

Nonlocality in continuous-variable quantum networks

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Quantum networks enable forms of nonlocality beyond the standard Bell scenario, with a multitude of potential applications. Continuous-variable (CV) platforms are particularly attractive for large-scale networks, offering deterministic entanglement generation and favorable prospects for long-distance distribution. Here, we present a formalism to study CV network nonlocality using pseudospin measurements. Considering the linear chain and star configurations, we derive the maximal violations of the corresponding network locality inequalities for arbitrary two-mode states. Using two-mode squeezed vacuum states, we show that the strength of nonlocality in the star configuration remains independent of the network size. Moreover, the nonlocal correlations persist even at arbitrarily high temperatures provided the squeezing exceeds a critical threshold. Further, we demonstrate non-Gaussianity as an enhancer of network nonlocality through illustrations of various classes of non-Gaussian resources. Remarkably, a coherent superposition of single-photon subtractions across modes achieves maximal violation for arbitrarily small squeezing. Finally, we provide schematics of an experimentally feasible implementation of CV network nonlocality based on the isomorphism between pseudospin and spatial parity observables.

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I. INTRODUCTION

Quantum nonlocality, manifested through violations of Bell inequalities [1], contradicts the classical notion of local realism [2] and lies at the heart of quantum information science [3]. Beyond its foundational significance, nonlocality enables practical tasks such as device-independent protocols [4–8], quantum cryptography [9–12], communication complexity [13], and randomness certification [14,15]. Rapid experimental progress has turned nonlocality into a reliable resource, with loophole-free demonstrations achieved on multiple platforms, including long-distance optical implementations on telecom networks [16–22]. Alongside these developments, nonlocality in continuous-variable (CV) systems remains as an active area of research [23–28]. CV systems are especially attractive due to their natural implementation in quantum optics [29–32].

Gaussian states, characterized by positive Wigner functions [33], are fundamental resources in CV quantum information [24]. Bell inequality violations for the two-mode squeezed vacuum (TMSV) state were first demonstrated using displaced parity measurements [34]. A major advance followed with the introduction of pseudospin observables, which enabled maximal Bell violations

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for Einstein-Podolsky-Rosen (EPR) states and provided an effective framework for probing nonlocality in CV systems [35]. Experimental demonstrations based on spatial parity measurements have further confirmed the nonlocal nature of Gaussian states [36]. The established isomorphism between spatial parity and pseudospin observables provides further motivation for exploring this approach.

On the other hand, the well-known limitations of Gaussian states and operations for tasks such as entanglement distillation and quantum error correction [37–42] have motivated increasing interest in non-Gaussian states. Examples of such non-Gaussian states, such as those generated via photon addition or subtraction, have been realized experimentally and enable advanced quantum applications through the help of features like enhanced entanglement, stronger Bell violations, and EPR steering [43–52]. These developments naturally motivate elevation of powerful CV nonlocal characteristics of Gaussian and non-Gaussian states beyond bipartite settings to the framework of quantum networks.

Quantum networks have emerged as a foundational setting for exploring nonlocal correlations that go beyond those captured by standard Bell-type experiments [53]. The motivation for studying network nonlocality is fundamentally different from that of standard Bell nonlocality. While Bell nonlocality concerns correlations generated by a single entangled source, network nonlocality is specifically designed to characterize correlations produced by multiple *independent* entangled sources distributing quantum states to nodes capable of performing joint measurements, allowing richer and stronger correlations than in bipartite settings [53–60]. Quantum networks can be used to generate long-distance entanglement distributions without direct interaction between endpoints using entanglement-swapping protocols essential for quantum repeaters [61–63], and form a fundamental backbone of scalable quantum technologies and the quantum internet [62–64]. Network nonlocality manifested in terms of quantum steering enables randomness certification [65]. Moreover, an analysis of network nonlocality-based security of quantum key distribution has been performed recently [66]. The potential for device-independent self-testing in the context of multiparty networks has been pointed out [67]. Certifying network nonlocality is expected to play an important role in the verification and benchmarking of future large-scale quantum communication networks, where independent sources constitute an essential physical resource.

CV systems offer certain key advantages in quantum communication, including deterministic entanglement generation and suitability for long-range distribution [23]. Despite such favorable prospects, and a background of a wide range of studies in discrete variable (DV) quantum networks [53], quantum nonlocality in CV networks remains largely unexplored, with only a limited number of studies in the literature [68–70]. It has been shown that Gaussian states with positive Wigner functions cannot exhibit network nonlocality under Gaussian phase-space measurements [68]. A one-way device-independent protocol for a specific tripartite CV entanglement-swapping network (which is simply the bilocal network) assisted by one-way classical communication has been introduced [69]. Further, a framework for witnessing CV network nonlocality with Gaussian states based on non-Gaussian generalized quasiprobability-function measurements has been proposed [70]. In the present work, we develop a fundamentally different pseudospin-based formalism to study CV network nonlocality. Our approach enables the derivation of an analytical criterion for the maximal violation of network-locality inequalities by arbitrary two-mode CV states. Moreover, unlike previous studies, our framework naturally incorporates both Gaussian and non-Gaussian resources, allows us to investigate thermal robustness and non-Gaussian enhancement of network nonlocality, and leads to an experimentally feasible implementation based on the correspondence between pseudospin and spatial parity observables.

In this work, we investigate nonlocality in quantum networks, focusing on two types of well-known open networks, viz., linear chain [58] and star networks [60] involving CV states. We provide a formalism for probing nonlocal features in these n -partite CV systems by employing pseudospin measurements. We derive explicit expressions for the maximal violation of the n -local inequalities that can be achieved by arbitrary two-mode CV states for both the above types of networks. As our first example, we consider the Gaussian TMSV state and show that it displays nonlocality in both the chain and star networks for any nonzero squeezing, with the violation of the corresponding

n -local inequality increasing monotonically and reaching its maximum in the infinite-squeezing limit. Moreover, for the star network with identically distributed states, we find that the maximal n -locality violation is independent of the network size.

We extend further the formalism developed here for studying quantum nonlocality in CV networks in several directions. We investigate the robustness of network nonlocality under local thermal noise, which, in general, degrades the nonlocal correlations. As a remarkable off-shoot of our study, it is observed that when the squeezing exceeds a critical value, the state can exhibit nonlocality for any arbitrarily high temperature. Moreover, we apply our formalism to examples of popular classes of non-Gaussian states such as the single-photon subtracted TMSV state, the entangled coherent state (ECS), and the CV Werner state. In addition to subtle features brought about by the interplay of mode operations, entanglement, and noise mixing, our results illustrate enhanced violation of network locality by incorporating non-Gaussianity compared to the magnitude of nonlocality obtained through the Gaussian TMSV state. We finally present a strategy for experimentally observing CV network nonlocality using spatial parity observables, by exploiting the well-established isomorphism between the single-mode multiphoton electromagnetic field space spanned by the Fock basis in pseudospin space and the single-photon multimode electromagnetic field space spanned by the spatial eigenmodes [36,71–73].

The remainder of the paper is organized as follows. In Sec. II, we provide a concise overview of the bilocal network scenario and the pseudospin framework for CV systems. Section III presents the general methodology used throughout this work, including the adaptation of DV formalism to CV systems and the strategy for optimizing the associated inequalities focusing on linear chain and star topologies. In Sec. IV, we probe network nonlocality using the TMSV states, followed by an examination of how local thermal noise affects its performance. Section V is devoted to studying non-Gaussian states. We highlight the impact of single-photon subtraction on TMSV states across different subtraction schemes, revealing how such operations can enhance nonlocality, along with displaying similar enhancement of nonlocality through the non-Gaussianity of entangled coherent and CV Werner states. In Sec. VI, we provide an experimental proposal to reveal CV network nonlocality in the spatial parity space. Finally, in Sec. VII, we summarize our main results and suggest potential directions for future research.

II. PRELIMINARIES

This section outlines the essential background for our study. We first recall the bilocal network, which is the simplest example of both linear chain and star-network configurations. This is followed by a discussion of pseudospin observables and their relevant properties, which play a central role in our analysis of network nonlocality with CV states.

A. Bilocal versus nonbilocal correlations

The bilocal network involves three spatially separated observers: Alice, Bob, and Charlie. The central party, Bob, shares an entangled state with Alice, distributed by a source S_1 , and another entangled state with Charlie, distributed by a second independent source S_2 .

The defining feature of the bilocal scenario is the independence of the two sources. In a classical hidden variable framework, this implies that the variables associated with S_1 and S_2 , denoted by Λ_1 and Λ_2 , are statistically independent. Any correlation that respects the bilocal structure must admit a probability decomposition of the form

$$p(A, B, C|x, y, z) = \iint d\Lambda_1 d\Lambda_2 q_1(\Lambda_1) q_2(\Lambda_2) p(A|x, \Lambda_1) p(B|y, \Lambda_1, \Lambda_2) p(C|z, \Lambda_2),$$

where x, y , and z denote the measurement settings chosen by Alice, Bob, and Charlie, respectively, and A, B , and C are their corresponding outcomes.

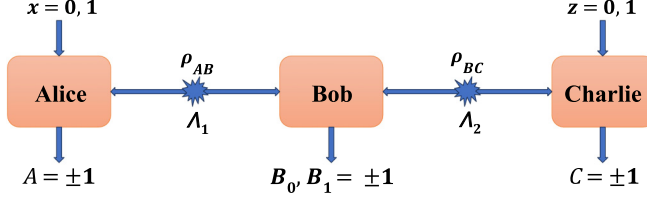


FIG. 1. Bilocal network scenario.

In the quantum realization, Bob receives two particles, one from each entangled pair, and performs a joint measurement. In this present work, we consider the scenario where Bob's measurement is fixed. Since Bob does not choose between different measurement settings, no measurement input (y) is associated with his station. Alice and Charlie each receive one particle from their respective pairs and perform local measurements with binary inputs $x, z \in \{0, 1\}$ and outputs $A_x, C_z \in \{\pm 1\}$. Bob's measurement yields two binary outcomes, denoted as $B_0, B_1 \in \{\pm 1\}$ (see Fig. 1).

Taking into account the bilocal decomposition, it can be shown that any bilocal hidden variable model must satisfy the following inequality [56]:

$$S_{\text{biloc}} = \sqrt{|I|} + \sqrt{|J|} \leq 2, \quad (1)$$

where the correlators I and J are defined as

$$\begin{aligned} I &= \langle (A_0 + A_1)B_0(C_0 + C_1) \rangle \\ &= \left\langle \sum_{x,z=0}^1 A_x B_0 C_z \right\rangle, \end{aligned} \quad (2)$$

$$\begin{aligned} J &= \langle (A_0 - A_1)B_1(C_0 - C_1) \rangle \\ &= \left\langle \sum_{x,z=0}^1 (-1)^{x+z} A_x B_1 C_z \right\rangle. \end{aligned} \quad (3)$$

Here, $\langle \cdot \rangle$ denotes the expectation value taken over many experimental runs.

Violation of the inequality in Eq. (1) provides evidence of nonbilocal correlations, that is, correlations that cannot be explained by any bilocal hidden variable model. This offers a refined notion of nonclassicality, going beyond standard Bell nonlocality by incorporating assumptions about source independence.

B. Pseudospin measurements

Pseudospin measurements were introduced to study Bell nonlocality with the TMSV state [35]. Later generalization has enabled one to capture nonlocal correlations using states with odd photon number difference between the two modes [74]. For a single-mode light field, the generalized pseudospin operators are defined in the Fock basis $\{|n\rangle\}$ as follows:

$$\begin{aligned} s_x &= \sum_{n=0}^{\infty} (|2n+q\rangle \langle 2n+q+1| + |2n+q+1\rangle \langle 2n+q|), \\ s_y &= i \sum_{n=0}^{\infty} (|2n+q\rangle \langle 2n+q+1| - |2n+q+1\rangle \langle 2n+q|), \\ s_z &= \sum_{n=0}^{\infty} (|2n+q+1\rangle \langle 2n+q+1| - |2n+q\rangle \langle 2n+q|), \end{aligned} \quad (4)$$

where q is an integer. These operators satisfy the standard $\mathfrak{su}(2)$ algebra, making them formally analogous to Pauli matrices σ^j , and are thus interpreted as effective spin-1/2 operators acting in the photon-number parity space.

The operator s_z for $q = 0$ corresponds to the negative of the parity operator $P = (-1)^N$, where N is the photon number operator, i.e., $s_z = -P$. The raising and lowering operators $s_{\pm} = \frac{1}{2}(s_x \pm is_y)$ serve as parity-flip operators. For $q = 0$, one can express s_- in terms of the annihilation operator as [35]

$$s_- = \frac{I + (-1)^N}{2\sqrt{N+1}} \hat{a},$$

where $\hat{a}$ is the annihilation operator. The commutation relations among the pseudospin operators are

$$[s_z, s_{\pm}] = \pm 2s_{\pm}, \quad [s_+, s_-] = s_z, \quad (5)$$

which mirror those of the standard spin-1/2 algebra. Hence, the vector operator $\vec{s} = (s_x, s_y, s_z)$ behaves analogously to the spin operator $\vec{\sigma}$ and can be viewed as generating rotations in the “parity spin” space of photons. In particular, rotations in this space can be described by the unitary operator

$$\begin{aligned} U(\tau, \hat{n}) &= \exp\left(-i\frac{\tau}{2} \hat{n} \cdot \vec{s}\right) \\ &= \cos\left(\frac{\tau}{2}\right) - i(\hat{n} \cdot \vec{s}) \sin\left(\frac{\tau}{2}\right), \end{aligned} \quad (6)$$

where τ is a rotation angle and $\hat{n}$ is an arbitrary unit vector.

III. FRAMEWORK

As explained in the previous section, the use of pseudospin operators allows for an effective connection between the infinite-dimensional Hilbert space of CV systems and the well-known two-dimensional (2D) formalism of spin-1/2 systems. This analogy facilitates the construction of Bell-type measurements and network local inequalities suited for CV states. We begin by defining parity-based Bell-like states and outlining partial Bell measurements, followed by the maximization strategy of the network locality inequalities for linear chain and star networks.

A. Partial Bell measurements

Following Ref. [75], we consider the parity eigenstates associated with mode j of the light field, defined as

$$|+\rangle_j \equiv \sum_{n=0}^{\infty} \mathcal{A}_n^{(j)} |2n+q+1\rangle_j, \quad |-\rangle_j \equiv \sum_{n=0}^{\infty} \mathcal{A}_n^{(j)} |2n+q\rangle_j, \quad (7)$$

where the coefficients $\mathcal{A}_n^{(j)}$ satisfy the normalization condition $\sum_{n=0}^{\infty} |\mathcal{A}_n^{(j)}|^2 = 1$. These parity-defined superpositions of Fock states form a qubitlike basis within an infinite-dimensional Hilbert space. The action of the pseudospin operators on these states is given by

$$s_{jz} |\pm\rangle_j = \pm |\pm\rangle_j, \quad s_{j\pm} |\pm\rangle_j = 0, \quad s_{j\pm} |\mp\rangle_j = |\pm\rangle_j. \quad (8)$$

Thus, $|\pm\rangle_j$ are eigenstates of s_{jz} with eigenvalues ± 1 , and $s_{j\pm}$ serve as raising and lowering operators in the parity space, mirroring the behavior of spin ladder operators in two-level systems.

Using the rotation operator defined in Eq. (6), one can construct eigenstates of arbitrary spin projections $\hat{n}_j \cdot \vec{s}_j$ for each mode. With this structure in place, one can define analogues of the Bell

states for CV systems:

$$\begin{aligned}
 |\Phi^+\rangle &= \frac{1}{\sqrt{2}}(|+\rangle_1 |+\rangle_2 + |-\rangle_1 |-\rangle_2), \\
 |\Phi^-\rangle &= \frac{1}{\sqrt{2}}(|+\rangle_1 |+\rangle_2 - |-\rangle_1 |-\rangle_2), \\
 |\Psi^+\rangle &= \frac{1}{\sqrt{2}}(|+\rangle_1 |-\rangle_2 + |-\rangle_1 |+\rangle_2), \\
 |\Psi^-\rangle &= \frac{1}{\sqrt{2}}(|+\rangle_1 |-\rangle_2 - |-\rangle_1 |+\rangle_2).
 \end{aligned} \tag{9}$$

These entangled states are defined entirely within an infinite-dimensional Fock space but retain the algebraic structure of qubit Bell states due to the pseudospin formalism.

We now define the partial Bell measurements performed by Bob using projective measurements in the pseudospin basis. Specifically, the measurement operators are

$$B_0 = (+1) \times (|\Phi^+\rangle \langle \Phi^+| + |\Phi^-\rangle \langle \Phi^-|) + (-1) \times (|\Psi^+\rangle \langle \Psi^+| + |\Psi^-\rangle \langle \Psi^-|) = s_z \otimes s_z, \tag{10}$$

$$B_1 = (+1) \times (|\Phi^+\rangle \langle \Phi^+| + |\Psi^+\rangle \langle \Psi^+|) + (-1) \times (|\Phi^-\rangle \langle \Phi^-| + |\Psi^-\rangle \langle \Psi^-|) = s_x \otimes s_x. \tag{11}$$

Importantly, B_0 and B_1 are separable operators composed of local pseudospin measurements. While they distinguish between certain pairs of Bell states, they do not implement a full Bell-state measurement, but rather a partial Bell measurement [55].

