatial and time discretization are $dx = 5 \times 10^{-2}$ and $dt = 10^{-3}$, respectively. Convergence of the results has been verified for smaller values of dx and dt for representative selected cases within the initial parameter space.

A. Dynamics

In order to investigate the dynamical behavior of the two-DBD and two-BDB systems in the IP and OP configurations, we consider the time evolution of the initial two-soliton state (11), in case of the two-DBD system, and Eq. (12), in case of the two-BDB system, for varying initial distances and q values.

For small $|q|$ values, both systems exhibit very similar dynamics, which is expected given that Eq. (13) tends to be symmetric upon exchange of the Ψ_0 and Ψ_1 components as $q \rightarrow 0$. For increasing $|q|$ values, the differences become visible, although also in this regime the qualitative characteristics of the dynamics remain the same. More specifically, a repulsive (attractive) effective interaction between the left and the right bright solitons occurs when the systems are in the IP (OP) configuration [see Figs. 1(b) and 1(e) for the OP configuration of the NS system], in agreement with previous results [25,26]. However, in case of the OP configuration, the spinor systems (F and AF) exhibit significantly different behavior from the NS one; that is, instead of forming a bound state, they appear to escape after one (or a few) collision(s) [see Figs. 1(a), 1(c), 1(d), and 1(f)]. Note that

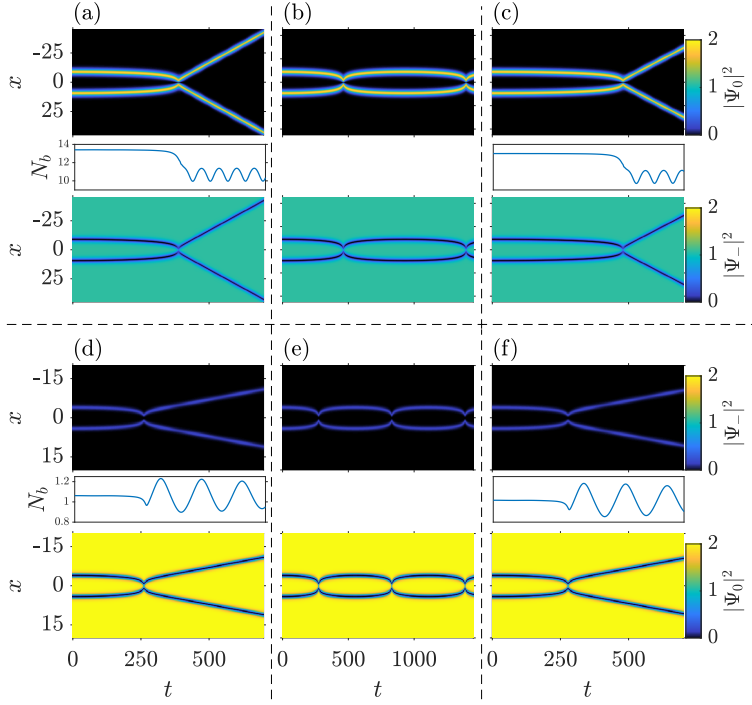


FIG. 1. Example of the spatiotemporal evolution of the densities of the dark ($|\Psi_0|^2$) and bright ($|\Psi_-|^2$) components of an OP two-DBD (two-BDB) [panels (a)–(c)] ([panels (d)–(f)]) system in case of the F [panels (a) and (d)], NS- [panels (b) and (e)] and AF [panels (c) and (f)] interactions, for $d_0 = 18$ and $q = -0.16$ ($d_0 = 8$ and $q = 0.835$) [panels (a)–(c)] ([panels (d)–(f)]). In case of the F and AF systems, the time evolution of the number of particles of the bright components (N_b) is depicted as well. The $|\Psi_+|^2$ density is not displayed here, since $\psi_+(t) = \psi_-(t)$, as discussed in Sec. II C.

instances where solitons undergo multiple collisions before escaping are rare and typically occur at shorter initial distances, where the two-soliton ansatz has limited validity.

