

Electrostatic self-energy of a uniformly charged regular polygon: Exact reduction, closed-form expressions, and convergence to the disk

J. Batle^{1,*} and O. Ciftja^{2,†}

¹CRISP—Centre de Recerca Independent de sa Pobla, E-07420 sa Pobla, Mallorca, Spain

²Department of Physics, *Prairie View A&M University*, Prairie View, Texas 77446, USA



(Received 5 June 2026; accepted 9 October 2026; published 23 October 2026)

We evaluate the electrostatic self-energy of a uniformly charged regular polygon, treated as a planar charge distribution embedded in three-dimensional space and interacting through the Coulomb potential. By exploiting the piecewise-linear nature of the boundary, the four-dimensional Coulomb integral can be reduced exactly to a finite sum of elementary sector contributions that is valid for any number of sides. This construction leads to closed-form expressions for the uniformly charged triangle, square, pentagon, and hexagon. The pentagon result, although involving a quartic number field, is remarkably compact. As the number of sides increases, the energy decreases monotonically toward the corresponding value for a uniformly charged disk. A boundary-perturbation analysis gives the leading relative correction, which decreases as the inverse square of the number of sides. The second variation of the energy produces a nonanalytic correction proportional to the inverse fourth power of the number of sides multiplied by its logarithm. This term provides a direct signature of the polygonal corners. The arithmetic structure of the exact results also explains why compact closed forms cease to be available beyond the uniformly charged hexagon. All results are validated by high-precision numerical calculations and provide useful benchmarks for studies of confined two-dimensional Coulomb systems.

DOI: [10.1103/c72v-54h8](https://doi.org/10.1103/c72v-54h8)

I. INTRODUCTION

Electrostatics has played a central role in the development of classical field theory, potential theory, and boundary-value methods, with problems involving the distribution of charge on conductors serving as important benchmarks for both analytical and numerical techniques [1–7]. These questions remain relevant in contemporary studies of confined Coulomb systems, where geometry and boundary effects strongly influence equilibrium configurations and energies [8–16]. Finding the equilibrium charge distribution on a conducting surface is a fundamental problem in electrostatics [2]. For closed conductors, the surface charge density is determined primarily by the local geometry, whereas for open conductors the presence of boundaries and edges plays a dominant role [3]. Among planar domains, the circular disk is unique in that its equilibrium charge density is known analytically in closed form [2]. Because there are very few other known exact results, the complementary problem of charges distributed uniformly over a finite region has often

*Contact author: batlequantum@gmail.com

†Contact author: ogciftja@pvamu.edu

been adopted as a useful and tractable model system [17]. Understanding how the electrostatic energy of such a uniform distribution varies with the shape of the confining boundary is of independent theoretical interest and is also important for interpreting numerical simulations of confined charged particles, where uniform background distributions frequently arise as reference states [18,19]. Two facts make the uniform charge density energy more than a model quantity. By Thomson's theorem, the equilibrium (equipotential) distribution minimizes the energy at fixed charge, so the uniform charge density energy is a rigorous upper bound on the energy of the corresponding conductor and calibrates numerical solutions of the conductor problem; in Monte Carlo or molecular-dynamics studies of charges confined to a two-dimensional (2D) polygonal region (typically, square/rectangle) with a neutralizing uniform background, the background self-energy enters as an additive constant that must be known exactly if correlation energies are to be extracted reliably.

Presently, we consider a regular n -gon of circumradius R carrying a uniform surface charge density and evaluate its electrostatic self-energy. The regular polygon is a planar (zero-thickness) 2D charge distribution embedded in ordinary three-dimensional space (3D), so two surface elements interact through the 3D Coulomb potential $\propto 1/|\mathbf{r} - \mathbf{r}'|$, where $\mathbf{r}$ ($\mathbf{r}'$) are position vectors. This is the setting of 2D electron systems, quantum dots, charged colloidal monolayers, and thin conducting plates, and it must be distinguished from the electrostatics of systems that are translationally invariant along the third direction (for instance, infinite line charges), for which the interaction is logarithmic. Our approach extends the vertical-slicing method previously developed for uniformly charged planar domains [20–22]. By first integrating over the transverse coordinate, the Coulomb interaction is expressed through an effective kernel that depends only on the longitudinal coordinates and the local strip width. This procedure reduces the original four-dimensional integral for the self-energy to a 2D form, substantially simplifying the calculation. For a regular polygon, the piecewise-linear nature of the boundary allows the integration domain to be partitioned into a finite set of affine sectors. Within each sector, the boundary is described by simple linear functions, enabling the remaining integrations to be carried out in a systematic manner. Exploiting the geometric symmetries of the polygon further reduces the computational effort and leads to compact expressions suitable for both analytical investigation and numerical evaluation.