B. Bilocal correlations with pseudospin observables

Let us choose an arbitrary vector living on the surface of a unit sphere $\vec{a} = (\sin\theta_a \cos\phi_a, \sin\theta_a \sin\phi_a, \cos\theta_a)$, where θ_a, ϕ_a are the polar and azimuthal angles. $\vec{a}$ may be defined as the direction along which we measure parity spin $\vec{s}$. With $\vec{s} = (s_x, s_y, s_z)$, we have

$$\vec{a} \cdot \vec{s} = s_z \cos\theta_a + \sin\theta_a (e^{i\phi_a} s_- + e^{-i\phi_a} s_+). \tag{12}$$

Using the commutation relations in Eq. (5), we have $(\vec{a} \cdot \vec{s})^2 = I$, which means that the outcome of the measurements of the Hermitian operator $\vec{a} \cdot \vec{s}$ is ± 1 . Thus, the bilocal inequality in Eq. (1) can be directly used in CV systems considering pseudospin measurements.

To simplify the analysis, we fix $\phi_a = \phi_c = 0$ for all the measurements performed by Alice and Charlie, thereby restricting all projective measurements by Alice and Charlie to the X - Z plane. Each measurement of Alice and Charlie can be characterized by one angle. The optimal settings are given by angles symmetric with respect to Z axis, $\pm\theta_1$ for Alice, $\pm\theta_2$ for Charlie. For all $x, z = 0, 1$, we get

$$\begin{aligned}
 \langle A_x B_0 C_z \rangle &= \langle [\cos(\theta_1) s_z + (-1)^x \sin(\theta_1) s_x] \otimes (s_z \otimes s_z) \\
 &\quad \otimes [\cos(\theta_2) s_z + (-1)^z \sin(\theta_2) s_x] \rangle_{\rho_{AB} \otimes \rho_{BC}},
 \end{aligned} \tag{13}$$

$$\begin{aligned}
 \langle A_x B_1 C_z \rangle &= \langle [\cos(\theta_1) s_z + (-1)^x \sin(\theta_1) s_x] \otimes (s_x \otimes s_x) \\
 &\quad \otimes [\cos(\theta_2) s_z + (-1)^z \sin(\theta_2) s_x] \rangle_{\rho_{AB} \otimes \rho_{BC}},
 \end{aligned} \tag{14}$$

using Eqs. (2) and (3), the bilocal expression can be written as

$$S(\theta_1, \theta_2) = \sqrt{|I(\theta_1, \theta_2)|} + \sqrt{|J(\theta_1, \theta_2)|}, \tag{15}$$

with respect to the settings specified by θ_1, θ_2 for a given state $\rho_{AB} \otimes \rho_{BC}$. Our goal is to maximize this expression over the measurement angles. The maximum value is given by

$$S^{\max} = \max_{\theta_1, \theta_2} S(\theta_1, \theta_2). \tag{16}$$

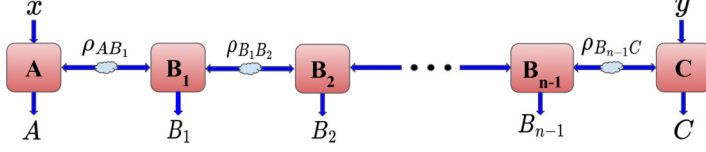


FIG. 2. Linear chain network scenario.

Note that the q parameters associated with the pseudospin operators may be chosen by examining the structure of the state. Once the q values are determined, the remaining task is to optimize over the measurement angles (θ_1, θ_2) .

C. Linear chain network scenario

We now consider a CV quantum network arranged in a linear chain configuration consisting of n independent sources. The extreme parties, Alice (A) and Charlie (C), are connected through a sequence of intermediate nodes $B_1, B_2, \dots, B_{n-1}$ (see Fig. 2). Each source distributes a two-mode state $\rho_{B_k B_{k+1}}$, with the identification $B_0 \equiv A$ and $B_n \equiv C$. The global state of the network is therefore

$$\rho = \bigotimes_{k=0}^{n-1} \rho_{B_k B_{k+1}}. \quad (17)$$

Alice and Charlie each perform one of two dichotomic pseudospin measurements, specified by the setting pairs $(\vec{a}, \vec{a}')$ and $(\vec{c}, \vec{c}')$, respectively. Each intermediate node B_k performs a fixed joint pseudospin measurement on the two modes received from the adjacent sources. These measurements are chosen such that, along one branch of the inequality, the correlations involve pseudospin s_z operators, while along the complementary branch they involve s_x operators.

Under the assumption of n -local hidden variables $\Lambda_1, \dots, \Lambda_n$, the observable correlations must satisfy the nonlinear n -local inequality [58],

$$S_{\text{chain}} = \sqrt{|I_n^{\text{chain}}|} + \sqrt{|J_n^{\text{chain}}|} \leq 2. \quad (18)$$

The correlators are defined as

$$I_n^{\text{chain}} = \langle (\vec{a} + \vec{a}') \cdot \vec{s} \otimes (s_z \otimes s_z)^{\otimes(n-1)} \otimes (\vec{c} + \vec{c}') \cdot \vec{s} \rangle, \quad (19)$$

$$J_n^{\text{chain}} = \langle (\vec{a} - \vec{a}') \cdot \vec{s} \otimes (s_x \otimes s_x)^{\otimes(n-1)} \otimes (\vec{c} - \vec{c}') \cdot \vec{s} \rangle. \quad (20)$$

Because the global state is a tensor product over independent sources and all observables are pseudospin operators, the correlators factorize into products of two-mode correlations. Introducing the pseudospin correlation matrices

$$t_{ij}^{(k)} = \text{Tr}[s_i \otimes s_j \rho_{B_k B_{k+1}}], \quad i, j \in \{1, 2, 3\}, \quad (21)$$

with $(s_1, s_2, s_3) = (s_x, s_y, s_z)$, Eqs. (19) and (20) reduce to

$$\begin{aligned} I_n^{\text{chain}} &= \text{Tr}[(\vec{a} + \vec{a}') \cdot \vec{s} \otimes s_z \rho_{B_0 B_1}] \times \left(\prod_{k=1}^{n-2} \text{Tr}[s_z \otimes s_z \rho_{B_k B_{k+1}}] \right) \times \text{Tr}[s_z \otimes (\vec{c} + \vec{c}') \cdot \vec{s} \rho_{B_{n-1} B_n}] \\ &= \sum_i (a_i + a'_i) t_{i3}^{(0)} \left(\prod_{k=1}^{n-2} t_{33}^{(k)} \right) \sum_j t_{3j}^{(n-1)} (c_j + c'_j), \end{aligned} \quad (22)$$

$$\begin{aligned} J_n^{\text{chain}} &= \text{Tr}[(\vec{a} - \vec{a}') \cdot \vec{s} \otimes s_x \rho_{B_0 B_1}] \times \left(\prod_{k=1}^{n-2} \text{Tr}[s_x \otimes s_x \rho_{B_k B_{k+1}}] \right) \text{Tr}[s_x \otimes (\vec{c} - \vec{c}') \cdot \vec{s} \rho_{B_{n-1} B_n}] \\ &= \sum_i (a_i - a'_i) t_{i1}^{(0)} \left(\prod_{k=1}^{n-2} t_{11}^{(k)} \right) \sum_j t_{1j}^{(n-1)} (c_j - c'_j). \end{aligned} \quad (23)$$

Before deriving the maximal n -locality violation, we first specify the measurement basis used at the intermediate nodes. For each source, we perform the singular value decomposition of the corresponding pseudospin correlation matrix. The pseudospin directions labeled by z and x are then chosen to coincide with the singular vectors associated with the two largest singular values. Equivalently, each intermediate node may be viewed as applying suitable local pseudospin basis rotations to the incoming modes before performing the standard partial Bell measurement defined by Eqs. (10) and (11). This is the CV analog of the optimization procedure employed in the DV framework [56]. Consequently, the optimization below is carried out over the measurement directions of the terminal parties, while the partial Bell measurements at the intermediate nodes are understood to be defined in these optimized local pseudospin bases.

With this construction, we now derive a closed-form expression for the maximal value of the n -locality expression achievable by arbitrary two-mode CV states under the restriction that all local measurements are pseudospin observables. The resulting criterion provides a direct CV analog of the well-known Horodecki criterion for Bell nonlocality [76], extending the correlation-matrix optimization method developed for two-qubit states [56] and subsequently generalized to n -local quantum networks [59,77].

Theorem 1. For a linear chain network probed by pseudospin measurements, the maximal value of the n -locality expression is

$$S_{\text{chain}}^{\max} = 2 \sqrt{\left(\prod_{k=0}^{n-1} v_1^{(k)}\right) + \left(\prod_{k=0}^{n-1} v_2^{(k)}\right)}, \quad (24)$$

where $v_1^{(k)} \geq v_2^{(k)} \geq v_3^{(k)} \geq 0$ are the singular values of the correlation matrix $t^{(k)}$ associated with the state $\rho_{B_k B_{k+1}}$. Consequently, the n -local inequality (18) is violated whenever

$$\prod_{k=0}^n v_1^{(k)} + \prod_{k=0}^n v_2^{(k)} > 1. \quad (25)$$

Proof. For each source k , we perform a singular-value decomposition of the correlation matrix $t^{(k)}$, with $R^{(k)} = \sqrt{(t^{(k)})^\dagger t^{(k)}}$. The non-negative eigenvalues of $R^{(k)}$ are denoted as $v_1^{(k)} \geq v_2^{(k)} \geq v_3^{(k)}$. The extremal parties Alice and Charlie can restrict their measurement directions to the two-dimensional subspaces spanned by the singular vectors corresponding to $v_1^{(0)}, v_2^{(0)}$ and $v_1^{(n-1)}, v_2^{(n-1)}$, respectively, without loss of generality. This simplification allows the use of the following parametrization:

$$\begin{aligned} \vec{a} &= (\sin \theta_1, 0, \cos \theta_1), \\ \vec{a}' &= (\sin \theta'_1, 0, \cos \theta'_1), \\ \vec{c} &= (\sin \theta_2, 0, \cos \theta_2), \\ \vec{c}' &= (\sin \theta'_2, 0, \cos \theta'_2). \end{aligned}$$

Under these choices, Eqs. (22) and (23) reduce to effective two-dimensional expressions,

$$I_n^{\text{chain}} = \left(\prod_{k=0}^{n-1} v_1^{(k)}\right) (\cos \theta_1 + \cos \theta'_1) (\cos \theta_2 + \cos \theta'_2), \quad (26)$$

$$J_n^{\text{chain}} = \left(\prod_{k=0}^{n-1} v_2^{(k)}\right) (\sin \theta_1 - \sin \theta'_1) (\sin \theta_2 - \sin \theta'_2). \quad (27)$$

The optimal angles are obtained by maximizing the n -locality expression with respect to the measurement parameters. Setting the partial derivatives of S_{chain} with respect to all angles to zero

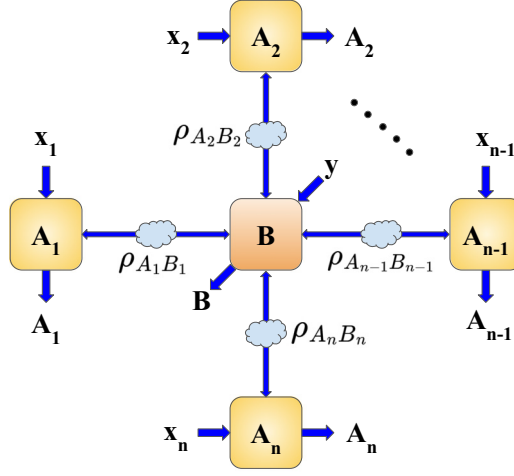


FIG. 3. Star-network scenario.

yields the following optimality conditions:

$$\theta'_1 = -\theta_1, \quad \theta'_2 = -\theta_2,$$

together with

$$\cos \theta_1 = \cos \theta_2 = \sqrt{\frac{\prod_{k=0}^{n-1} v_1^{(k)}}{\prod_{k=0}^{n-1} v_1^{(k)} + \prod_{k=0}^{n-1} v_2^{(k)}}}.$$

Substituting these values into Eq. (18) yields the maximal value $S_{\text{chain}}^{\text{max}}$ in Eq. (24). $\blacksquare$

In the symmetric case where all sources distribute identical states, $t^{(k)} = t$ for all k , the violation condition simplifies to

$$(v_1)^n + (v_2)^n > 1, \quad (28)$$

This condition shows that network nonlocality weakens with increasing chain length. Since $v_1, v_2 \leq 1$ for physical states, the contributions $(v_1)^n$ and $(v_2)^n$ either or both decrease monotonically as n grows. Consequently, noisy sources with $v_1, v_2 < 1$ cease to violate the inequality beyond a certain network size.

D. Star-network scenario

We now extend the correlation matrix optimization method to n -partite star network [59,60,77] probed by pseudospin measurements on CV systems. This scenario generalizes the bilocal configuration and consists of n independent sources, each distributing a two-mode CV state between one peripheral party A_i and a central node B (see Fig. 3).

The global state of the network is assumed to be a tensor product of independent bipartite states,

$$\rho = \bigotimes_{i=1}^n \rho_{A_i B_i}, \quad (29)$$

where the states $\rho_{A_i B_i}$ are arbitrary two-mode CV states. Each peripheral party A_i performs one of two dichotomic pseudospin measurements,

$$A_{x_i}^i = \vec{a}_{x_i}^i \cdot \vec{s}, \quad x_i \in \{0, 1\}, \quad (30)$$

while the central node performs separable joint measurements of the form

$$B_y = \bigotimes_{i=1}^n B_y^i, \quad B_y^i = \vec{b}_y^i \cdot \vec{s}, \quad y \in \{0, 1\}, \quad (31)$$

with $\vec{s} = (s_x, s_y, s_z)$ denoting the pseudospin operators.

Under the assumption of n -local hidden variables, the local correlations observed in this network satisfy the nonlinear star-network inequality [60]

$$\mathcal{S}_{\text{star}} = |I_n^{\text{star}}|^{1/n} + |J_n^{\text{star}}|^{1/n} \leq 2, \quad (32)$$

where

$$I_n^{\text{star}} = \sum_{x_1, \dots, x_n} \langle A_{x_1}^1 \cdots A_{x_n}^n B_0 \rangle, \quad (33)$$

$$J_n^{\text{star}} = \sum_{x_1, \dots, x_n} (-1)^{\sum_i x_i} \langle A_{x_1}^1 \cdots A_{x_n}^n B_1 \rangle. \quad (34)$$

Because the global state is a tensor product over independent sources and the measurement at the central node is separable, the correlators factorize as

$$\langle A_{x_1}^1 \cdots A_{x_n}^n B_y \rangle = \prod_{i=1}^n \langle A_{x_i}^i B_y^i \rangle. \quad (35)$$

For each bipartite state $\rho_{A_i B_i}$, we define the pseudospin correlation matrix

$$t_{jk}^{(i)} = \text{Tr}[s_j \otimes s_k \rho_{A_i B_i}], \quad j, k \in \{x, y, z\}. \quad (36)$$

Let $R^{(i)} = \sqrt{(t^{(i)})^T t^{(i)}}$ and denote its eigenvalues by $v_1^{(i)} \geq v_2^{(i)} \geq v_3^{(i)} \geq 0$. Now, we derive a closed-form expression for the maximal value of the n -locality expression achievable by arbitrary two-mode CV states, in the star network.

Theorem 2. For an n -partite CV star network probed by pseudospin measurements and described by the global state $\rho = \bigotimes_{i=1}^n \rho_{A_i B_i}$, the maximal value of the star-network expression $\mathcal{S}_{\text{star}}$ defined in Eq. (32) is given by

$$\mathcal{S}_{\text{star}}^{\text{max}} = 2 \sqrt{\left(\prod_{i=1}^n v_1^{(i)} \right)^{2/n} + \left(\prod_{i=1}^n v_2^{(i)} \right)^{2/n}}, \quad (37)$$

where $v_1^{(i)} \geq v_2^{(i)} \geq v_3^{(i)} \geq 0$ are the singular values of $t^{(i)}$. Consequently, the star-network inequality is violated whenever

$$\left(\prod_{i=1}^n v_1^{(i)} \right)^{2/n} + \left(\prod_{i=1}^n v_2^{(i)} \right)^{2/n} > 1. \quad (38)$$

Proof. For each source i , we first perform the singular-value decomposition of the correlation matrix $t^{(i)}$, with $R^{(i)} = \sqrt{(t^{(i)})^T t^{(i)}}$. Let $v_1^{(i)} \geq v_2^{(i)} \geq v_3^{(i)} \geq 0$ denote the eigenvalues of $R^{(i)}$. As in the proof of Theorem 1, the local pseudospin directions labeled by z and x are chosen to coincide with the singular vectors corresponding to $v_1^{(i)}$ and $v_2^{(i)}$, respectively. Equivalently, each local measurement at the central node may be viewed as applying an appropriate local pseudospin basis rotation before performing the standard measurements s_z and s_x . Consequently, without loss of generality, the local measurements performed at the central node may be taken to be $B_0 = s_z^{\otimes n}$, $B_1 = s_x^{\otimes n}$, where the pseudospin directions x and z denote the optimized local bases determined by the singular-value decomposition. The remaining optimization concerns only the measurement directions of the peripheral parties.