The dynamics of the OP spinor systems includes also significant changes of the individual particle numbers of the $m_F = 0, \pm 1$ components. More specifically, as the left and right solitons deform while approaching the collision point, intercomponent particle exchange can be identified [Figs. 1(a), 1(c), 1(d), and 1(f)], associated with the activation of the spin-mixing terms of the Hamiltonian, followed by postcollision oscillatory behavior, a characteristic signature of the breathing motion of the solitons. Notably, in systems with small $|q|$ values, a permanent intercomponent mass transfer occurs, since the postcollision average particle number of the bright component N_b lies clearly below the initial value [Figs. 1(a) and 1(c)]. The described spin-mixing-induced particle exchange is a central feature of the spinorial system whose role, to the best of our knowledge, has not been previously appreciated.

Finally, we emphasize that the spin-mixing-induced escape constitutes a robust physical feature of the OP spinor dynamics and is not a consequence of the initial superposition ansatz. Indeed, the escape behavior is consistently observed for sufficiently large initial separations, where the overlap between the two solitary waves is negligible and the condition $d_0 > 4D^{-1}$ is well satisfied. This robustness extends throughout the investigated q range. Deviations from the escape dynamics are only encountered for a limited number of cases with very small initial separations, where the initial overlap is already significant and the two-soliton ansatz itself becomes less reliable.

B. Effective description

In what follows, we attempt to capture the dynamical behavior of the above studied systems, and more importantly the ability of the OP spinor solitons to escape, adopting an effective classical particle approach. More specifically, we employ the adiabatic approximation of the Lagrangian theory for solitons, according to which the soliton parameters become slowly varying functions of time, but the functional form of the soliton remains unchanged [43].

In order to build the two-soliton ansatz, we recall Sec. II C where we have shown that, in case of the IP and OP systems, Eq. (1) can be reduced to Eq. (13). Therefore, a DBD (BDB) system reduces effectively to a DB system with $\mu_d = \mu_1 = \mu - q$, $\mu_b = \mu_0 = \mu$ ($\mu_d = \mu_0 = \mu$, $\mu_b = \mu_1 = \mu - q$). Note that the quadratic Zeeman term q acts as an effective chemical potential on the Ψ_1 component. After rescaling space-time coordinates as $t \rightarrow \mu_d t$, $x \rightarrow \sqrt{\mu_d} x$ and the densities $|\Psi|^2 \rightarrow \mu_d^{-1} |\Psi|^2$, the ansatz for the dark and bright components of the two-soliton state reads

$$\begin{aligned} \Psi_d(x, t) &= (\nu \tanh(D\tilde{x}_-) + i\lambda)(\nu \tanh(D\tilde{x}_+) - i\lambda), \\ \Psi_b(x, t) &= \frac{\sqrt{DN_b}}{2} \sec h(D\tilde{x}_-) e^{i(x\frac{\xi}{2} + \varphi(t))} + \frac{\sqrt{DN_b}}{2} \sec h(D\tilde{x}_+) e^{i\Delta\theta} e^{i(-x\frac{\xi}{2} + \varphi(t))}, \end{aligned} \quad (14)$$

where $\tilde{x}_{\pm}(t) = x \pm \frac{\xi(t)}{2}$, with $\xi(t)$ being the distance between the soliton centers and $\Delta\theta$ the phase difference between left and right bright solitons of the same hyperfine component. Within the adiabatic Lagrangian approach, the soliton parameters become time-dependent variational parameters. The Lagrangian density reads