For the regular triangle and square, our analysis reproduces previously known exact results [17,18]. More significantly, it yields new closed-form expressions for the pentagon and hexagon. Among these cases, the pentagon is particularly noteworthy because its geometry introduces a substantially richer algebraic structure than that encountered for polygons whose exact energies can be expressed within simple quadratic number fields. Despite this added complexity, the final result reduces to a remarkably compact expression involving only a small set of logarithmic constants associated with the golden ratio and the number 5. We also investigate the large- n limit, where a regular polygon approaches a circular disk. In this regime, we recover the exact energy of a uniformly charged disk and derive the leading finite- n correction by means of a boundary-perturbation analysis. This correction shows that the electrostatic self-energy decreases monotonically as the number of sides increases, converging smoothly to the circular value. In other words, among regular polygons of fixed circumradius, shapes with more sides provide progressively better approximations to the electrostatic behavior of a uniformly charged disk.

Beyond the leading correction, we examine higher-order terms in the asymptotic expansion and find that the convergence to the disk is not purely algebraic. Instead, the fourth-order contribution contains a logarithmic dependence on the number of sides, reflecting the persistent influence of the polygon's corners even when the boundary is nearly circular. This logarithmic term is a signature of the boundary singularities, and both its coefficient and the accompanying constant are obtained in closed form from the second variation of the electrostatic energy about the disk (Appendix B). Special values of the Riemann zeta function enter for a transparent reason: The n -fold symmetry restricts the boundary corrugation to the angular harmonics k , $k = 1, 2, \dots$, with amplitudes proportional to k^{-2} , while the second variation of the Coulomb energy grows logarithmically with the harmonic index; summing over k therefore produces $\sum_k k^{-4} = \zeta(4)$ and $\sum_k k^{-4} \ln k = -\zeta'(4)$, which fixes

the logarithmic amplitude $-\frac{3}{2}\zeta(4) = -\pi^4/60$ and its $\zeta'(4)$ companion. The results presented here are relevant to a broad range of confined Coulomb systems in both classical and quantum settings. Examples include electrons confined in quantum dots [23], trapped-ion assemblies, and colloidal particles manipulated by optical trapping techniques [24]. In addition, the exact expressions and asymptotic formulas derived in this work provide valuable benchmarks for numerical approaches, including Monte Carlo simulations of finite charged systems, where reliable analytical reference values are often scarce.

The paper is organized as follows. We begin in Sec. II by formulating the problem and deriving the reduced kernel representation of the electrostatic self-energy. Section III develops the affine-sector framework that underlies the exact evaluation of the relevant integrals. Applications to specific regular polygons and the extension to arbitrary n are presented in Secs. IV and V. The asymptotic approach to the circular disk limit is analyzed in Sec. VI, while numerical tests of the analytical formulas are reported in Sec. VII. Section VIII summarizes the results and discusses extensions of the method to irregular polygons, nonuniform densities, and the equipotential (conductor) problem.

II. FORMULATION AND EXACT KERNEL

We consider a regular n -gon of circumradius R and area A_n containing a uniformly distributed charge Q . The corresponding surface charge density is therefore constant throughout the polygonal domain:

$$\rho_0 = \frac{Q}{A_n}, \quad (1)$$

where

$$A_n = \frac{n}{2} R^2 \sin\left(\frac{2\pi}{n}\right). \quad (2)$$

The electrostatic self-energy stored in the object is

$$U_n = \frac{k_e}{2} \rho_0^2 \iint_{\Omega_n} \iint_{\Omega_n} \frac{d^2r d^2r'}{|\mathbf{r} - \mathbf{r}'|}, \quad (3)$$

where $\mathbf{r}$ ($\mathbf{r}'$) are 2D vectors, k_e is Coulomb's electric constant, and Ω_n is the polygonal domain of integration (with area A_n). The kernel $1/|\mathbf{r} - \mathbf{r}'|$ is the standard Coulomb potential restricted to the plane $z = 0$ that contains the charge. It is not the Green's function $-\ln|\mathbf{r} - \mathbf{r}'|$ of the 2D Laplacian, which would describe charges extended uniformly along the third z direction. Because $1/|\mathbf{r} - \mathbf{r}'|$ is integrable over a bounded planar domain, the self-energy in Eq. (3) is finite. The disk ($n \rightarrow \infty$) is the classical example: For uniform surface charge density, it gives $U_\infty = 8k_e Q^2 / (3\pi R)$ [Eq. (31)], and for the equipotential disk the surface charge density $\sigma \propto (R^2 - r^2)^{-1/2}$ and capacitance $C = 8\epsilon_0 R$ follow from the same kernel [2]. We orient the polygon so that one vertex lies on the positive x axis and the figure is symmetric about that axis,

$$\Omega_n = \{(x, y) : -F(x) \leq y \leq F(x)\}, \quad (4)$$

where $F(x) \geq 0$ is the piecewise-linear upper boundary.