As in the proof of Theorem 1, we restrict the measurements of the peripheral parties in the XZ plane. Thus, we have

$$\vec{a}_0^i = (\sin \theta_i, 0, \cos \theta_i), \quad \vec{a}_1^i = (\sin \theta'_i, 0, \cos \theta'_i).$$

Under these choices, the two-body correlators entering the star-network expression reduce to

$$\langle A_0^i B_0^i \rangle + \langle A_1^i B_0^i \rangle = v_1^{(i)} (\cos \theta_i + \cos \theta'_i), \quad (39)$$

$$\langle A_0^i B_1^i \rangle - \langle A_1^i B_1^i \rangle = v_2^{(i)} (\sin \theta_i - \sin \theta'_i), \quad (40)$$

Using Eqs. (33) and (34), we get

$$I_n^{\text{star}} = \left(\prod_{i=1}^n v_1^{(i)} (\cos \theta_i + \cos \theta'_i) \right), \quad (41)$$

$$J_n^{\text{star}} = \left(\prod_{i=1}^n v_2^{(i)} (\sin \theta_i - \sin \theta'_i) \right). \quad (42)$$

Therefore,

$$\mathcal{S}_{\text{star}} = \left[\left| \prod_{i=1}^n v_1^{(i)} (\cos \theta_i + \cos \theta'_i) \right|^{1/n} + \left| \prod_{i=1}^n v_2^{(i)} (\sin \theta_i - \sin \theta'_i) \right|^{1/n} \right]. \quad (43)$$

Maximizing $\mathcal{S}_{\text{star}}$ with respect to the measurement angles $\{\theta_i, \theta'_i\}$ yields the stationarity conditions

$$\tan \theta'_i = -\tan \theta_i, \quad (44)$$

whose only maximizing solution is

$$\theta'_i = -\theta_i, \quad (45)$$

since the other solution, $\theta'_i = -\theta_i + \pi$ gives $(\cos \theta_i + \cos \theta'_i) = 0$ and $(\sin \theta_i - \sin \theta'_i) = 0$, corresponding to a vanishing value of $\mathcal{S}_{\text{star}}$. The remaining stationarity conditions are

$$\tan^2 \theta_j = \left| \frac{\delta_2}{\delta_1} \prod_{i=1}^n \tan \theta_i \right|^{1/n} \quad (46)$$

$$= \tan^2 \theta_k, \quad \forall j, k, \quad (47)$$

where $\delta_1 = \prod_{i=1}^n v_1^{(i)}$, $\delta_2 = \prod_{i=1}^n v_2^{(i)}$. Solving the above stationarity conditions yields

$$\tan^2 \theta_j = \left(\frac{\delta_2}{\delta_1} \right)^{2/n}, \quad \forall j. \quad (48)$$

Consequently,

$$|\cos \theta_j| = \frac{\delta_1^{1/n}}{\sqrt{\delta_1^{2/n} + \delta_2^{2/n}}}, \quad |\sin \theta_j| = \frac{\delta_2^{1/n}}{\sqrt{\delta_1^{2/n} + \delta_2^{2/n}}},$$

for every j . Substituting these optimal angles into Eq. (43) gives

$$\mathcal{S}_{\text{star}}^{\text{max}} = 2 \sqrt{\left(\prod_{i=1}^n v_1^{(i)} \right)^{2/n} + \left(\prod_{i=1}^n v_2^{(i)} \right)^{2/n}},$$

which is precisely Eq. (37). ■

In the symmetric case, where all sources distribute identical states, i.e., $t^{(i)} = t$ for all i , Eq. (37) reduces to

$$\mathcal{S}_{\text{star}}^{\text{max}} = 2\sqrt{v_1^2 + v_2^2},$$

and the violation criterion becomes

$$v_1^2 + v_2^2 > 1. \quad (49)$$

In particular, unlike the linear-chain network, in the symmetric case the maximal violation in star network is independent of the number of parties.

E. Comparison with two-qubit systems

For two-qubit systems, violation of network local inequalities in the linear chain and star networks based on Pauli operators offer necessary and sufficient conditions for detecting nonlocality [56]. This is due to the fact that Pauli operators span the full operator space. In contrast, pseudospin operators in CV systems form only a subspace of all dichotomic observables. Hence, the pseudospin-based network local inequalities used here provide a sufficient but not necessary condition for nonlocality. Obtaining a necessary condition would require optimization over all dichotomic observables in CV systems, a task that is practically infeasible as the basis for constructing any dichotomic observable is also infinite. Nevertheless, pseudospin observables have proven to be effective and one of the best options in various studies, achieving Tsirelson's bound for EPR states [35] where other methods [34] fall short, and have been successfully extended to broader classes of states [74,78].

IV. NETWORK NONLOCALITY WITH TWO-MODE SQUEEZED VACUUM STATES

Gaussian states play a central role in the analysis of CV quantum systems [24]. Within this class, the most general form of pure states is given by displaced squeezed states [23]. However, for the purpose of investigating quantum correlations, it is well known that the displacement operation does not affect a state's nonlocal features [23]. Therefore, without any loss of generality, we restrict our analysis to TMSV, excluding displacements. This state is obtained by squeezing the two-mode vacuum state $(|0, 0\rangle)$ using the two-mode squeezing operator $\mathcal{S}(\zeta) = \exp(\zeta \hat{a}_1^\dagger \hat{a}_2^\dagger - \zeta^* \hat{a}_1 \hat{a}_2)$, where $\zeta = r \exp(i\phi)$ and $r > 0$ are the squeezing parameters. This state can be produced in a nondegenerate optical parametric amplifier. Considering $\phi = 0$, the state is represented as [23]

$$|\zeta\rangle = \mathcal{S}(\zeta) |0, 0\rangle = \sqrt{1 - \lambda^2} \sum_{n=0}^{\infty} \lambda^n |n, n\rangle, \quad (50)$$

where $\lambda = \tanh r \in [0, 1]$ and $|n, n\rangle = |n\rangle \otimes |n\rangle$ are usual two-mode Fock states. Pseudospin observables with $q_i = 0$ can be chosen to show maximal violation with TMSV [35].

First, we consider the bilocal network scenario where the states shared between Alice and Bob ($|\psi_{AB}\rangle$), and the state shared between Bob and Charlie ($|\psi_{BC}\rangle$) are TMSVs with squeezing parameters r_1 and r_2 . We represent them as $|\zeta\rangle_1$ and $|\zeta\rangle_2$, respectively. Here, we consider pseudospin observables with all $q_i = 0$. Using Eqs. (13) and (14), we get

$$\langle A_x B_0 C_z \rangle_{|\zeta\rangle_1 \otimes |\zeta\rangle_2} = \cos(\theta_1) \cos(\theta_2), \quad (51)$$

$$\langle A_x B_1 C_z \rangle_{|\zeta\rangle_1 \otimes |\zeta\rangle_2} = K(r_1) K(r_2) \sin(\theta_1) \sin(\theta_2), \quad (52)$$

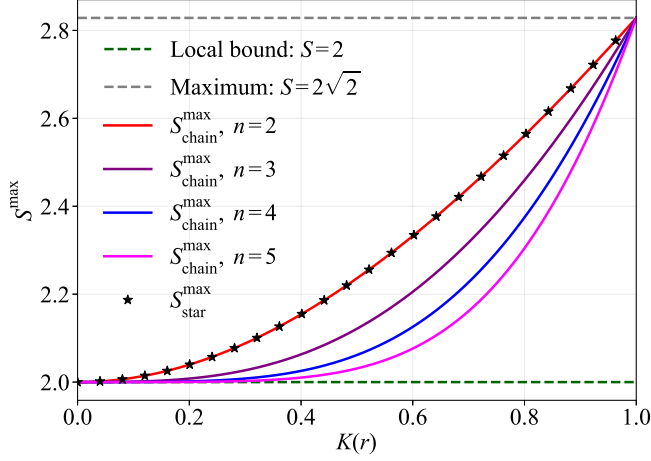


FIG. 4. Maximal n -locality expressions for linear-chain and star-network configurations using TMSV states as a function of the squeezing parameter $K = \tanh(2r)$, assuming equal squeezing $r_i = r$ for all sources. The violations increase monotonically with squeezing and approach the maximal quantum value $2\sqrt{2}$ in the limit $K \rightarrow 1$. The horizontal dashed lines at $S = 2$ (green) and $S = 2\sqrt{2}$ (gray) denote the classical and maximal quantum bounds, respectively.

where $K(r_1) = \tanh(2r_1)$ and $K(r_2) = \tanh(2r_2)$. Using Eqs. (2) and (3), we can calculate the values of I , J , and S ,

$$\begin{aligned} I_{\zeta_1, \zeta_2} &= 4 \cos(\theta_1) \cos(\theta_2), \\ J_{\zeta_1, \zeta_2} &= 4K(r_1)K(r_2) \sin(\theta_1) \sin(\theta_2) \\ &= 4L^2 \sin(\theta_1) \sin(\theta_2), \\ S_{\zeta_1, \zeta_2} &= 2[\sqrt{\cos(\theta_1) \cos(\theta_2)} + L\sqrt{\sin(\theta_1) \sin(\theta_2)}], \end{aligned} \quad (53)$$

where $L^2 = K(r_1)K(r_2)$. The optimal measurement settings, which maximize the value of S_ζ in Eq. (15), are given by

$$\begin{aligned} \theta_1 = \theta_2 = \theta &= \arctan(L). \\ \text{Hence, } \cos \theta &= \frac{1}{\sqrt{1+L^2}} \quad \text{and} \quad \sin \theta = \frac{L}{\sqrt{1+L^2}}, \end{aligned} \quad (54)$$

such that the maximal bilocality quantity $S^{\max}$ reduces to

$$S_{\zeta_1, \zeta_2}^{\max} = 2\sqrt{1+L^2}. \quad (55)$$

Thus, the bilocality inequality [Eq. (1)] is always violated by TMSV provided $r_1, r_2 \neq 0$, and the degree of violation is completely specified by the squeezing parameters r_1, r_2 . In the infinite squeezing limit, $r_1, r_2 \rightarrow \infty$, $K(r_1), K(r_2) \rightarrow 1$, both $|\zeta_1\rangle$ and $|\zeta_2\rangle$ become the original EPR states, for which we have

$$S_{\zeta_1, \zeta_2}^{\max}(r_1 \rightarrow \infty, r_2 \rightarrow \infty) = 2\sqrt{2}. \quad (56)$$

This indicates that the normalized version of original EPR states can maximally violate the bilocality inequality (see Fig. 4).

A. Linear-chain and star networks with TMSV states

We now investigate n -local correlations in the linear chain and the star networks, assuming that each source distributes a TMSV state with identical squeezing parameter r . For each bipartite source, the correlations are characterized by the correlation matrix $t^{(i)}$ with elements

$$t_{jk}^{(i)} = \text{Tr}[s_j \otimes s_k |\zeta\rangle\langle\zeta|], \quad j, k \in \{x, y, z\}, \quad (57)$$

where $\{s_x, s_y, s_z\}$ denote the pseudospin operators.

The nonzero eigenvalues of the associated matrix $R^{(i)} = \sqrt{(t^{(i)})^T t^{(i)}}$ are found to be

$$\mu_1 = 1, \quad \mu_2 = K(r) = \tanh(2r), \quad (58)$$

which completely determine the maximal violation of the n -local network inequalities following the prescription introduced in Sec. III.

Using these eigenvalues, we evaluate the maximal values of the n -locality expressions for both linear chain and star-network configurations, assuming equal squeezing for all sources. The resulting violations are shown in Fig. 4 as functions of the squeezing parameter $K(r)$. In both network topologies, the violation increases monotonically with increasing squeezing and approaches the maximal quantum value $2\sqrt{2}$ in the limit of infinite squeezing ($K \rightarrow 1$).

However, the dependence on the network size is qualitatively different for the two topologies. For identical TMSV sources, substituting $\mu_1 = 1$ and $\mu_2 = K(r)$ into the maximal expressions derived in Sec. III shows that the linear-chain configuration yields

$$S_{\text{chain}}^{\max} = 2\sqrt{1 + K(r)^n}, \quad (59)$$

so that the contribution of the second singular value is raised to a power that grows with the number of links. Since $0 \leq K(r) < 1$ for finite squeezing, the term $K(r)^n$ decreases monotonically with increasing n , leading to a progressive reduction of the attainable violation. This reflects the multiplicative attenuation of correlations as they propagate through successive independent links.

In contrast, for the star configuration, Eq. (37) gives

$$S_{\text{star}}^{\max} = 2\sqrt{1 + K(r)^2}, \quad (60)$$

because $(\prod_{i=1}^n \mu_1)^{2/n} = 1$ and $(\prod_{i=1}^n \mu_2)^{2/n} = K(r)^2$, so the explicit n -dependence cancels. Consequently, the maximal violation is determined solely by the squeezing parameter and remains unchanged as the number of outer parties increases. This explains why the star-network result yields a single n -independent curve in Fig. 4, coinciding with the bilocal network scenario. The distinct scaling with network size revealed above directly leads to the following observations, which summarize the qualitative behavior of the maximal violations for the two topologies.

Observation 1. For linear chain network with identically distributed nonmaximally entangled states, the maximal violation of the n -locality inequality decreases as the size of the network increases.

Observation 2. For star networks with identically distributed states, the maximal n -locality violation is independent of the network size.

B. Resistance to thermal noise

The discussions so far have assumed an idealized setting, where the TMSV states are unaffected by any form of environmental disturbance. However, in realistic experimental conditions, the presence of noise is almost unavoidable. We take into account an example of such imperfection by introducing local thermal noise into the system. We explicitly investigate how this noise influences the strength of bilocality violation, showing that it generally diminishes the violation and can, in certain regimes, even eliminate it altogether. Our results obtained below can be readily generalized to the n -local network scenario, retaining their characteristic features, since the bilocal network forms the basic building block of both the linear chain and star network configurations.

Consider the noisy TMSV state,

$$\rho_* = p |\zeta\rangle \langle \zeta| + (1-p) \left(\sum_{n=0}^{\infty} \gamma_n |n\rangle \langle n| \otimes \sum_{m=0}^{\infty} \eta_m |m\rangle \langle m| \right), \quad (61)$$

where $0 \leq p \leq 1$ and $\sum_{n=0}^{\infty} \gamma_n = \sum_{m=0}^{\infty} \eta_m = 1$. The local thermal noise parameters are given by

$$\begin{aligned} \gamma_n &= (1 - e^{-\beta_1}) e^{-\beta_1 n}, \\ \eta_m &= (1 - e^{-\beta_2}) e^{-\beta_2 m}. \end{aligned} \quad (62)$$

Here, $\beta_i = \frac{1}{k_B T_i}$ representing the inverse temperature of the thermal noise associated with i th mode, k_B is the Boltzmann constant, and the magnitude of the associated energies is assumed to be 1.

We consider the bilocality scenario, where $\rho_{AB} = \rho_{BC} = \rho_*$, with all states having same squeezing value r , and all pseudospin observables with $q_i = 0$. Using Eqs. (13) and (14), we get

$$\langle A_x B_0 C_z \rangle_{\rho_* \otimes \rho_*} = \mathcal{N}_1^2 \cos(\theta_1) \cos(\theta_2), \quad (63)$$

$$\langle A_x B_1 C_z \rangle_{\rho_* \otimes \rho_*} = \mathcal{N}_2^2 \sin(\theta_1) \sin(\theta_2), \quad (64)$$

where

$$\begin{aligned} \mathcal{N}_1 &= \left[p + (1-p) \tanh \frac{\beta_1}{2} \tanh \frac{\beta_2}{2} \right], \\ \mathcal{N}_2 &= pK(r), \end{aligned} \quad (65)$$

with $K(r) = \tanh(2r)$. Using Eqs. (2) and (3), we can calculate the values of I , J , and S ,

$$\begin{aligned} I_{\rho_* \otimes \rho_*} &= 4 \cos(\theta_1) \cos(\theta_2) \mathcal{N}_1^2, \\ J_{\rho_* \otimes \rho_*} &= 4 \sin(\theta_1) \sin(\theta_2) \mathcal{N}_2^2, \\ S_{\rho_* \otimes \rho_*} &= 2[\mathcal{N}_1 \sqrt{\cos(\theta_1) \cos(\theta_2)} + \mathcal{N}_2 \sqrt{\sin(\theta_1) \sin(\theta_2)}]. \end{aligned} \quad (66)$$

The optimal measurement settings, which maximize the value of $S_{\rho_* \otimes \rho_*}$, are given by

$$\theta_1 = \theta_2 = \theta = \arctan \left(\frac{\mathcal{N}_2}{\mathcal{N}_1} \right), \quad (67)$$

such that the maximal bilocality quantity $S_{\rho_*, \rho_*}^{\max}$ reduces to

$$S_{\rho_*, \rho_*}^{\max} = 2\sqrt{\mathcal{N}_1^2 + \mathcal{N}_2^2} = 2\sqrt{p^2 K^2 + \left[p + (1-p) \tanh \frac{\beta_1}{2} \tanh \frac{\beta_2}{2} \right]^2}. \quad (68)$$

Thus, violation of the bilocality inequality occurs when

$$p > \frac{-\mathcal{E} + \sqrt{\mathcal{E}^2 - 4\mathcal{D}\mathcal{F}}}{2\mathcal{D}}, \quad (69)$$

where

$$\mathcal{D} = K^2 + (1 - \epsilon)^2, \quad \mathcal{E} = 2\epsilon(1 - \epsilon), \quad \mathcal{F} = \epsilon^2 - 1,$$

with $\epsilon = \tanh \frac{\beta_1}{2} \tanh \frac{\beta_2}{2}$.