$$\begin{aligned} \mathcal{L}\{\Psi\} &= \frac{i}{2}(\Psi_d^* \partial_t \Psi_d - \Psi_d \partial_t \Psi_d^*) \left(1 - \frac{1}{|\Psi_d|^2}\right) + \frac{i}{2}(\Psi_b^* \partial_t \Psi_b - \Psi_b \partial_t \Psi_b^*) - \frac{1}{2}|\partial_x \Psi_d|^2 - \frac{1}{2}|\partial_x \Psi_b|^2 \\ &\quad - \frac{1}{2}(|\Psi_d|^2 + |\Psi_b|^2 - 1)^2 + (\tilde{\mu} - 1)|\Psi_b|^2 - g_s(|\Psi_d|^2 |\Psi_b|^2 + \text{Re}\{(\Psi_d^*)^2 \Psi_b^2\}), \end{aligned} \quad (15)$$

from which the averaged Lagrangian is obtained

$$L = \int_{-\infty}^{\infty} \mathcal{L} dx = \mathcal{T} - (2E_1 + E_{dd} + E_{bb} + E_{db} + E_s), \quad (16)$$

where

$$\begin{aligned} \mathcal{T} &\approx 2\dot{\xi} \left[-\nu\lambda + \arctan\left(\frac{\nu}{\lambda}\right) \right] - \frac{N_b}{4} \xi \ddot{\xi} - N_b \dot{\varphi} [1 + 2D \cos(\Delta\theta) \xi e^{-D\xi}] \\ &\quad - \frac{4\nu\dot{\lambda}}{D} [1 + 2(1 - 2D\xi) e^{-2D\xi}], \\ E_1 &= \frac{2\nu^2}{3} \left(D - \frac{N_b}{2} + \frac{\nu^2}{D} \right) + \frac{N_b}{16} \dot{\xi}^2 + \frac{DN_b}{12} \left(D + \frac{N_b}{2} \right) + \frac{N_b}{2} (1 - \tilde{\mu}), \\ E_{dd} &\approx 16 \left[-\frac{7}{3D} + \frac{4D}{3} + (1 - D^2) \xi \right] e^{-2D\xi}, \\ E_{bb} &\approx DN_b \left[2D \left(1 - \frac{1}{2} D\xi \right) + N_b \right] \cos(\Delta\theta) e^{-D\xi} - DN_b^2 [1 + 2 \cos^2(\Delta\theta)] (1 - D\xi) e^{-2D\xi}, \\ E_{db} &\approx 4N_b \left[\frac{1}{2} (1 - \tilde{\mu}) D\xi - 1 \right] \cos(\Delta\theta) e^{-D\xi} + \left(\frac{40}{3} - 8D\xi \right) N_b e^{-2D\xi}, \\ E_s &\approx \frac{g_s}{3} N_b [1 + \cos(\Delta\theta) \cos(\Delta\theta + 2\varphi)], \end{aligned}$$

and $\tilde{\mu} \equiv \frac{\mu_b}{\mu_d}$ is the rescaled chemical potential. In the above expression, $\mathcal{T}$ collects all contributions involving time derivatives of the fields, E_1 is the (potential) energy of a single DB soliton, E_{dd} , E_{bb} , and E_{db} account for the interaction between the two solitons, with E_{dd} denoting the—generically repulsive—interaction between the two darks, E_{bb} the interaction between the two brights, and E_{db} the interaction energy between the dark soliton of one component and the bright soliton in the other component, while E_s accounts for the spin interactions. For the bright soliton interaction, we notice the dependence on $\cos(\Delta\theta)$, which crucially changes the nature of the force as $\Delta\theta$ varies from 0 to π .

Since we consider slowly moving, well-separated solitons, and a small spin-coupling parameter g_s , we retain only terms up to order $\mathcal{O}(g_s)$, $\mathcal{O}(e^{-2D\xi})$, and $\mathcal{O}(\dot{\xi}^2)$, and neglect higher powers as well as mixed products of these small quantities. Within this approximation, the spin interaction acts as an internal contribution for each soliton, independent of both the soliton-soliton interaction [terms $\mathcal{O}(g_s e^{-D\xi})$ and higher are neglected] and the soliton's motion [terms $\mathcal{O}(g_s \dot{\xi})$ and higher are neglected]. Likewise, the soliton-soliton interaction is independent of the soliton's motion [terms $\mathcal{O}(\dot{\xi} e^{-D\xi})$ and higher are neglected].