To visualize the effect of local thermal noise on bilocality violation, we examine the maximal bilocality expression $S_{\rho_*, \rho_*}^{\max}$ across relevant state and noise parameters. In Fig. 5, we fix the thermal environment to $\beta_1 = \beta_2 = 1$ and study the dependence of $S_{\rho_*, \rho_*}^{\max}$ on the noise probability p and the entanglement parameter $K = \tanh(2r)$. The plot identifies the critical boundary separating bilocal and nonbilocal regions, illustrating how squeezing and noise jointly determine the network behavior.

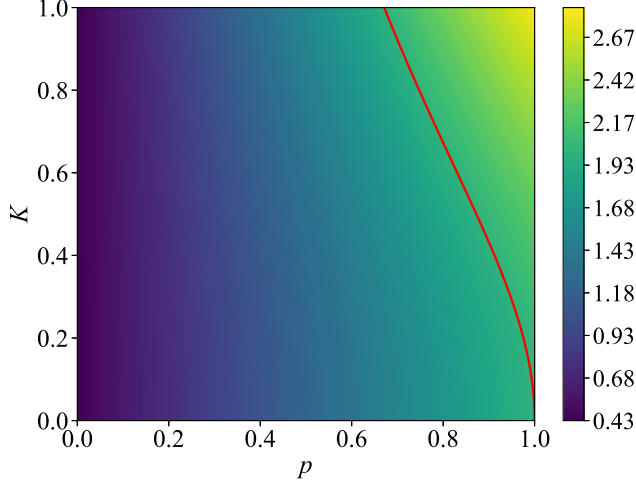


FIG. 5. $S_{\rho_*, \rho_*}^{\max}$ is plotted as a function of p and K , for fixed inverse temperature parameters $\beta_1 = \beta_2 = 1$. The red contour line corresponds to $S_{\rho_*, \rho_*}^{\max} = 2$, separating the bilocal ($S_{\rho_*, \rho_*}^{\max} \leq 2$) and nonbilocal ($S_{\rho_*, \rho_*}^{\max} > 2$) regions in the parameter space.

In Fig. 6, we instead vary the inverse temperature $\beta_1 = \beta_2 = \beta$, revealing how thermal fluctuations influence the bilocality expression. As β decreases (higher temperature), the violation diminishes and eventually disappears for certain states, demonstrating the fragility of nonclassical correlations under thermal noise. Conversely, stronger entanglement and lower noise fractions allow the system to retain nonbilocal over a wider temperature range. These features motivate the following observations summarizing the thermal suppression and robustness of bilocality violations.

Observation 3. As β decreases (i.e., as the temperature increases), nonbilocal is suppressed (see Fig. 6).

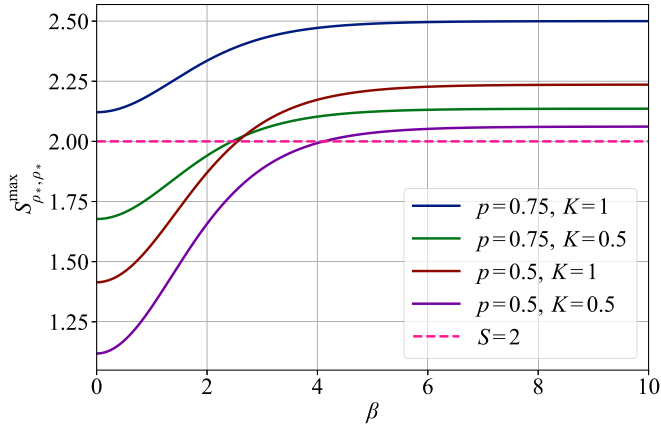


FIG. 6. Variation of the maximized bilocal expression $S_{\rho_*, \rho_*}^{\max}$ for the noisy TMSV state as a function of the inverse temperature parameter β , for different values of (p, K) . The plot illustrates that at low β (i.e., high temperature), nonbilocal correlations are suppressed and disappear below a certain threshold. As β increases (low-temperature regime), the impact of thermal noise diminishes and the nonbilocal becomes more robust.

Observation 4. Bilocality violation is guaranteed independently of the thermal parameters whenever

$$K > \sqrt{\frac{1}{p^2} - 1}. \quad (70)$$

Under this condition, the bilocality expression remains violated throughout the entire physically accessible range of noise temperatures (β_1, β_2), ensuring persistence of nonbilocal correlations even in the presence of arbitrarily strong thermal noise (see Appendix B 1 for the derivation).

V. NETWORK NONLOCALITY WITH NON-GAUSSIAN STATES

Despite their broad applicability, Gaussian states have a key limitation: Measurement statistics involving only Gaussian operations can always be efficiently simulated classically, where non-Gaussian states with negative Wigner function play the crucial nonclassical role [79–81]. This underscores the importance of non-Gaussian states, which are crucial for achieving universal CV quantum computation [82]. Here, we explore nonbilocality using three prominent classes of non-Gaussian states: single-photon subtracted two-mode squeezed vacuum state, entangled coherent state, and CV Werner state.

A. Enhanced bilocality violation for photon-subtracted squeezed vacuum states

Violation of bilocality inequality with TMSV has been studied in Sec. IV. TMSV can be de-Gaussified through local photon addition or subtraction on individual modes. From an experimental perspective, photon subtraction is generally more feasible than photon addition, as the latter typically requires an auxiliary photon pumping apparatus [83]. Although both operations can yield enhanced nonlocality in theory, photon subtraction is often the method of choice in practical implementations. In what follows, we investigate how single-photon subtraction influences the extent of violation of bilocality inequality.

We consider three types of photon-subtracted states, viz., (1) asymmetric photon subtraction (single photon from one mode), (2) symmetric photon subtraction (single photon from both modes), and (3) coherent superposition of photon subtraction across modes. In each case, we analyze two network configurations, viz., (1) hybrid scenario where Alice and Bob share the photon-subtracted state, while Bob and Charlie share a TMSV state, and (2) symmetric scenario where both shared states are photon-subtracted and identical. We assume the squeezing parameter r to be the same for all involved states. The corresponding maximization procedures follow the prescription detailed in Sec. III.

1. Asymmetric photon subtraction: Single-mode operation

We first study the impact of asymmetric photon subtraction on the violation of the bilocality inequality. Specifically, we consider the case where a single photon is subtracted from one mode of a TMSV state.

Applying the annihilation operator ($\mathbb{I} \otimes \hat{b}$) on the second mode of a TMSV yields the normalized state

$$\begin{aligned} |\xi_1\rangle &= \left(\frac{1 - \lambda^2}{\lambda} \right) \sum_{n=1}^{\infty} \lambda^n \sqrt{n} |n, n-1\rangle \\ &= (1 - \lambda^2) \sum_{n=0}^{\infty} \lambda^n \sqrt{n+1} |n+1, n\rangle, \end{aligned} \quad (71)$$

where $\lambda = \tanh r$ and $\hat{b}$ is the annihilation operator for the second mode.

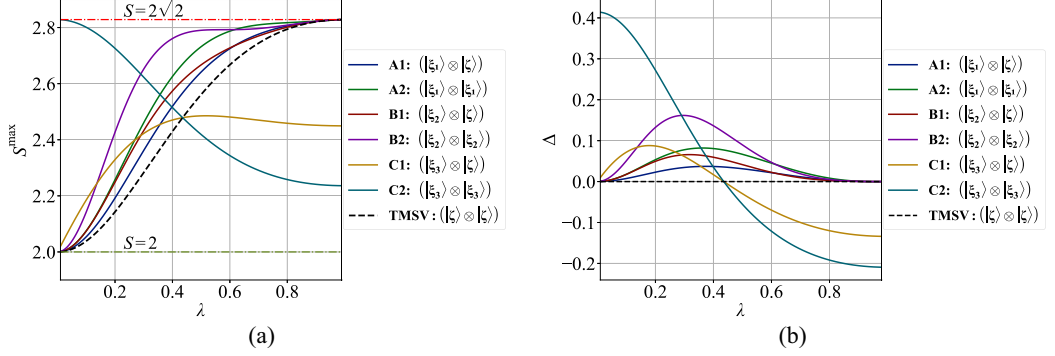


FIG. 7. Maximal bilocality violation for different configurations involving single-photon-subtracted squeezed vacuum states as a function of the squeezing parameter λ , with the TMSV case taken as a reference. (a) Maximized bilocality expression $S^{\max}$, showing that configurations A1, A2, B1, and B2 exhibit enhanced violations over the full range of λ , while configurations C1 and C2 show enhancement only up to $\lambda = 0.4354$. In configuration C2, where both entangled pairs are in the state $|\xi_3\rangle$, the maximal quantum value $2\sqrt{2}$ is attained in the limit $\lambda \rightarrow 0$. (b) Relative enhancement Δ of the bilocality violation with respect to the TMSV reference scenario.

Configuration A1: In the first scenario, Alice and Bob share the photon-subtracted state $|\xi_1\rangle$, while Bob and Charlie share the standard TMSV state $|\zeta\rangle$. The maximal value of the bilocality expression is given by

$$S_{\xi_1, \zeta}^{\max} = 2\sqrt{v_1^{(a)}\mu_1 + v_2^{(a)}\mu_2}. \quad (72)$$

Here, $\{\mu_i\}$ and $\{v_i^{(a)}\}$ are two largest eigenvalues of the corresponding correlation matrices for the states $|\zeta\rangle$ and $|\xi_1\rangle$, respectively. For further details including expressions of all parameters, see Appendix A 1.

Configuration A2: In the second configuration, both shared states are identical and given by the photon-subtracted state $|\xi_1\rangle$. In this case, the maximal value of the bilocality expression is given by

$$S_{\xi_1, \xi_1}^{\max} = 2\sqrt{(v_1^{(a)})^2 + (v_2^{(a)})^2}. \quad (73)$$

Our results for bilocality violation versus the squeezing parameter for both the above cases are presented in Fig. 7(a).

Observation 5. The configuration A2 leads to more bilocality violation compared to the configuration A1 for any finite squeezing.

2. Symmetric photon subtraction: Dual-mode operation

We next investigate the impact of symmetric photon subtraction on the maximal violation of the bilocality inequality. Specifically, we investigate the case where a single photon is subtracted from both modes of a TMSV.

Applying the annihilation operators $(\hat{a} \otimes \hat{b})$ on the TMSV yields the normalized two-mode state:

$$\begin{aligned} |\xi_2\rangle &= \left(\frac{1 - \lambda^2}{\lambda\sqrt{1 + \lambda^2}} \right) \sqrt{1 - \lambda^2} \sum_{n=1}^{\infty} \lambda^n n |n-1, n-1\rangle \\ &= (1 - \lambda^2) \sqrt{\frac{1 - \lambda^2}{1 + \lambda^2}} \sum_{n=0}^{\infty} \lambda^n (n+1) |n, n\rangle, \end{aligned} \quad (74)$$

where $\lambda = \tanh r$, and $\hat{a}, \hat{b}$ denote annihilation operators for the first and second modes, respectively.

Configuration B1: In this configuration, the entangled state shared between Alice and Bob is $|\xi_2\rangle$, while Bob and Charlie share the original TMSV state $|\zeta\rangle$. The resulting maximal bilocality expression is given by

$$S_{\xi_2, \zeta}^{\max} = 2\sqrt{v_1^{(s)}\mu_1 + v_2^{(s)}\mu_2}. \quad (75)$$

Configuration B2: Here, both shared states are taken to be $|\xi_2\rangle$, making this a fully symmetric setting. The maximum bilocality expression in this configuration is

$$S_{\xi_2, \xi_2}^{\max} = 2\sqrt{(v_1^{(s)})^2 + (v_2^{(s)})^2}. \quad (76)$$

Our results for bilocality violation versus the squeezing parameter for both the above cases are presented in Fig. 7(a). For a more detailed discussion, see Appendix A 1.

Observation 6. Symmetric photon subtraction (B2) produces a stronger violation of the bilocality inequality than the asymmetric scheme (A2) for $0 < \lambda < 0.616$. Beyond this crossover point, i.e., for larger squeezing parameters, the ordering reverses, and the asymmetric scheme yields the stronger violation. An analogous behavior is observed when comparing configurations B1 and A1 (see Appendix B 2 for the derivation).

Explanation. This crossover can be understood by examining how photon subtraction reshapes the photon-number distribution of the TMSV state in a way that affects pseudospin correlations. At low squeezing, the TMSV state is only weakly entangled and dominated by low photon-number components, so symmetric subtraction enhances correlated higher-photon sectors in both modes and generates stronger non-Gaussian correlations, leading to the ordering $B2 > A2$ and $B1 > A1$. As the squeezing increases, however, the TMSV already possesses strong multiphoton correlations, and the nonlocality test becomes sensitive not simply to entanglement strength, but to how different number sectors contribute to the pseudospin observables. In this regime, symmetric subtraction redistributes weight too strongly toward very high photon numbers, which are not optimally aligned with these correlators, whereas single-mode subtraction introduces a milder imbalance that preserves useful correlations while still inducing non-Gaussianity, resulting in a reverse of the order, i.e., $A2 > B2$ and $A1 > B1$ beyond the crossover. Importantly, both photon-subtracted states continue to outperform the Gaussian TMSV even at large squeezing, reflecting the well-known advantage of non-Gaussian features for pseudospin-based nonlocality tests, where higher-order coherences produced by subtraction enable stronger violations than those achievable with Gaussian states alone.

Observation 7. The asymmetric subtraction applied to both states (A2) still outperforms the symmetric subtraction applied to only one state (B1), highlighting the benefit of distributing non-Gaussianity across the network [see Fig. 7(a)].

3. Coherent superposition of photon subtractions across modes

Here, we consider an explicit example of a non-Gaussian state that is obtained with coherent superposition of two asymmetric single-photon subtracted states. This photon subtraction operation can be written as

$$\mathcal{G} = a \otimes I + (-1)^k I \otimes b, \quad (77)$$

where $k \in \{0, 1\}$. It is also assumed that the specific mode from which the photon is subtracted remains unknown. The state is represented as [28,84]

$$\begin{aligned} |\xi_3\rangle &= \left(\frac{1-\lambda^2}{\sqrt{2}}\right) \sum_{n=1}^{\infty} \lambda^{n-1} \sqrt{n} [|n-1, n\rangle + (-1)^k |n, n-1\rangle] \\ &= \left(\frac{1-\lambda^2}{\sqrt{2}}\right) \sum_{n=0}^{\infty} \lambda^n \sqrt{n+1} [|n, n+1\rangle + (-1)^k |n+1, n\rangle], \end{aligned} \quad (78)$$

with $\lambda = \tanh r$, $r > 0$. Here, we consider the state with $k = 0$.

It is known that CV entangled states of the form $|\psi\rangle = \sum_{n=0}^{\infty} C_n |n, n + \bar{n}\rangle$ or $|\psi\rangle = \sum_{n=0}^{\infty} C_n |n + \bar{n}, n\rangle$ do not exhibit a violation of the Clauser–Horne–Shimony–Holt (CHSH) inequality when $\bar{n}$ is an odd integer and the pseudospin measurement parameters q_i are chosen to be zero [74]. However, we demonstrate that this limitation does not hold when coherent superpositions of such states are considered.

Configuration C1: In this configuration, the entangled state shared between Alice and Bob is $|\xi_3\rangle$, while Bob and Charlie share the original TMSV state $|\zeta\rangle$. The resulting maximal bilocality expression is given by (see Appendix A 1)

$$S_{\xi_3, \zeta}^{\max} = 2\sqrt{v_1^{(c)}\mu_1 + v_2^{(c)}\mu_2}. \quad (79)$$

Configuration C2: Here, both shared states are taken to be $|\xi_3\rangle$, making this a fully symmetric setting. The maximum bilocality expression in this configuration is given by

$$S_{\xi_3, \xi_3}^{\max} = 2\sqrt{(v_1^{(c)})^2 + (v_2^{(c)})^2}. \quad (80)$$

It may be noted from Fig. 7(a) that all photon-subtracted cases yield higher bilocality violation compared to the TMSV in the low squeezing regime. Moreover, the case with coherent superposition of subtraction of photons (C2) performs better in bilocality violation compared to the other two cases of symmetric and asymmetric photon subtraction for low squeezing [see Fig. 7(a)].

Observation 8. Both configurations C1 and C2 exhibit stronger bilocality violations than the TMSV state within the squeezing range $0 < \lambda < 0.4354$, with configuration C2 providing the largest enhancement. Beyond this threshold, neither configuration yields an advantage over the TMSV, and configuration C2 performs worse than C1 (see Appendix B 3 for details).

Explanation. This observation provides a concrete example that non-Gaussian operations do not universally enhance nonlocal correlations. In the low-squeezing regime, the TMSV state itself carries only weak nonlocal correlations, and the observed enhancement arises primarily from the action of the non-Gaussian operation $\mathcal{G}$. The coherent subtraction reshapes the photon-number amplitudes and introduces interference between neighboring sectors, which are strongly aligned with the structure probed by pseudospin measurements, thereby increasing the violation. As the squeezing increases, however, the underlying Gaussian correlations of the TMSV become dominant and already support substantial violations. In this regime, the additional reshaping induced by $\mathcal{G}$ no longer matches the measurement structure optimally and instead redistributes weight into sectors that contribute less effectively to the correlators. Consequently, the advantage diminishes and eventually disappears, illustrating that the usefulness of non-Gaussian operations for nonlocality tests is regime-dependent rather than universal.

Observation 9. For the configuration C2 with the state $|\xi_3\rangle$, the bilocality expression attains the maximum, $S_{\xi_3, \xi_3}^{\max} = 2\sqrt{2}$, in the limit $\lambda \rightarrow 0$, corresponding to arbitrarily small squeezing. This indicates a maximal violation of the bilocality inequality (as well as CHSH inequality with $|\xi_3\rangle$) despite the absence of squeezing [see Fig. 7(a)].

Explanation. It is important to emphasize that $|\xi_3\rangle$ is not a maximally entangled state in the full infinite-dimensional Hilbert space when $\lambda \rightarrow 0$, as the von Neumann entropy remains finite when traced over one mode. This apparent paradox is resolved by noting that, in the limit $\lambda \rightarrow 0$, the state $|\xi_3\rangle$ effectively resides in a two-dimensional subspace. Within this subspace, it is indeed a maximally entangled state, as it takes the form

$$|\xi_3(\lambda \rightarrow 0)\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle), \quad (81)$$

which belongs to the parity-based Bell states defined in Eq. (9), with $\mathcal{A}_0 = \mathcal{B}_0 = 1$ and $\mathcal{A}_n = \mathcal{B}_n = 0$, for any $n \neq 0$. Since pseudospin measurements probe precisely this parity structure, the bilocality expression reaches its algebraic maximum even in the absence of squeezing. Thus the maximal violation originates not from large CV entanglement, but from the effective qubit embedding induced by the coherent superpositions of photon subtraction at arbitrarily small squeezing.

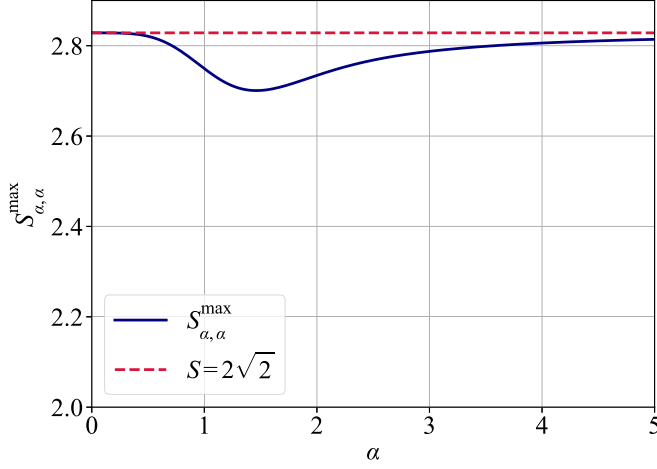


FIG. 8. Variation of the maximized bilocality expression $S_{\alpha, \alpha}^{\max}$ for ECS as a function of α (blue curve). $S_{\alpha, \alpha}^{\max}$ reaches maximum ($=2\sqrt{2}$) when $\alpha \rightarrow 0$ or $\alpha \rightarrow \infty$.

The enhancement in the maximal violation of the bilocality inequality due to photon subtraction can be quantified using the measure

$$\Delta = \frac{S_{\psi_{AB}, \phi_{BC}}^{\max} - S_{\zeta, \zeta}^{\max}}{S_{\zeta, \zeta}^{\max}}. \quad (82)$$

In this expression, $S_{\psi_{AB}, \phi_{BC}}^{\max}$ denotes the maximal bilocality violation for a given configuration of non-Gaussian entangled states $|\psi_{AB}\rangle \otimes |\phi_{BC}\rangle$, while $S_{\zeta, \zeta}^{\max}$ corresponds to the maximal violation for the reference configuration comprising TMSV states, $|\zeta\rangle \otimes |\zeta\rangle$. This measure allows us to systematically compare the advantage gained through non-Gaussian operations [see Fig. 7(b)].

B. Entangled coherent states

The entangled coherent state (ECS) [85] is an important non-Gaussian CV entangled state. Such states, also labeled as “cat” states, are regarded to be of foundational as well as practical relevance [86,87]. Several applications in quantum information processing have been studied utilizing ECS [88,89]. Here, we consider the asymmetric ECS (|ECS>), which can be defined as [90]

$$|\text{ECS}\rangle_{\alpha} = \mathcal{N}(|\alpha\rangle - |\alpha\rangle - |-\alpha\rangle|\alpha\rangle), \quad (83)$$

where $|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$, $|\alpha\rangle$ is a coherent state with $\alpha \neq 0$, we have considered α to be real for simplicity. $\mathcal{N}$ is a normalization factor given by

$$\mathcal{N} = \frac{1}{\sqrt{2(1 - e^{-4\alpha^2})}}. \quad (84)$$

The expression for the maximum bilocality $S^{\max}$ is given by

$$S_{\alpha, \beta}^{\max} = 2\sqrt{1 + Q^2(\alpha, \beta)}, \quad (85)$$

where Q is a function of the coherent state parameters α, β (see Appendix A 2 for the derivation).

Observation 10. ECS always violates the bilocality inequality Eq. (1) as $0 < Q(\alpha, \beta) < 1$, and the degree of violation is completely specified by magnitude of $Q(\alpha, \beta)$. $Q(\alpha, \beta)$ approaches 1 when $\alpha, \beta \rightarrow 0$ (but $\alpha, \beta \neq 0$) giving maximal violation of the bilocality inequality, and when $\alpha, \beta \rightarrow \infty$ (see Fig. 8).

Explanation. The behavior of $S_{\alpha,\beta}^{\max}$ (the scenario with $\alpha = \beta$ is shown in Fig. 8) can be understood from the limiting forms of the entangled coherent state. For very small coherent amplitudes,

$$|\alpha\rangle \approx e^{-\alpha^2/2}(|0\rangle + \alpha|1\rangle).$$

Substituting this expansion into Eq. (83) and retaining the leading nonvanishing terms, we obtain

$$|\text{ECS}\rangle_\alpha \approx \frac{1}{\sqrt{2}}(|10\rangle - |01\rangle) + O(\alpha^2).$$

Thus, in the limit $\alpha \rightarrow 0$ (with $\alpha \neq 0$), the ECS approaches the maximally entangled single-photon Bell state. Consequently, the bilocality expression reaches its maximal value,

$$S_{\alpha,\alpha}^{\max} \rightarrow 2\sqrt{2}.$$

In the opposite limit $\alpha \rightarrow \infty$, the overlap between the coherent states vanishes,

$$\langle\alpha| - \alpha\rangle = e^{-2\alpha^2} \rightarrow 0.$$

Therefore, $|\alpha\rangle$ and $|- \alpha\rangle$ become effectively orthogonal states. In this regime, the ECS approaches

$$|\text{ECS}\rangle_\alpha \rightarrow \frac{1}{\sqrt{2}}(|\alpha\rangle| - \alpha\rangle - |- \alpha\rangle|\alpha\rangle),$$

which is again a maximally entangled state in the subspace spanned by $\{|\alpha\rangle, |- \alpha\rangle\}^{\otimes 2}$. As a result, the maximal bilocality violation $2\sqrt{2}$ is recovered in the large- α limit.

For intermediate values of α , however, the coherent states are not fully orthogonal. The ECS therefore lies in a crossover regime between the two limiting maximally entangled structures discussed above. This behavior is captured by the parameter $Q(\alpha, \beta)$ appearing in Eq. (85). While $Q(\alpha, \beta) \rightarrow 1$ as $\alpha, \beta \rightarrow 0$ and as $\alpha, \beta \rightarrow \infty$, it decreases below unity for intermediate amplitudes. Since $S_{\alpha,\beta}^{\max} = 2\sqrt{1 + Q^2(\alpha, \beta)}$, the reduction of Q directly leads to the dip observed in Fig. 8, with the minimum occurring around $\alpha \simeq 1.4$. The precise location of the minimum follows from the competition between these two limiting behaviors and is obtained numerically.

C. CV Werner state

In the DV regime, the Werner state [91] is a convex mixture of a maximally entangled state and the maximally mixed state:

$$\rho_{\text{Werner}} = p |\Psi_d\rangle \langle \Psi_d| + \frac{1-p}{d^2} I_1 \otimes I_2, \quad 0 \leq p \leq 1, \quad (86)$$

where $|\Psi_d\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^d |i\rangle_1 |i\rangle_1$. Extending this concept to CV systems is nontrivial due to the absence of a proper maximally mixed state in infinite dimensions. An operational analog can be defined by mimicking the DV structure. Since the reduced states of a TMSV are thermal states, a natural CV analog involves mixing the TMSV with a product of two identical thermal states [92],

$$\rho_T = (1 - \lambda_2^2)^2 \sum_{m,n=0}^{\infty} \lambda_2^{2(m+n)} |m\rangle \langle m| \otimes |n\rangle \langle n|, \quad (87)$$

where $\lambda_2 = \tanh s$ and $\langle n \rangle_T = \sinh^2 s$.

Thus, the CV Werner state takes the form

$$\rho_W = p \rho_{\text{TMSV}} + (1-p) \rho_T, \quad 0 \leq p \leq 1, \quad (88)$$

where $\rho_{\text{TMSV}} = |\zeta\rangle \langle \zeta|$, and $|\zeta\rangle$ is the TMSV state (50). This defines a class of states parametrized by (p, r, s) . For simplicity, we take $r = s$, reducing to two parameters (p, r) . In the limit $r \rightarrow \infty$,

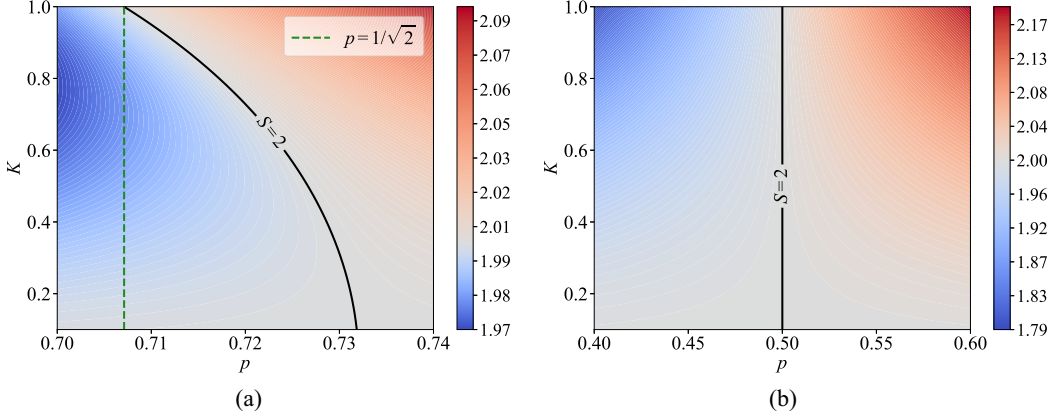


FIG. 9. Maximal bilocality value for a CV Werner state as a function of the noise parameter p and squeezing parameter K . (a) Symmetric configuration with $p_1 = p_2 = p$. (b) Asymmetric configuration with $p_1 = 1$ and $p_2 = p$. In both cases, the black curve corresponds to $S^{\max} = 2$, separating the nonbilocal region from the bilocal one. For $K = 1$, violation in panel (a) requires $p > 1/\sqrt{2}$, while in panel (b) it occurs for any nonzero K when $p > 1/2$.

the CV Werner state approaches a mixture of a maximally entangled EPR state and a maximally mixed state, closely resembling the DV case.

We analyze the bilocality violation in the following two scenarios:

Case 1: When both Werner-type resource states have equal noise parameters, i.e., $p_1 = p_2 = p$. We obtain (see Appendix A 3 for further details)

$$S_{\rho_{W_1}, \rho_{W_2}}^{\max}(p_1 = p_2 = p) = 2 \times \{p^2(1 + K^2) + 2p(1 - p)(1 - K^2) + (1 - p)^2(1 - K^2)^2\}^{\frac{1}{2}}. \quad (89)$$

Violation of the bilocality inequality occurs when

$$p > \frac{-E + \sqrt{E^2 - 4D(F-1)}}{2D},$$

where $D = 3K^2 - 1 + (1 - K^2)^2$, $E = 2K^2(1 - K^2)$, $F = (1 - K^2)^2 - 1$.

Observation 11. For the extremal case $K = 1$, with $p_1 = p_2 = p$, CV Werner state shows non-bilocality when $p > \frac{1}{\sqrt{2}}$, matching the known threshold for nonbilocality with two-qubit Werner states [see Fig. 9(a)].

Case 2: Now, we set $p_1 = 1$ and $p_2 = p$, we find

$$S_{\rho_{W_1}, \rho_{W_2}}^{\max}(p_1 = 1, p_2 = p) = 2\sqrt{1 + (2p - 1)K^2}. \quad (90)$$

Observation 12. With $p_1 = 1$, $p_2 = p$, CV Werner states violate bilocality inequality whenever $p > 1/2$, for any nonzero squeezing (i.e., $K > 0$) [see Fig. 9(b)].

VI. SCHEMATICS FOR EXPERIMENTAL REALIZATION

In this section, we outline an experimentally feasible approach to probing nonbilocal correlations using spatial parity observables. Our proposal is inspired by the spatial-parity Bell experiments of Ref. [36], where Bell nonlocality was demonstrated using spatially entangled two-photon EPR states and spatial parity measurements on the transverse spatial modes of single photons. At first sight, this implementation appears different from the theoretical framework developed in the preceding sections, where network nonlocality is analyzed using multiphoton two-mode CV states expressed in the Fock basis. The purpose of this section is to clarify this connection. In particular, these two seemingly different physical descriptions are related by a well-established Hilbert-space

isomorphism between the single-mode multiphoton Fock space and the single-photon multimode spatial Hilbert space [71]. Under this correspondence, the pseudospin operators and measurement scheme developed in the previous sections are mapped directly onto experimentally accessible spatial parity observables.

We present a brief overview of the Hilbert-space isomorphism underlying this correspondence. By expanding the transverse spatial wavefunction of a single photon in an orthonormal basis of spatial modes, one obtains a representation that is mathematically equivalent to the Fock basis of a single bosonic mode. The even and odd parity sectors then define an effective qubit on which the pseudospin operators act exactly as in the multiphoton Fock-space description.

Consider a single photon state in the spatial domain, $|\Psi\rangle = \int dx \psi(x) |x\rangle$, with $\int dx |\psi(x)|^2 = 1$. This state is formally infinite dimensional, the transverse profile $\psi(x)$ can be expanded in an orthonormal functional basis $\{\phi_n(x)\}$ as $\psi(x) = \sum_n c_n \phi_n(x)$, with $|\Psi\rangle = \sum_n c_n |n\rangle$, which enables the association $|n\rangle = \int dx \phi_n(x) |x\rangle$. A suitable choice for the discrete orthonormal basis $\{\phi_n(x)\}$ is provided by the set of Hermite-Gaussian functions. A noteworthy property of this basis is the alternating parity of its elements: Functions with even indices are even, while those with odd indices are odd. Mathematically, this can be expressed as

$$\phi_{2n}(-x) = \phi_{2n}(x), \quad \phi_{2n+1}(-x) = -\phi_{2n+1}(x), \quad \forall n \in \mathbb{N}.$$

A key observation is that the wavefunction can be decomposed into its even, i.e., $\psi_e(x) = \frac{1}{2}[\psi(x) + \psi(-x)]$ and odd, i.e., $\psi_o(x) = \frac{1}{2}[\psi(x) - \psi(-x)]$ parts, which are orthogonal, $\int dx \psi_e^*(x) \psi_o(x) = 0$. This naturally defines a two-dimensional “parity qubit” with logical basis states

$$|e\rangle \propto \sum_n c_{2n} |2n\rangle, \quad |o\rangle \propto \sum_n c_{2n+1} |2n+1\rangle, \quad (91)$$

so that the original state may be expressed in the 2D space of spatial parity as $|\Psi\rangle = e|e\rangle + o|o\rangle$, where e and o are amplitudes. With this association, it is possible to decompose the state into even ($|e\rangle$) and odd ($|o\rangle$) spatial parity components.