The variational parameters are the soliton distance $\xi(t)$, and the phase $\varphi(t)$ and particle number $N_b(t)$ of the bright component, while $\lambda(t)$ and $D(t)$ are constrained by the solitonic relations:

$$D^2 = 1 - \lambda^2 - \frac{DN_b}{4}, \quad \lambda = \frac{v\dot{\xi}}{2D}, \quad v^2 + \lambda^2 = 1. \quad (18)$$

For slow moving solitons, the above system of equations can be approximately expressed in terms of the free variational parameters as follows:

$$D = -\frac{N_b}{8} + \sqrt{1 + \lambda^2 + \left(\frac{N_b}{8}\right)^2}, \quad \lambda \approx \frac{\dot{\xi}}{2D^{(0)}}, \quad (19)$$

where

$$D^{(0)} = -\frac{N_b}{8} + \sqrt{1 + \left(\frac{N_b}{8}\right)^2}. \quad (20)$$

In order to get these approximations, we neglect terms $\mathcal{O}(\dot{\xi}^3)$ and higher. Note here that in the absence of soliton-soliton interaction, the phase $\varphi(t)$ and particle number $N_b(t)$ of the bright component form a canonical pair.

The time evolution of the soliton parameters can be obtained by applying the Euler-Lagrange equations

$$\partial_\varphi L - \frac{d}{dt}(\partial_{\dot{\varphi}} L) = 0, \quad (21)$$

$$\partial_{N_b} L - \frac{d}{dt}(\partial_{\dot{N}_b} L) = 0, \quad (22)$$

and

$$\partial_\xi L - \frac{d}{dt}(\partial_{\dot{\xi}} L) + \frac{d^2}{dt^2}(\partial_{\ddot{\xi}} L) = 0, \quad (23)$$

which lead to the evolution equations

$$\dot{N}_b \approx -\frac{2}{3} g_s N_b \cos(\Delta\theta) \sin(\Delta\theta + 2\varphi), \quad (24)$$

$$\dot{\varphi} \approx \frac{1}{2} \left(D^2 - \frac{\dot{\xi}^2}{4} \right) + \tilde{\mu} - 1, \quad (25)$$

and

$$\ddot{\xi} \approx \left[\frac{8N_b(D^{(0)})^4 \cos(\Delta\theta)}{8 - D^{(0)}N_b} \right] e^{-D^{(0)}\xi} + \left[\frac{4D^{(0)}(-272 + D^{(0)}(96\xi + 104N_b + D^{(0)}(176 - 96D^{(0)}\xi - 48N_b\xi + 3(2 + \cos(2\Delta\theta))(-3 + 2D^{(0)}\xi)N_b^2)))}{3(-8 + D^{(0)}N_b)} \right] \times e^{-2D^{(0)}\xi}. \quad (26)$$

Note that terms $\mathcal{O}(g_s)$ and $\mathcal{O}(e^{-D\xi})$ have been neglected from Eq. (25), since the latter is already $\mathcal{O}(1)$ and therefore interaction and spinor terms serve as small corrections. On the other hand, terms $\mathcal{O}(g_s)$ are neglected from Eq. (26) since they would lead to spurious self-acceleration terms, indicating forces acting on the solitons even at large distances, where the interaction between them is practically negligible. At these distances, the solitary waves evolve as individual, independent entities and hence such terms are artifacts of the ansatz.

To assess the validity of the suggested model, the system of equations [Eqs. (24)–(26)] has been evaluated numerically using a fourth-order Runge-Kutta integrator. The initial parameters are $\xi(t=0) = d_0$, where d_0 is the initial distance between the solitons, $\dot{\xi}(t=0) = 0$, $\varphi(t=0) = 0$, and $(\varphi(t=0) = \frac{\pi}{2})$ in case of the F (AF) system, and $N_b(t=0) = \frac{4(1-2(1-\bar{\mu}))}{\sqrt{2(1-\bar{\mu})}}$. The latter is obtained from Eq. (25) by imposing $\dot{\varphi} = 0$ in case of a stationary state in order to fulfill Eq. (2) [note that the latter requires fixing $\varphi(t)$ to a specific value]. This approach provides $N_b(t=0)$ values that deviate from those obtained from our numerical simulations by less than 5% in most cases.