In the spatial representation, the pseudospin operators ($q_i = 0$) acting on $\psi(x)$ are defined as [71]

$$\bar{s}_k = \iint dx dx' h_k(x, x') |x\rangle \langle x'|, \quad k = x, y, z, \quad (92)$$

where $h_k(x, x')$ is the impulse response function given by

$$\begin{aligned} h_x(x, x') &= \sum_n [\phi_{2n+1}(x) \phi_{2n}^*(x') + \phi_{2n}(x) \phi_{2n+1}^*(x')], \\ h_y(x, x') &= i \sum_n [\phi_{2n+1}(x) \phi_{2n}^*(x') - \phi_{2n}(x) \phi_{2n+1}^*(x')], \\ h_z(x, x') &= \sum_n [\phi_{2n}(x) \phi_{2n}^*(x') - \phi_{2n+1}(x) \phi_{2n+1}^*(x')]. \end{aligned} \quad (93)$$

In the basis of spatial eigenmodes, using $|n\rangle = \int dx \phi_n(x) |x\rangle$, the spatial parity operators defined in Eq. (92) can be directly identified with the pseudospin operators introduced in Eq. (4). Acting with a spatial parity operator on the state $|\Psi\rangle$ yields

$$\bar{s}_k |\Psi\rangle = \bar{s}_k \left(\sum_n c_n |n\rangle \right) = \sum_n c'_n |n\rangle = \int dx \psi'(x) |x\rangle, \quad (94)$$

where the transformed spatial wavefunction is given by

$$\psi'(x) = \int dx' h_k(x, x') \psi(x'). \quad (95)$$

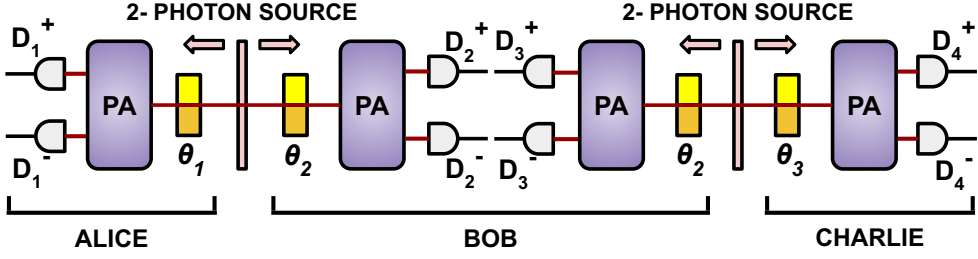


FIG. 10. Schematic configuration for testing nonbilocal correlations using spatial parity measurements. PA denotes the parity analyzer, θ the parity rotator, and D the detector.

This correspondence establishes a direct operational link between pseudospin transformations and experimentally accessible operations in spatial parity space, thereby enabling the implementation of pseudospin measurements through spatial parity observables.

The above correspondence extends naturally to bipartite states. Consider an arbitrary two-mode CV state expressed in the Fock basis,

$$|\Psi\rangle = \sum_{m,n=0}^{\infty} c_{mn} |m, n\rangle. \quad (96)$$

Using the correspondence $|n\rangle = \int dx \phi_n(x)|x\rangle$, this state is mapped onto the two-photon state

$$|\Psi\rangle = \iint dx_1 dx_2 \psi(x_1, x_2) |x_1, x_2\rangle, \quad (97)$$

whose spatial wavefunction is

$$\psi(x_1, x_2) = \sum_{m,n=0}^{\infty} c_{mn} \phi_m(x_1) \phi_n(x_2). \quad (98)$$

Therefore, every two-mode CV state considered in the present work possesses a unique isomorphic representation as a spatially entangled two-photon state. The experimental proposal presented here exploits this Hilbert-space isomorphism to implement the pseudospin measurements through spatial parity observables.

The experimental elements [36] required for probing spatial parity consist of two basic components: a *parity rotator* and a *parity analyzer*. The *parity rotator* implements a rotation in the even-odd spatial parity subspace and is realized using a phase plate that introduces a relative phase θ between the $x \geq 0$ and $x < 0$ halves of the transverse spatial plane. This operation is formally equivalent to an $SO(2)$ rotation acting on the effective two-dimensional parity space. The *parity analyzer* performs a projective measurement in the even-odd basis and is implemented using a parity sensitive Mach-Zehnder interferometer, in which a spatial flipper (a spatial flipper implements a transverse spatial inversion of the optical field, $\phi(x) \rightarrow \phi(-x)$) is inserted in one interferometer arm, rendering the interference outcome parity dependent.

A schematic of the bilocal measurement configuration is shown in Fig. 10. Two independent photon pair sources distribute entangled photons between Alice and Bob, and between Bob and Charlie, respectively. Alice applies a parity rotation characterized by the angle θ_1 , followed by a parity analyzer. Bob receives two photons, one from each source, and independently applies parity rotations characterized by the angle θ_2 to both photons, followed by separate parity analyzers. Charlie applies a parity rotation characterized by the angle θ_3 , followed by a parity analyzer. Each parity analyzer yields a dichotomic outcome, where an even-parity detection event is assigned the value $+1$ and an odd-parity detection event is assigned the value -1 .

The experimental data required for the bilocality test consist of coincidence counts corresponding to all possible combinations of parity outcomes. For each setting of the rotation angles $(\theta_1, \theta_2, \theta_3)$, one records coincidence counts

$$N(a, b_1, b_2, c | \theta_1, \theta_2, \theta_3),$$

where $a, b_1, b_2, c \in \{+1, -1\}$ denote the parity outcomes at Alice, Bob, and Charlie, respectively. For instance, the outcome $(+, +, +, -)$ corresponds to even-parity detections at Alice and Bob's detectors and an odd-parity detection at Charlie, while $(-, -, -, -)$ corresponds to odd-parity detections at all four detectors.

From these coincidence counts, the correlation function is defined as

$$E(\theta_1, \theta_2, \theta_3) = \frac{\sum_{a, b_1, b_2, c = \pm 1} a b_1 b_2 c N(a, b_1, b_2, c | \theta_1, \theta_2, \theta_3)}{\sum_{a, b_1, b_2, c = \pm 1} N(a, b_1, b_2, c | \theta_1, \theta_2, \theta_3)}. \quad (99)$$

The bilocality inequality involves two nonlinear combinations of such correlators, denoted by I and J , which are constructed from correlation functions evaluated at different measurement settings. Explicitly,

$$\begin{aligned} I &= \sum_{x, z=0,1} E'(\theta_1^{(x)}, \theta_2, \theta_3^{(z)}), \\ J &= \sum_{x, z=0,1} (-1)^{x+z} E(\theta_1^{(x)}, \theta_2, \theta_3^{(z)}), \end{aligned} \quad (100)$$

where $\theta_1^{(x)}$ and $\theta_3^{(z)}$ denote two measurement settings for Alice and Charlie, respectively, while Bob performs fixed parity measurements. These quantities are then combined according to the bilocality inequality in Eq. (1).

Since the spatial parity rotators and analyzers required for these measurements have already been implemented experimentally in Bell tests [36], the above procedure provides a direct and experimentally grounded route for probing nonbilocal correlations using spatial parity observables.

VII. CONCLUSIONS

Quantum networks provide a powerful setting for investigating nonlocal correlations that go beyond the conventional Bell framework. These networks facilitate the distribution of entanglement over long distances without requiring direct interaction between distant nodes, making them highly suitable for practical quantum architectures and repeaters [61, 62]. The ability of network nonlocality to certify quantum correlations generated by multiple independent sources fundamentally distinguishes it from standard Bell nonlocality. Such certification is expected to play an important role in the characterization and verification of future quantum communication networks and quantum repeater architectures. While the study of nonlocal correlations in DV quantum networks has seen substantial progress [53], CV networks have hitherto received rather limited attention [68–70]. This is despite the notable advantages offered by CV systems, including near-deterministic state preparation, experimental accessibility, and their compatibility with long-distance communication [23].

In this work, we have proposed a formalism to probe CV network nonlocality using pseudospin measurements. Our work takes a critical step toward understanding and leveraging nonlocal correlations in CV-based quantum networks. A comparison of our approach with the existing literature on CV network nonlocality is presented in Appendix C. Specifically, we have investigated here network nonlocality corresponding to the scenarios of two basic open network topologies, viz., the linear chain network and the star network. As a first application of the pseudospin framework in networks, we have presented a study of the Gaussian two-mode squeezed vacuum state, showing reasonable persistence of n -locality violation even in the presence of thermal noise. We have further demonstrated network nonlocality for a wide range of CV resources, i.e., several categories of

non-Gaussian states such as photon-subtracted squeezed states, entangled coherent states, and CV Werner-class states. Our results substantiate the role of non-Gaussianity as a powerful enabler of network nonlocality. Moreover, by relating pseudospin measurements to spatial parity observables, we have proposed an experimentally grounded framework for a practical demonstration of CV network nonlocality.

To summarize, we highlight some of the significant results on network nonlocality with CV systems that have been revealed through our analysis here. First, it is shown that the strength of nonlocality in the star-network configuration with TMSV states stays independent of the network size, a result with potentially wide applicability in information processing protocols based on scaled-up quantum networks [93]. Another key result of our analysis is the robustness of CV network nonlocality against local thermal noise: We have shown that nonbilocal correlations can persist even at arbitrarily high temperatures, provided the squeezing exceeds a critical threshold. Next, we have established non-Gaussianity to enhance network nonlocality beyond the Gaussian paradigm, with specific illustrations across several classes of non-Gaussian states. In particular, we have found that a coherent superposition of single-photon subtractions across modes yields maximal bilocality violation even in the absence of squeezing, a regime of direct experimental relevance.

Our findings contribute significantly to the understanding of network nonlocality in infinite-dimensional systems and open up several promising avenues for future research. Adaptation of our analysis to other measurement schemes might offer deeper insights into the structure of nonlocal correlations in the CV regime. Though we have studied the effects of thermal noise in this work, the impact of other kinds of environmental effects on the robustness of network nonlocality merit further exploration. Another interesting direction would be systematic investigations of the role of non-Gaussian operations on network nonlocality, and their interplay with resources such as entanglement and squeezing. Finally, the present work has focused on open quantum-network topologies, namely, linear-chain and star networks, for which the corresponding network-locality inequalities admit the correlation-matrix decomposition underlying our Horodecki-type optimization. Extending this framework to closed network topologies, such as the triangle network, remains an interesting open problem. Developing suitable CV network-locality criteria and optimization methods for such closed networks constitutes an important direction for future research.

DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX A: NETWORK NONLOCALITY WITH NON-GAUSSIAN STATES

1. Photon-subtracted squeezed vacuum state

a. Asymmetric photon subtraction: Single mode operation

Applying the annihilation operator ($\mathbb{I} \otimes \hat{b}$) on the second mode of a TMSV yields the unnormalized state

$$|\xi_1\rangle \propto \sum_{n=1}^{\infty} \lambda^n \sqrt{n} |n, n-1\rangle, \quad (\text{A1})$$

where $\lambda = \tanh r$, and $\hat{b}$ is the annihilation operator for the second mode. Upon normalization and relabeling, this becomes

$$|\xi_1\rangle = (1 - \lambda^2) \sum_{n=0}^{\infty} \lambda^n \sqrt{n+1} |n+1, n\rangle. \quad (\text{A2})$$

For the state $|\xi_1\rangle$, the optimal pseudospin measurement settings correspond to the parameters $q = 1$ for the first mode and $q = 0$ for the second mode.

Configuration A1: In the first scenario, Alice and Bob share the photon-subtracted state $|\xi_1\rangle$, while Bob and Charlie share the standard TMSV state $|\zeta\rangle$. That is,

$$\rho_{AB} = |\xi_1\rangle\langle\xi_1|, \quad \rho_{BC} = |\zeta\rangle\langle\zeta|.$$

The two largest eigenvalues of the correlation matrices R^{AB} and R^{BC} (see Sec. VI) in this case are found to be

$$v_1^{(a)} = 1, \quad v_2^{(a)} = 2\lambda(1 - \lambda^2)^2 \sum_{n=0}^{\infty} \lambda^{4n} \sqrt{(2n+1)(2n+2)}, \quad (\text{A3})$$

$$\mu_1 = 1, \quad \mu_2 = \frac{2\lambda}{1 + \lambda^2} = K(r). \quad (\text{A4})$$

Thus, the maximal value of the bilocality expression becomes

$$S_{\xi_1, \zeta}^{\max} = 2\sqrt{v_1^{(a)}\mu_1 + v_2^{(a)}\mu_2}. \quad (\text{A5})$$

Configuration A2: In the second configuration, both shared states are identical and given by the photon-subtracted state $|\xi_1\rangle$, i.e.,

$$\rho_{AB} = \rho_{BC} = |\xi_1\rangle\langle\xi_1|.$$

In this case, the maximal value of the bilocality expression is

$$S_{\xi_1, \xi_1}^{\max} = 2\sqrt{(v_1^{(a)})^2 + (v_2^{(a)})^2}. \quad (\text{A6})$$

b. Symmetric photon subtraction: Dual-mode operation

Applying the annihilation operators ($\hat{a} \otimes \hat{b}$) on the TMSV yields the normalized two-mode state

$$\begin{aligned} |\xi_2\rangle &= \left(\frac{1 - \lambda^2}{\lambda\sqrt{1 + \lambda^2}} \right) \sqrt{1 - \lambda^2} \sum_{n=1}^{\infty} \lambda^n n |n-1, n-1\rangle \\ &= (1 - \lambda^2) \sqrt{\frac{1 - \lambda^2}{1 + \lambda^2}} \sum_{n=0}^{\infty} \lambda^n (n+1) |n, n\rangle, \end{aligned} \quad (\text{A7})$$

where $\lambda = \tanh r$, and $\hat{a}, \hat{b}$ denote annihilation operators for the first and second modes, respectively.

For the state $|\xi_2\rangle$, the optimal pseudospin measurement settings correspond to the parameters $q_i = 0$ for both modes.

Configuration B1: In this configuration, the entangled state shared between Alice and Bob is $|\xi_2\rangle$, while Bob and Charlie share the original TMSV state $|\zeta\rangle$:

$$\rho_{AB} = |\xi_2\rangle\langle\xi_2|, \quad \rho_{BC} = |\zeta\rangle\langle\zeta|.$$

The two largest eigenvalues of the corresponding correlation matrix R^{AB} are given by

$$v_1^{(s)} = 1, \quad v_2^{(s)} = \frac{2\lambda(1 - \lambda^2)^3(6\lambda^4 + 2)}{(1 + \lambda^2)(1 - \lambda^4)^3}. \quad (\text{A8})$$

The resulting maximal bilocality expression becomes

$$S_{\xi_2, \zeta}^{\max} = 2\sqrt{v_1^{(s)}\mu_1 + v_2^{(s)}\mu_2}. \quad (\text{A9})$$

Configuration B2: Here, both shared states are taken to be $|\xi_2\rangle$, making this a fully symmetric setting

$$\rho_{AB} = \rho_{BC} = |\xi_2\rangle \langle \xi_2|.$$

The maximum bilocality expression in this configuration becomes

$$S_{\xi_2, \xi_2}^{\max} = 2\sqrt{(v_1^{(s)})^2 + (v_2^{(s)})^2}. \quad (\text{A10})$$

c. Coherent superposition of photon subtractions across modes

The state is represented as [28,84]

$$\begin{aligned} |\xi_3\rangle &= \frac{1}{\sqrt{2} \sinh r} \sum_{n=1}^{\infty} \lambda^n \sqrt{n} [|n-1, n\rangle + (-1)^k |n, n-1\rangle] \\ &= \frac{1}{\sqrt{2}} (1 - \lambda^2) \sum_{n=0}^{\infty} \lambda^n \sqrt{n+1} [|n, n+1\rangle + (-1)^k |n+1, n\rangle], \end{aligned} \quad (\text{A11})$$

with $\lambda = \tanh r$, $r > 0$. Here, we consider the state with $k = 0$. In order to reveal the nonbilocality nature of the state $|\xi_3\rangle$, we evaluate the bilocality expression using pseudospin observables with $q_i = 0$.