To quantitatively compare the predicted trajectories $\xi(t)$ with the original soliton trajectories $d(t)$, we introduce the following fidelity measure, based on the symmetric mean absolute percentage error (sMAPE) metric [44]

$$F = 1 - \frac{1}{\mathcal{N}} \sum_{i=1}^{\mathcal{N}} \frac{|\dot{d}(t_i) - \dot{\xi}(t_i)|}{|\dot{d}(t_i)| + |\dot{\xi}(t_i)|}. \quad (27)$$

Note that we use the corresponding velocities instead of the trajectories themselves to avoid the potential contribution of a constant offset and to capture the dynamical character of the matching of the respective trajectories.

For the NS systems in the OP configuration, the number of time points ($\mathcal{N}$) considered in the evaluation is determined by the criterion that the solitons have experienced at most three collisions. In all other scenarios, $\mathcal{N}$ corresponds to the time point when the soliton acceleration drops below 10^{-4} , signifying negligible interaction and indicating that the solitons move at constant velocity.

In Fig. 2 (Fig. 3), we compare the effective model with our numerical simulations for the IP (OP) system. In the IP case, we can observe a very reasonable agreement of the solitonic repulsive trajectories within the spinor system. While the effect of the pairwise solitonic repulsion seems to be weakest in the NS case, it appears to be captured well in all three scenarios (F, NS, and AF, respectively, in the figure). In Fig. 4, we depict the time evolution of the number of particles of the bright components (N_b) in case of the spinor OP systems. It is once again clear that the exchange of particles between the components and the activation of the spin-mixing terms is critical for observing the resulting phenomenology. The aggregated results for the fidelities for both systems and all available initial parameter sets (d_0, q) are shown in Figs. 5 and 6. Note here that we consider the comparison to be unsuccessful in cases involving spinor OP systems where the effective model incorrectly forecasts a bound state within the simulation time frame. These cases are highlighted in red in Fig. 6 and we set $F = 0$. Additionally, note that there are two distinct (d_0, q) areas of missing data (colored black): one for small (d_0, q) values, where the initial overlap of the two-soliton system is significant, and thus these cases are excluded, and another for large (d_0, q) values, where the interaction between solitons is negligible within the fixed simulation time.

From the presented results, it is clear that in the case of IP configurations the model yields significantly better results as compared to OP configurations. This difference can be attributed to the

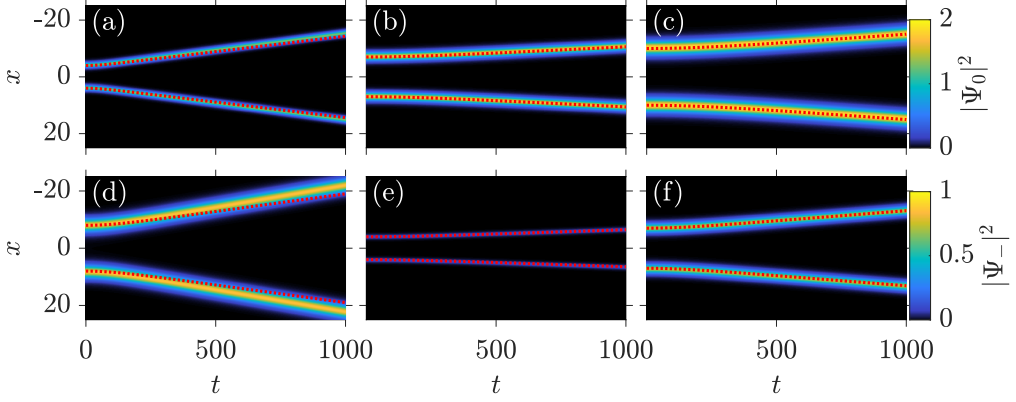


FIG. 2. Characteristic results of the comparison between the spatiotemporal evolution of the IP systems and the trajectories predicted from the model (red dotted lines). The depicted cases: (a) two-DBD F system with $q = -0.72$ and $d_0 = 8$ ($F = 0.96$), (b) two-DBD NS system with $q = -0.32$ and $d_0 = 14$ ($F = 1$), (c) two-DBD AF system with $q = -0.12$ and $d_0 = 20$ ($F = 0.98$), (d) two-BDB F system with $q = 0.145$ and $d_0 = 16$ ($F = 0.86$), (e) two-BDB NS system with $q = 0.895$ and $d_0 = 8$ ($F = 1$), and (f) two-BDB AF system with $q = 0.26$ and $d_0 = 14$ ($F = 0.99$).

fact that the soliton overlap remains small throughout the dynamics in the repulsive IP configuration, whereas the attractive OP interaction inevitably leads to large overlaps around collision points, in turn invalidating the assumptions associated with our model.

For the OP configuration, we observe that the model does a good job at capturing numerous of its qualitative characteristics. As Figs. 3 and 4 illustrate, there are scenarios where the evolution of the bright atom number exchange is reasonably accurate. In these cases, the particle model closely follows the PDE. In other cases, e.g., Fig. 4(c), the sensitivity of the exchange does not enable the accurate prediction thereof and ultimately leads into a failed diagnosis of the resulting dynamics. The NS case is again immune to such features, due to its conservation of N_b and is thus generally well captured.

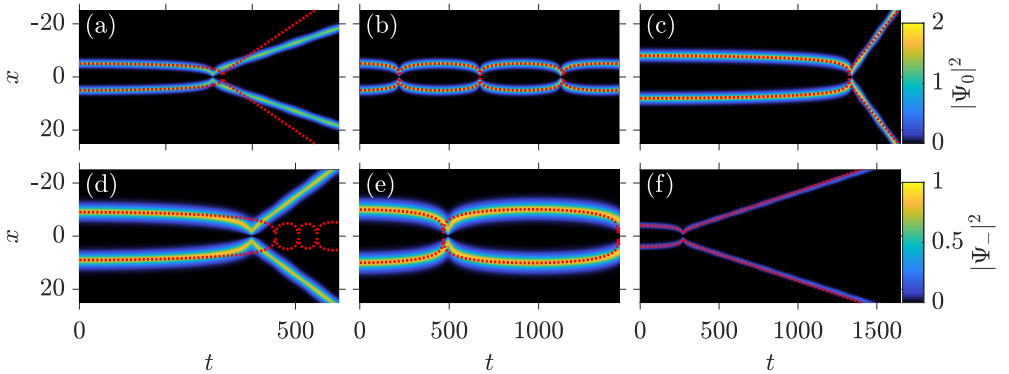


FIG. 3. Same as Fig. 2 but for the OP configuration. The depicted cases: (a) two-DBD F system with $q = -0.67$ and $d_0 = 10$ ($F = 0.71$), (b) two-DBD NS system with $q = -0.56$ and $d_0 = 10$ ($F = 0.96$), (c) two-DBD AF system with $q = -0.36$ and $d_0 = 16$ ($F = 0.88$), (d) two-BDB F system with $q = 0.16$ and $d_0 = 18$ ($F = 0$), (e) two-BDB NS system with $q = 0.12$ and $d_0 = 20$ ($F = 0.80$), (f) two-BDB AF system with $q = 0.835$ and $d_0 = 8$ ($F = 0.98$).