Configuration C1: In this configuration, the entangled state shared between Alice and Bob is $|\xi_3\rangle$, while Bob and Charlie share the original TMSV state $|\zeta\rangle$:

$$\rho_{AB} = |\xi_3\rangle \langle \xi_3|, \quad \rho_{BC} = |\zeta\rangle \langle \zeta|.$$

The two largest eigenvalues of the corresponding correlation matrix R^{BC} are given by

$$v_1^{(c)} = 1, \quad v_2^{(c)} = \frac{1 + \lambda^4}{(1 + \lambda^2)^2}. \quad (\text{A12})$$

The resulting maximal bilocality expression becomes

$$S_{\xi_3, \zeta}^{\max} = 2\sqrt{v_1^{(c)} \mu_1 + v_2^{(c)} \mu_2}. \quad (\text{A13})$$

Configuration C2: Here, both shared states are taken to be $|\xi_3\rangle$, making this a fully symmetric setting:

$$\rho_{AB} = \rho_{BC} = |\xi_3\rangle \langle \xi_3|.$$

The maximum bilocality expression in this configuration becomes

$$S_{\xi_3, \xi_3}^{\max} = 2\sqrt{(v_1^{(c)})^2 + (v_2^{(c)})^2}. \quad (\text{A14})$$

2. Entangled coherent state

The ECS can be defined as [90]

$$|\text{ECS}\rangle_{\alpha} = \mathcal{N}(|\alpha\rangle - |\alpha\rangle - |\alpha\rangle|\alpha\rangle), \quad (\text{A15})$$

where $|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$, $|\alpha\rangle$ is a coherent state with $\alpha \neq 0$, we have considered α to be real for simplicity. $\mathcal{N}$ is a normalization factor given by

$$\mathcal{N} = \frac{1}{\sqrt{2(1 - e^{-4\alpha^2})}}. \quad (\text{A16})$$

We consider the bilocal network scenario where the states shared between Alice and Bob ($|\psi_{AB}\rangle$), and the state shared between Bob and Charlie ($|\psi_{BC}\rangle$) both are two entangled coherent states, $|\text{ECS}\rangle_{\alpha}$

and $|\text{ECS}\rangle_\beta$, respectively. Using Eqs. (13) and (14), we get

$$\langle A_x B_0 C_z \rangle_{|\text{ECS}\rangle_\alpha \otimes |\text{ECS}\rangle_\beta} = \cos(\theta_1) \cos(\theta_2), \quad (\text{A17})$$

$$\langle A_x B_1 C_z \rangle_{|\text{ECS}\rangle_\alpha \otimes |\text{ECS}\rangle_\beta} = Q^2(\alpha, \beta) \sin(\theta_1) \sin(\theta_2), \quad (\text{A18})$$

where

$$Q^2(\alpha, \beta) = \frac{\left(\sum_{n,m=0}^{\infty} \frac{\alpha^{4n+1}}{\sqrt{(2n)!(2n+1)!}} \frac{\beta^{4m+1}}{\sqrt{(2m)!(2m+1)!}} \right)^2}{\cosh(\alpha^2) \sinh(\alpha^2) \cosh(\beta^2) \sinh(\beta^2)}. \quad (\text{A19})$$

Using Eqs. (2) and (3), we can calculate the values of I , J , and S ,

$$\begin{aligned} I_{\alpha,\beta} &= 4 \cos(\theta_1) \cos(\theta_2), \\ J_{\alpha,\beta} &= 4Q^2(\alpha, \beta) \sin(\theta_1) \sin(\theta_2), \\ S_{\alpha,\beta} &= 2[\sqrt{\cos(\theta_1) \cos(\theta_2)} + Q(\alpha, \beta) \sqrt{\sin(\theta_1) \sin(\theta_2)}]. \end{aligned} \quad (\text{A20})$$

The optimal measurement settings, which maximize the value of S_ζ , also the violation of the bilocality inequality considered in Eq. (1), are given by

$$\theta_1 = \theta_2 = \theta = \arctan(Q).$$

$$\text{Hence, } \cos \theta = \frac{1}{\sqrt{1+Q^2}} \quad \text{and} \quad \sin \theta = \frac{Q}{\sqrt{1+Q^2}}, \quad (\text{A21})$$

such that the maximal bilocality quantity $S^{\max}$ reduces to

$$S_{\alpha,\beta}^{\max} = 2\sqrt{1+Q^2}. \quad (\text{A22})$$

3. CV Werner state

The CV Werner state takes the form

$$\rho_W = p \rho_{\text{TMSV}} + (1-p) \rho_T, \quad 0 \leq p \leq 1, \quad (\text{A23})$$

where $\rho_{\text{TMSV}} = |\zeta\rangle\langle\zeta|$, and $|\zeta\rangle$ is the TMSV state (50). This defines a class of states parametrized by (p, r, s) . For simplicity, we take $r = s$, reducing to two parameters (p, r) .

We now consider the bilocal network scenario where the states shared between Alice and Bob ($|\psi_{AB}\rangle$), and the state shared between Bob and Charlie ($|\psi_{BC}\rangle$) both are CV Werner states with parameters (p_1, r) and (p_2, r) , respectively. Using Eqs. (13) and (14), we get

$$\langle A_x B_0 C_z \rangle_{\rho_{W_1} \otimes \rho_{W_2}} = T_I \cos(\theta_1) \cos(\theta_2), \quad (\text{A24})$$

$$\langle A_x B_1 C_z \rangle_{\rho_{W_1} \otimes \rho_{W_2}} = T_J \sin(\theta_1) \sin(\theta_2), \quad (\text{A25})$$

where

$$\begin{aligned} T_I &= p_1 p_2 + (1-K^2)(p_1 - 2p_1 p_2 + p_2) + (1-p_1)(1-p_2)(1-K^2)^2, \\ T_J &= p_1 p_2 K^2, \end{aligned} \quad (\text{A26})$$

with $K = \tanh 2r$. Using Eqs. (2) and (3), we can calculate the values of I , J , and S ,

$$\begin{aligned} I_{\rho_{W_1} \otimes \rho_{W_2}} &= 4T_I \cos(\theta_1) \cos(\theta_2), \\ J_{\rho_{W_1} \otimes \rho_{W_2}} &= 4T_J \sin(\theta_1) \sin(\theta_2), \\ S_{\rho_{W_1} \otimes \rho_{W_2}} &= 2[\sqrt{T_I \cos(\theta_1) \cos(\theta_2)} + \sqrt{T_J \sin(\theta_1) \sin(\theta_2)}]. \end{aligned} \quad (\text{A27})$$

The optimal measurement settings that maximize the violation of the bilocality inequality considered in Eq. (1) are given by

$$\theta_1 = \theta_2 = \theta = \arctan \left(\sqrt{\frac{T_J}{T_I}} \right), \quad (\text{A28})$$

$$(\text{A29})$$

such that the maximal bilocality quantity $S^{\max}$ reduces to

$$S_{\rho_{W_1}, \rho_{W_2}}^{\max} = 2\sqrt{T_I + T_J}. \quad (\text{A30})$$

APPENDIX B: DERIVATIONS SUPPORTING THE OBSERVATIONS

1. Derivations supporting Observation 4

The bilocality inequality violation condition for the noisy TMSV state can be expressed as

$$\mathcal{Q}(\epsilon) = \mathcal{D}p^2 + \mathcal{E}p + \mathcal{F} > 0, \quad (\text{B1})$$

with

$$\mathcal{D} = K^2 + (1 - \epsilon)^2, \quad (\text{B2})$$

$$\mathcal{E} = 2\epsilon(1 - \epsilon), \quad (\text{B3})$$

$$\mathcal{F} = \epsilon^2 - 1, \quad (\text{B4})$$

and

$$\epsilon = \tanh \left(\frac{\beta_1}{2} \right) \tanh \left(\frac{\beta_2}{2} \right). \quad (\text{B5})$$

Since $\beta_i = 1/T_i$ and $T_i > 0$, we have

$$0 \leq \epsilon \leq 1. \quad (\text{B6})$$

Substituting the coefficients into Eq. (B1) yields

$$\begin{aligned} [K^2 + (1 - \epsilon)^2]p^2 + 2\epsilon(1 - \epsilon)p + \epsilon^2 - 1 &> 0 \\ \Rightarrow (1 - p)^2\epsilon^2 + 2p(1 - p)\epsilon + p^2(K^2 + 1) - 1 &> 0. \end{aligned} \quad (\text{B7})$$

Thus, $\mathcal{Q}(\epsilon)$ is convex in ϵ , since $(1 - p)^2 \geq 0$. The stationary point is determined by imposing $\partial \mathcal{Q}(\epsilon)/\partial \epsilon = 0$,

$$\epsilon_* = -\frac{p}{1 - p}, \quad (\text{B8})$$

which lies outside the physical domain (B6). Therefore, the minimum of $\mathcal{Q}(\epsilon)$ within the accessible parameter range occurs at the boundary $\epsilon = 0$, when temperature of either one or both noises is infinity. Evaluating Eq. (B7),

$$\mathcal{Q}_{\min} = \mathcal{Q}(0) = p^2(K^2 + 1) - 1. \quad (\text{B9})$$

A temperature-independent sufficient condition ensuring $\mathcal{Q}(\epsilon) > 0$ for all β_1, β_2 is therefore

$$p^2(K^2 + 1) - 1 > 0 \quad \Rightarrow \quad K > \sqrt{\frac{1}{p^2} - 1}, \quad (\text{B10})$$

which establishes Observation 4.

2. Derivation supporting Observation 6

To compare the strength of bilocality violations produced by symmetric and asymmetric photon subtraction, we note that for all configurations considered here the maximal bilocality expressions depend monotonically on the second singular value of the corresponding correlation matrix.

For configurations A1 and B1, one has

$$S_{\xi_1, \zeta}^{\max} = 2\sqrt{1 + v_2^{(a)}(\lambda)\mu_2(\lambda)}, \quad (\text{B11})$$

$$S_{\xi_2, \zeta}^{\max} = 2\sqrt{1 + v_2^{(s)}(\lambda)\mu_2(\lambda)}, \quad (\text{B12})$$

while for configurations A2 and B2,

$$S_{\xi_1, \xi_1}^{\max} = 2\sqrt{1 + (v_2^{(a)}(\lambda))^2}, \quad (\text{B13})$$

$$S_{\xi_2, \xi_2}^{\max} = 2\sqrt{1 + (v_2^{(s)}(\lambda))^2}. \quad (\text{B14})$$

Since $\mu_2(\lambda) > 0$ and the square-root function is monotonic, both comparisons reduce to determining the sign of

$$\Delta(\lambda) = v_2^{(s)}(\lambda) - v_2^{(a)}(\lambda). \quad (\text{B15})$$

Substituting

$$v_2^{(a)}(\lambda) = 2\lambda(1 - \lambda^2)^2 \sum_{n=0}^{\infty} \lambda^{4n} \sqrt{(2n+1)(2n+2)},$$

$$v_2^{(s)}(\lambda) = \frac{2\lambda(1 - \lambda^2)^3(6\lambda^4 + 2)}{(1 + \lambda^2)(1 - \lambda^4)^3},$$

we evaluate their difference numerically. An exact analytical evaluation of $v_2^{(a)}(\lambda)$ is hindered by the square-root structure within the summation. To estimate it accurately, we use a numerical truncation technique: the summation is carried out up to $n = N$ terms, and convergence is confirmed if the difference between partial sums with N and $N + 1$ terms falls below 10^{-20} . Once this criterion is met, we consider the sum to have converged with sufficient precision. Computing $v_2^{(a)}(\lambda)$ to convergence and solving $\Delta(\lambda) = 0$ yields a single crossing point

$$\lambda_c \approx 0.616. \quad (\text{B16})$$

We find

$$v_2^{(s)}(\lambda) > v_2^{(a)}(\lambda) \quad \text{for} \quad 0 \leq \lambda \leq \lambda_c, \quad (\text{B17})$$

with the ordering reversed beyond this value.

Consequently, symmetric photon subtraction produces a larger maximal bilocality violation than the asymmetric scheme for both A1 versus B1 and A2 versus B2 whenever $0 < \lambda < 0.616$, establishing Observation 6.

3. Derivation supporting Observation 8

Observation 8 states that configurations C1 and C2 exhibit enhancement in violation over TMSV up in the squeezing parameter range, $0 < \lambda < 0.4354$. To justify the observation, we compare the maximal violations with that of the TMSV case. For TMSV states, the two largest singular values are $\mu_1 = 1$ and $\mu_2 = K(r) = \frac{2\lambda}{1+\lambda^2}$, so that

$$S_{\text{biloc\_TMSV}}^{\max} = 2\sqrt{1 + \left(\frac{2\lambda}{1 + \lambda^2}\right)^2}. \quad (\text{B18})$$

For the photon-subtracted configurations, we have

$$\nu_1^{(c)} = 1, \quad \nu_2^{(c)} = \frac{1 + \lambda^4}{(1 + \lambda^2)^2}. \quad (\text{B19})$$

Configurations C1 and C2 yield

$$S_{\xi_3, \zeta}^{\max} = 2\sqrt{1 + \left(\frac{1 + \lambda^4}{(1 + \lambda^2)^2}\right)\left(\frac{2\lambda}{1 + \lambda^2}\right)}, \quad (\text{B20})$$

$$S_{\xi_3, \xi_3}^{\max} = 2\sqrt{1 + \left(\frac{1 + \lambda^4}{(1 + \lambda^2)^2}\right)^2}. \quad (\text{B21})$$

Hence, enhancement over the TMSV value occurs whenever

$$\frac{1 + \lambda^4}{(1 + \lambda^2)^2} > \frac{2\lambda}{1 + \lambda^2} \Rightarrow \lambda^4 - 2\lambda^3 - 2\lambda + 1 = \mathcal{L}(\lambda) > 0. \quad (\text{B22})$$

The boundary between enhancement and no enhancement is determined by

$$\mathcal{L}(\lambda) = \lambda^4 - 2\lambda^3 - 2\lambda + 1 = 0. \quad (\text{B23})$$

The only physically relevant root in the interval $0 < \lambda < 1$ is

$$\lambda_* = \frac{1 + \sqrt{3} - 12^{1/4}}{2} \approx 0.4354. \quad (\text{B24})$$

$\mathcal{L}(\lambda)$ is positive in the range $0 < \lambda < \lambda_*$ and negative in the range $\lambda_* < \lambda < 1$. Hence, configurations C1 and C2 exhibit an advantage over the TMSV resource whenever

$$0 < \lambda < \lambda_* \approx 0.4354. \quad (\text{B25})$$

Thus, within this squeezing range the maximal bilocal violation produced by the coherent photon-subtracted resources exceeds that of the TMSV state, establishing the observation.

APPENDIX C: COMPARISON WITH PREVIOUS STUDIES ON NONLOCALITY IN CONTINUOUS VARIABLE NETWORKS

Since the present work develops a pseudospin-based formalism for characterizing network nonlocality, it is useful to briefly compare it with other approaches towards CV network nonlocality.

Guerra Copete [68] investigated network nonlocality in CV systems from the perspective of Gaussian measurements and positive Wigner functions. A central conclusion of that work is that Gaussian states admitting positive Wigner representations cannot demonstrate network nonlocality under phase-space Gaussian measurements, thereby motivating the necessity of non-Gaussian measurements.

Jiang *et al.* [69] introduced a one-way device-independent protocol for witnessing network nonlocality in an all-optical CV entanglement-swapping experiment. Their analysis focuses on a specific bilocal network under one-way classical communication and derives Bell-type inequalities tailored to that operational scenario.

Yan *et al.* [70] have developed a framework for CV network nonlocality based on generalized quasiprobability-function measurements. Their formalism establishes nonlinear Bell-type inequalities for several network topologies, including chain, star, tree, and cyclic networks, and is primarily formulated for Gaussian states.

The present work differs substantially from these studies. Rather than constructing Bell inequalities from quasiprobability functions or considering specific communication-assisted protocols, we formulate CV network nonlocality entirely within the pseudospin framework. This enables us to derive analytical Horodecki-type criteria for arbitrary two-mode continuous-variable states in both linear-chain and star networks. Furthermore, our approach naturally applies to both Gaussian

TABLE I. Comparison of the present work with previous studies on CV network nonlocality. The table highlights the differences in the underlying measurement framework, network topology, class of CV states considered, and the principal contributions of each work.

Reference	Measurement framework	Networks	States	Main contribution
Guerra Copete [68]	Gaussian phase-space measurements	Bilocal scenario	Gaussian states	Gaussian states cannot exhibit network nonlocality under Gaussian measurements.
Jiang <i>et al.</i> [69]	All-optical measurements with one-way classical communication	Bilocal scenario	EPR states	All-optical device-independent one-way network nonlocality.
Yan <i>et al.</i> [70]	Generalized non-Gaussian quasiprobability functions	Chain, star, tree, and cyclic networks	Gaussian states	Nonlinear Bell inequalities based on quasiprobability function measurements.
Present work	Pseudospin observables	Linear chain and star networks	Arbitrary two-mode states	Horodecki-type analytical criteria for maximal network nonlocality, studies thermal robustness and non-Gaussian enhancement.

and non-Gaussian resources, allowing us to study thermal robustness, quantify the enhancement arising from photon subtraction and other non-Gaussian states, and propose an experimentally feasible realization based on the well-established isomorphism between pseudospin and spatial parity observables. For clarity, the main differences are summarized in Table I.

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- [1] J. S. Bell, On the Einstein Podolsky Rosen paradox, *Phys. Phys. Fiz.* **1**, 195 (1964).
 - [2] A. Einstein, B. Podolsky, and N. Rosen, Can quantum-mechanical description of physical reality be considered complete? *Phys. Rev.* **47**, 777 (1935).
 - [3] N. Brunner, D. Cavalcanti, S. Pironio, V. Scarani, and S. Wehner, Bell nonlocality, *Rev. Mod. Phys.* **86**, 419 (2014).
 - [4] M. McKague and M. Mosca, Generalized self-testing and the security of the 6-state protocol, in *Theory of Quantum Computation, Communication, and Cryptography: 5th Conference*, Lecture Notes in Computer Sciences, Vol. 6519 (Springer, Berlin, Heidelberg, 2011), pp. 113–130.
 - [5] M. McKague, T. H. Yang, and V. Scarani, Robust self testing of the singlet, *J. Phys. A: Math. Theor.* **45**, 455304 (2012).
 - [6] I. Supic and J. Bowles, Self-testing of quantum systems: A review, *Quantum* **4**, 337 (2020).
 - [7] S. Goswami, B. Bhattacharya, D. Das, S. Sasmal, C. Jebaratnam, and A. S. Majumdar, One-sided device-independent self-testing of any pure two-qubit entangled state, *Phys. Rev. A* **98**, 022311 (2018).
 - [8] Z. Bian, A. S. Majumdar, C. Jebarathinam, K. Wang, L. Xiao, X. Zhan, Y. Zhang, and P. Xue, Experimental demonstration of one-sided device-independent self-testing of any pure two-qubit entangled state, *Phys. Rev. A* **101**, 020301 (2020).