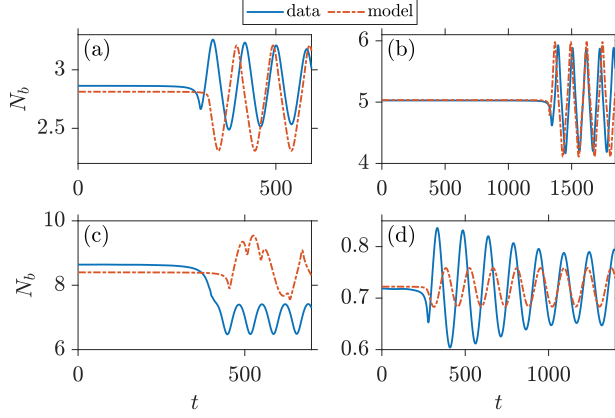


FIG. 4. Characteristic results of the comparison between the time evolution of the number of particles of the bright components (N_b) and the model predictions in case of the OP system. The depicted cases: (a) two-DBD F system with $q = -0.67$ and $d_0 = 10$, (b) two-DBD AF system with $q = -0.36$ and $d_0 = 16$, (c) two-BDB F system with $q = 0.16$ and $d_0 = 18$ ($F = 0$), and (d) two-BDB AF system with $q = 0.835$ and $d_0 = 8$.

The reduced accuracy observed in some OP systems can be attributed primarily to the limitations of the variational two-soliton ansatz underlying the effective model. Within the adiabatic Lagrangian framework, the individual soliton profiles are assumed to retain their single-soliton functional form throughout the dynamics, while the two-soliton state is constructed by adding the bright components and multiplying and normalizing the dark components. This approximation becomes progressively less accurate as the overlap between the two solitary waves increases during the collision, leading to profile deformations that cannot be represented within the chosen ansatz. This trend is particularly evident for small values of $|q|$, where the solitary waves are substantially wider and therefore experience stronger overlap and mutual deformation. Consequently, the effective particle description becomes less accurate in this regime.

In addition to this systematic trend, a small number of outlier cases remain in the F system, where the effective model fails despite the relatively localized nature of the solitons. Our analysis suggests that these cases may be attributed to an additional limitation associated with the construction of the variational ansatz itself. The latter is based on the exact NS (Manakov) soliton profiles, which also

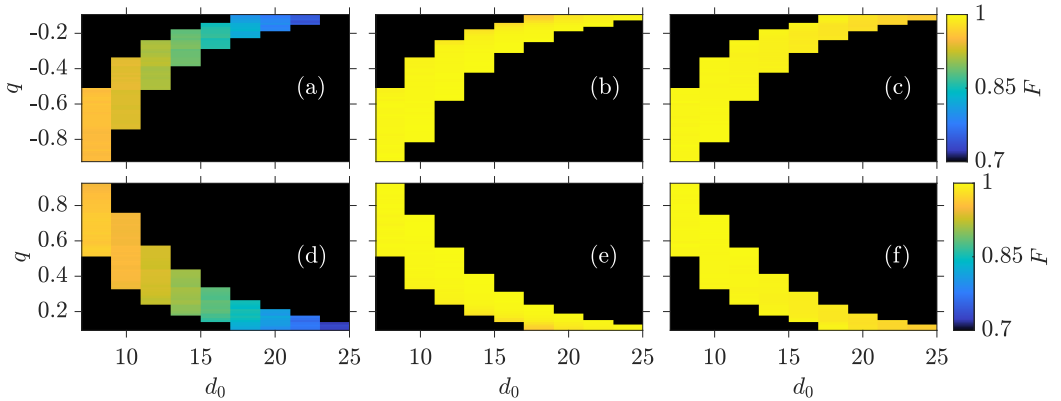


FIG. 5. Fidelities of the IP systems: (a) two-DBD F system, (b) two-DBD NS system, (c) two-DBD AF system, (d) two-BDB F system, (e) two-BDB NS system, and (f) two-BDB AF system.

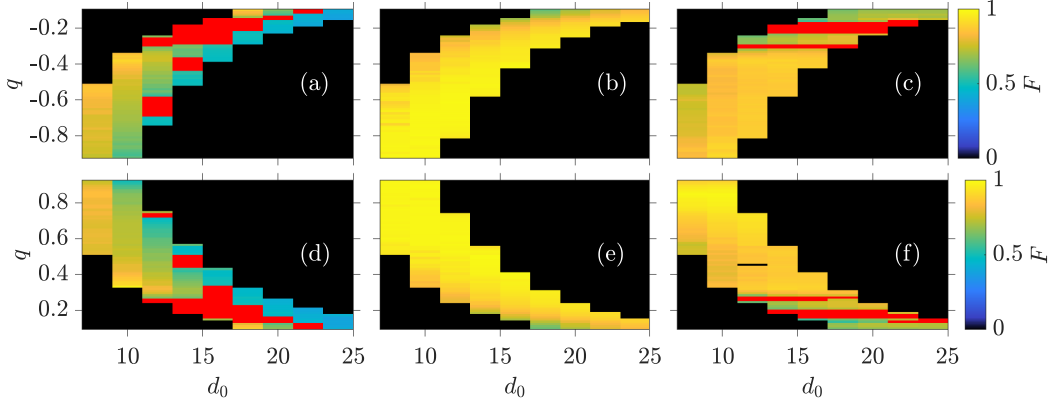


FIG. 6. Same as Fig. 5 but for the OP configuration.

reproduce the stationary density profiles of the symmetric AF states because the spin-dependent contributions cancel for the corresponding relative phase. In the F system, however, the spin-dependent energy remains finite and negative, so that the true stationary soliton profiles generally acquire corrections with respect to their NS counterparts. These corrections are not included in the present zeroth-order ansatz and, although they are expected to remain small for the weak spin coupling considered here, they may become relevant during collisions and contribute to the isolated discrepancies observed in the F system. A more detailed investigation of these cases and of the role of corrections to the underlying soliton ansatz would constitute an interesting direction for future work.

IV. CONCLUSIONS AND FUTURE DIRECTIONS

In this study, we investigated the interactions between pairs of DBD and BDB spinor solitons in free space. We explored the dependence of these interactions on the phase differences among the constituent bright solitons and examined the influence of spin interactions on the dynamics. To accomplish this, we created stationary single-soliton solutions that we used as building blocks to initialize the two-soliton system. Based on the symmetries of the system, we identified the different case scenarios and considered the time evolution of the initial two-soliton states for varying initial parameters.

Our numerical results confirm that IP configurations are repulsive, while OP configurations are attractive at larger distances but become repulsive at shorter distances, consistent with previous studies [25,26]. In the out-of-phase scenario, spin interactions were found to break the bound state, enabling solitons to escape and causing particle exchange among the $m_F = 0, \pm 1$ components. We furthermore attempted to capture the spinor dynamics by considering an effective classical model based on a Lagrangian approach. The accuracy of the proposed model was tested by direct comparison between the soliton trajectories and the model predictions, employing a fidelity measure based on the comparison between the data and predicted velocities using the sMAPE metric.

Despite its limitations, our proposed effective description successfully captured the dynamical behavior of interacting solitons in the majority of cases, providing a reasonable approximation of the soliton trajectories and the interspecies particle-exchange dynamics, offering insights into the complex nature of soliton interactions in spinor BECs. The most accurate case was the nonspinorial one, where atom numbers were conserved. Subsequently, we also generally captured the repulsive interaction between IP bright solitary waves. The most challenging scenario, due to the exchange of particles between the components, was realized in the setting of OP solitonic interactions.

Future work may focus on refining the model to address its limitations and extend its applicability to more diverse and dynamically sensitive scenarios. In this context, alternative two-soliton ansätze,

based entirely on numerical methods, might better capture the dynamics. Indeed, this problem may constitute an interesting setting for the potential implementation of discovery techniques of effective ordinary differential equations via machine learning methods, for which numerous techniques exist (such as SINDy [45], DeepXDE [46], DUE [47], among many others). Naturally, generalizing also the relevant considerations to higher-dimensional spinor systems involving vortices and bright solitary waves will be of particular interest in its own right.

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

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