- [9] A. Acin, N. Brunner, N. Gisin, S. Massar, S. Pironio, and V. Scarani, Device-independent security of quantum cryptography against collective attacks, *Phys. Rev. Lett.* **98**, 230501 (2007).
- [10] M. Farkas, Unbounded device-independent quantum key rates from arbitrarily small nonlocality, *Phys. Rev. Lett.* **132**, 210803 (2024).
- [11] C. Portmann and R. Renner, Security in quantum cryptography, *Rev. Mod. Phys.* **94**, 025008 (2022).
- [12] F. Grasselli, G. Murta, H. Kampermann, and D. Bruß, Entropy bounds for multiparty device-independent cryptography, *PRX Quantum* **2**, 010308 (2021).
- [13] H. Buhrman, R. Cleve, S. Massar, and R. de Wolf, Nonlocality and communication complexity, *Rev. Mod. Phys.* **82**, 665 (2010).
- [14] S. Pironio *et al.*, Random numbers certified by Bell's theorem, *Nature (London)* **464**, 1021 (2010).
- [15] A. Acín, S. Massar, and S. Pironio, Randomness versus nonlocality and entanglement, *Phys. Rev. Lett.* **108**, 100402 (2012).
- [16] L. K. Shalm *et al.*, Strong loophole-free test of local realism, *Phys. Rev. Lett.* **115**, 250402 (2015).
- [17] M. Giustina, A. Mech, S. Ramelow, B. Wittmann, J. Kofler, J. Beyer, A. E. Lita, B. Calkins, T. Gerrits, S. Woo Nam, R. Ursin, C. Wittmann, and A. Zeilinger, Bell violation using entangled photons without the fair-sampling assumption, *Nature (London)* **497**, 227 (2013).
- [18] B. Hensen, H. Bernien, A. E. Dréau, A. Reiserer, N. Kalb, M. S. Blok, J. Ruitenbergh, R. F. L. Vermeulen, R. N. Schouten, C. Abellán, W. Amaya, V. Pruneri, M. W. Mitchell, M. Markham, D. J. Twitchen, D. Elkouss, S. Wehner, T. H. Taminiau, and R. Hanson, Loophole-free Bell inequality violation using electron spins separated by 1.3 kilometres, *Nature (London)* **526**, 682 (2015).
- [19] Y. P. Zhong, H. S. Chang, K. J. Satzinger, M. Pechal, A. Bienfait, M. H. Chou, A. N. Cleland, M. Freedman, D. P. Pappas, L. Frunzio, M. H. Devoret, and A. N. Cleland, Violating Bell's inequality with remotely connected superconducting qubits, *Nat. Phys.* **15**, 741 (2019).
- [20] D.-K. Shin, B. M. Henson, S. S. Hodgman, and A. G. Truscott, Bell correlations between spatially separated pairs of atoms, *Nat. Commun.* **10**, 4447 (2019).
- [21] P. R. Tapster, J. G. Rarity, and P. C. M. Owens, Violation of Bell's inequality over 4 km of optical fiber, *Phys. Rev. Lett.* **73**, 1923 (1994).
- [22] W. Tittel, J. Brendel, H. Zbinden, and N. Gisin, Violation of Bell inequalities by photons more than 10 km apart, *Phys. Rev. Lett.* **81**, 3563 (1998).
- [23] S. L. Braunstein and P. van Loock, Quantum information with continuous variables, *Rev. Mod. Phys.* **77**, 513 (2005).
- [24] G. Adesso, S. Ragy, and A. R. Lee, Continuous variable quantum information: Gaussian states and beyond, *Open Syst. Inf. Dyn.* **21**, 1440001 (2014).
- [25] S. Adhikari, A. S. Majumdar, and N. Nayak, Broadcasting of continuous-variable entanglement, *Phys. Rev. A* **77**, 042301 (2008).
- [26] S. Adhikari, B. Chakraborty, A. S. Majumdar, and S. Vaidya, Quantum entanglement in a noncommutative system, *Phys. Rev. A* **79**, 042109 (2009).
- [27] P. Chowdhury, A. S. Majumdar, and G. S. Agarwal, Nonlocal continuous-variable correlations and violation of Bell's inequality for light beams with topological singularities, *Phys. Rev. A* **88**, 013830 (2013).
- [28] P. Chowdhury, T. Pramanik, A. S. Majumdar, and G. S. Agarwal, Einstein-Podolsky-Rosen steering using quantum correlations in non-Gaussian entangled states, *Phys. Rev. A* **89**, 012104 (2014).
- [29] G. S. Agarwal, *Quantum Optics* (Cambridge University Press, Cambridge, UK, 2012).
- [30] C. C. Gerry and P. L. Knight, *Introductory Quantum Optics* (Cambridge University Press, Cambridge, UK, 2023).
- [31] D. Leibfried, D. M. Meekhof, B. E. King, C. H. Monroe, W. M. Itano, and D. J. Wineland, Experimental determination of the motional quantum state of a trapped atom, *Phys. Rev. Lett.* **77**, 4281 (1996).
- [32] D. T. Smithey, M. Beck, M. G. Raymer, and A. Faridani, Measurement of the Wigner distribution and the density matrix of a light mode using optical homodyne tomography: Application to squeezed states and the vacuum, *Phys. Rev. Lett.* **70**, 1244 (1993).
- [33] E. Wigner, On the quantum correction for thermodynamic equilibrium, *Phys. Rev.* **40**, 749 (1932).

- [34] K. Banaszek and K. Wódkiewicz, Nonlocality of the Einstein-Podolsky-Rosen state in the Wigner representation, *Phys. Rev. A* **58**, 4345 (1998).
- [35] Z.-B. Chen, J.-W. Pan, G. Hou, and Y.-D. Zhang, Maximal violation of Bell's inequalities for continuous variable systems, *Phys. Rev. Lett.* **88**, 040406 (2002).
- [36] T. Yarnall, A. F. Abouraddy, B. E. A. Saleh, and M. C. Teich, Experimental violation of Bell's inequality in spatial-parity space, *Phys. Rev. Lett.* **99**, 170408 (2007).
- [37] E. M. Rains, Rigorous treatment of distillable entanglement, *Phys. Rev. A* **60**, 173 (1999).
- [38] E. M. Rains, Bound on distillable entanglement, *Phys. Rev. A* **60**, 179 (1999).
- [39] C. H. Bennett, D. P. DiVincenzo, J. A. Smolin, and W. K. Wootters, Mixed-state entanglement and quantum error correction, *Phys. Rev. A* **54**, 3824 (1996).
- [40] G. Giedke and J. I. Cirac, Characterization of Gaussian operations and distillation of Gaussian states, *Phys. Rev. A* **66**, 032316 (2002).
- [41] J. Eisert, S. Scheel, and M. B. Plenio, Distilling Gaussian states with Gaussian operations is impossible, *Phys. Rev. Lett.* **89**, 137903 (2002).
- [42] S. L. Braunstein, Error correction for continuous quantum variables, *Phys. Rev. Lett.* **80**, 4084 (1998).
- [43] M. Walschaers, Non-Gaussian quantum states and where to find them, *PRX Quantum* **2**, 030204 (2021).
- [44] C. Navarrete-Benlloch, R. García-Patrón, J. H. Shapiro, and N. J. Cerf, Enhancing quantum entanglement by photon addition and subtraction, *Phys. Rev. A* **86**, 012328 (2012).
- [45] S.-Y. Lee, S.-W. Ji, H.-J. Kim, and H. Nha, Enhancing quantum entanglement for continuous variables by a coherent superposition of photon subtraction and addition, *Phys. Rev. A* **84**, 012302 (2011).
- [46] S. Roy, T. Chanda, T. Das, A. Sen(De), and U. Sen, Response in the violation of the Bell inequality to imperfect photon addition and subtraction in noisy squeezed states of light, *Phys. Rev. A* **98**, 052131 (2018).
- [47] A. Zavatta, S. Viciani, and M. Bellini, Quantum-to-classical transition with single-photon-added coherent states of light, *Science* **306**, 660 (2004).
- [48] V. Parigi, A. Zavatta, M. S. Kim, and M. Bellini, Probing quantum commutation rules by addition and subtraction of single photons to/from a light field, *Science* **317**, 1890 (2007).
- [49] O. Morin, K. Huang, J. Liu, H. L. Jeannic, C. Fabre, and J. Laurat, Remote creation of hybrid entanglement between particle-like and wave-like optical qubits, *Nat. Photonics* **8**, 570 (2014).
- [50] J. P. Olson, K. P. Seshadreesan, K. R. Motes, P. P. Rohde, and J. P. Dowling, Sampling arbitrary photon-added or photon-subtracted squeezed states is in the same complexity class as boson sampling, *Phys. Rev. A* **91**, 022317 (2015).
- [51] P. Chowdhury, T. Pramanik, and A. S. Majumdar, Stronger steerability criterion for more uncertain continuous-variable systems, *Phys. Rev. A* **92**, 042317 (2015).
- [52] A. G. Maity, S. Datta, and A. S. Majumdar, Tighter Einstein-Podolsky-Rosen steering inequality based on the sum-uncertainty relation, *Phys. Rev. A* **96**, 052326 (2017).
- [53] A. Tavakoli, A. Pozas-Kerstjens, M.-X. Luo, and M.-O. Renou, Bell nonlocality in networks, *Rep. Prog. Phys.* **85**, 056001 (2022).
- [54] C. Branciard, N. Gisin, and S. Pironio, Characterizing the nonlocal correlations created via entanglement swapping, *Phys. Rev. Lett.* **104**, 170401 (2010).
- [55] C. Branciard, D. Rosset, N. Gisin, and S. Pironio, Bilocal versus nonbilocal correlations in entanglement-swapping experiments, *Phys. Rev. A* **85**, 032119 (2012).
- [56] N. Gisin, Q. Mei, A. Tavakoli, M. O. Renou, and N. Brunner, All entangled pure quantum states violate the bilocality inequality, *Phys. Rev. A* **96**, 020304 (2017).
- [57] S. Gupta, S. Datta, and A. S. Majumdar, Preservation of quantum nonbilocal correlations in noisy entanglement-swapping experiments using weak measurements, *Phys. Rev. A* **98**, 042322 (2018).
- [58] K. Mukherjee, B. Paul, and D. Sarkar, Correlations in n -local scenario, *Quantum Info. Proc.* **14**, 2025 (2015).
- [59] A. Kundu, M. K. Molla, I. Chattopadhyay, and D. Sarkar, Maximal qubit violation of n -local inequalities in a quantum network, *Phys. Rev. A* **102**, 052222 (2020).
- [60] A. Tavakoli, P. Skrzypczyk, D. Cavalcanti, and A. Acín, Nonlocal correlations in the star-network configuration, *Phys. Rev. A* **90**, 062109 (2014).

- [61] M. Żukowski, A. Zeilinger, M. A. Horne, and A. K. Ekert, “Event-ready-detectors” Bell experiment via entanglement swapping, *Phys. Rev. Lett.* **71**, 4287 (1993).
- [62] K. Azuma, S. E. Economou, D. Elkouss, P. Hilaire, L. Jiang, H.-K. Lo, and I. Tzitrin, Quantum repeaters: From quantum networks to the quantum internet, *Rev. Mod. Phys.* **95**, 045006 (2023).
- [63] S. Wehner, D. Elkouss, and R. Hanson, Quantum internet: A vision for the road ahead, *Science* **362**, eaam9288 (2018).
- [64] H. J. Kimble, The quantum internet, *Nature (London)* **453**, 1023 (2008).
- [65] S. Sarkar, Network quantum steering enables randomness certification without seed randomness, *Quantum* **8**, 1419 (2024).
- [66] K. Mukherjee, Full network nonlocality based security in quantum key distribution, [arXiv:2603.20035](https://arxiv.org/abs/2603.20035).
- [67] S. Munshi and A. K. Pan, Device-independent full network nonlocality for arbitrary-party and unbounded-input scenario, *Phys. Rev. Lett.* **134**, 210203 (2025).
- [68] J. C. Guerra Copete, Network nonlocality with continuous variables, Ph.D. thesis, ICFO—Institut de Ciències Fotòniques, Barcelona, Spain, 2021.
- [69] J.-L. Jiang, X.-Z. Liu, X. Yang, X. Ding, D. Zhang, and M.-X. Luo, One-way network nonlocality of continuous variable entangled networks, *Adv. Quantum Technol.* **8**, e2500147 (2025).
- [70] T. Yan, J. Hou, X. Qi, and K. He, Witnessing nonlocality in quantum networks of continuous-variable systems by generalized quasiprobability functions, *Sci. China Inf. Sci.* **69**, 180503 (2026).
- [71] A. F. Abouraddy, T. Yarnall, B. E. A. Saleh, and M. C. Teich, Violation of Bell’s inequality with continuous spatial variables, *Phys. Rev. A* **75**, 052114 (2007).
- [72] T. Yarnall, A. F. Abouraddy, B. E. A. Saleh, and M. C. Teich, Synthesis and analysis of entangled photonic qubits in spatial-parity space, *Phys. Rev. Lett.* **99**, 250502 (2007).
- [73] K. H. Kagalwala, G. Di Giuseppe, A. F. Abouraddy, and B. E. A. Saleh, Single-photon three-qubit quantum logic using spatial light modulators, *Nat. Commun.* **8**, 739 (2017).
- [74] B. Zhang, Z.-R. Zhong, and S.-B. Zheng, Quantum nonlocality for two-mode correlated states based on generalized pseudospin operators, *Phys. Lett. A* **375**, 1765 (2011).
- [75] Z.-B. Chen and Y.-D. Zhang, Greenberger-Horne-Zeilinger nonlocality for continuous-variable systems, *Phys. Rev. A* **65**, 044102 (2002).
- [76] R. Horodecki, P. Horodecki, and M. Horodecki, Violating Bell inequality by mixed spin-12 states: Necessary and sufficient condition, *Phys. Lett. A* **200**, 340 (1995).
- [77] F. Andreoli, G. Carvacho, L. Santodonato, R. Chaves, and F. Sciarrino, Maximal qubit violation of n -locality inequalities in a star-shaped quantum network, *New J. Phys.* **19**, 113020 (2017).
- [78] M. M. Dorantes and J. L. Lucio M, Generalizations of the pseudospin operator to test the Bell inequality for the TMSV state, *J. Phys. A: Math. Theor.* **42**, 285309 (2009).
- [79] S. D. Bartlett, B. C. Sanders, S. L. Braunstein, and K. Nemoto, Efficient classical simulation of continuous variable quantum information processes, *Phys. Rev. Lett.* **88**, 097904 (2002).
- [80] A. Mari and J. Eisert, Positive Wigner functions render classical simulation of quantum computation efficient, *Phys. Rev. Lett.* **109**, 230503 (2012).
- [81] S. Chakrabarty, B. Mallick, A. G. Maity, and A. S. Majumdar, Operational detection of Wigner negativity in arbitrary quantum states from few copies, [arXiv:2606.26084](https://arxiv.org/abs/2606.26084).
- [82] S. Lloyd and S. L. Braunstein, Quantum computation over continuous variables, *Phys. Rev. Lett.* **82**, 1784 (1999).
- [83] M. S. Kim, Recent developments in photon-level operations on travelling light fields, *J. Phys. B: At. Mol. Opt. Phys.* **41**, 133001 (2008).
- [84] B. Mallick, S. Chakrabarty, S. Mukherjee, A. G. Maity, and A. S. Majumdar, Efficient detection of nonclassicality using moments of the Wigner function, *Phys. Rev. A* **111**, 032406 (2025).
- [85] B. C. Sanders, Entangled coherent states, *Phys. Rev. A* **45**, 6811 (1992).
- [86] P. Nandi, N. Debnath, S. Kala, and A. S. Majumdar, Magnetically induced Schrödinger cat states: The shadow of a quantum space, *Phys. Rev. A* **110**, 032204 (2024).
- [87] D. Hoshi, T. Nagase, S. Kwon, D. Iyama, T. Kamiya, S. Fujii, H. Mukai, S. Ahmed, A. F. Kockum, S. Watabe, F. Yoshihara, and J.-S. Tsai, Entangling Schrödinger’s cat states by bridging discrete- and continuous-variable encoding, *Nat. Commun.* **16**, 1309 (2025).

- [88] S. J. van Enk and O. Hirota, Entangled coherent states: Teleportation and decoherence, [Phys. Rev. A **64**, 022313 \(2001\)](#).
- [89] P. De Fabritiis, F. M. Guedes, G. Peruzzo, and S. P. Sorella, Entangled coherent states and violations of Bell-CHSH inequalities, [Phys. Lett. A **486**, 129111 \(2023\)](#).
- [90] H. Jeong, W. Son, M. S. Kim, D. Ahn, and Č. Brukner, Quantum nonlocality test for continuous-variable states with dichotomic observables, [Phys. Rev. A **67**, 012106 \(2003\)](#).
- [91] R. F. Werner, Quantum states with Einstein-Podolsky-Rosen correlations admitting a hidden-variable model, [Phys. Rev. A **40**, 4277 \(1989\)](#).
- [92] L. Mišta, R. Filip, and J. Fiurášek, Continuous-variable Werner state: Separability, nonlocality, squeezing, and teleportation, [Phys. Rev. A **65**, 062315 \(2002\)](#).
- [93] B. Kubala, A. Sauer, A. Tarantola, D. Fabian, A. Ginter, O. Kulikovska, F. D. Pumpo, J. Seiler, W. P. Schleich, and M. Zimmermann, Advanced quantum communication and quantum networks—From basic research to future applications, [arXiv:2602.05781](#).