Writing $\mathbf{r} = (x, y)$, $\mathbf{r}' = (x', y')$, and $d = |x - x'|$, the inner double integral over $y \in [-a, a]$ and $y' \in [-b, b]$ (with $a = F(x)$ and $b = F(x')$) is elementary. The convolution of the two intervals has the trapezoidal weight $w(u) = \min(a+b - |u|, 2 \min(a, b))$ for $|u| \leq a + b$, so

$$g(a, b, d) = \int_{-a}^a \int_{-b}^b \frac{dy dy'}{\sqrt{d^2 + (y - y')^2}} = 2 \int_0^{a+b} \frac{w(u) du}{\sqrt{d^2 + u^2}}. \quad (5)$$

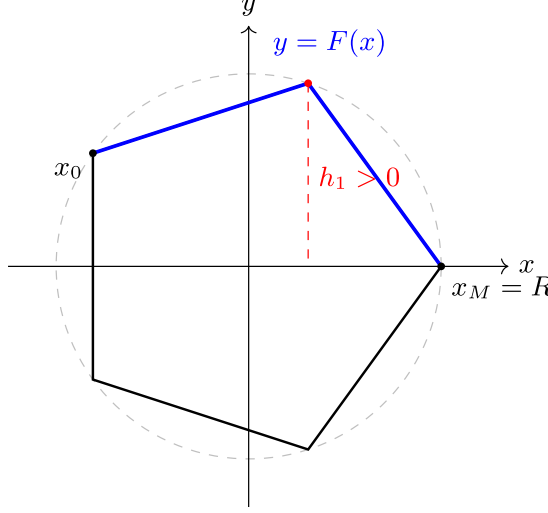


FIG. 1. Geometry for $n = 5$. The upper boundary $y = F(x)$ (blue) is piecewise linear. It meets the axis only at the vertex $x_M = R$; at the left end x_0 (black) the polygon closes with a vertical edge; every interior vertex sits at a strictly positive height (red), so its affine piece has a nonzero intercept that must be retained.

Carrying out the integral gives the exact kernel

$$g(a, b, d) = 2 \left[(a + b) \sinh^{-1} \frac{a + b}{d} - (a - b) \sinh^{-1} \frac{a - b}{d} - \sqrt{d^2 + (a + b)^2} + \sqrt{d^2 + (a - b)^2} \right]. \quad (6)$$

The two square-root terms and the overall factor of 2 are essential: They originate from the finite width of the strips and cannot be dropped. With Eq. (6), the self-energy reduces to a twofold integral,

$$U_n = \frac{k_e}{2} \rho_0^2 \int dx \int dx' g(F(x), F(x'), |x - x'|). \quad (7)$$

As $d \rightarrow 0$ with $a, b \rightarrow F(x) > 0$, one has $g \sim 4F(x) \ln(1/d)$, an integrable logarithmic singularity, so Eq. (7) converges absolutely. It is convenient to use the dimensionless coefficient

$$C_n \equiv \frac{U_n R}{k_e Q^2} = \frac{R}{2A_n^2} \int dx \int dx' g(F(x), F(x'), |x - x'|), \quad (8)$$

which depends only on n .

III. AFFINE-SECTOR DECOMPOSITION

A. Geometry: Intercepts must be kept

For a regular n -gon the upper boundary consists of $M = \lfloor n/2 \rfloor$ straight segments (Fig. 1). Let the vertices with $y \geq 0$, ordered by increasing x , be $x_0 < x_1 < \dots < x_M$, with x_0 the leftmost vertex and $x_M = R$ the vertex on the axis. On sector i ,

$$F(x) = m_i x + c_i, \quad x \in [x_{i-1}, x_i], \quad (9)$$

with slope m_i and intercept c_i . Denote the end-point heights as

$$h_{i-1} = F(x_{i-1}), \quad h_i = F(x_i), \quad \Delta_i = x_i - x_{i-1}. \quad (10)$$

A crucial geometric fact governs the whole calculation: F vanishes at the vertex $x_M = R$ and, for even n , also at x_0 ; for odd n the polygon closes on the left with a vertical edge, and $F(x_0) = R \sin(\pi/n) > 0$. At every interior vertex the boundary is at a strictly positive height $h_i > 0$, since the polygon is convex and only the vertices lying on the symmetry axis touch it. Consequently the intercept c_i is nonzero for all interior sectors, and the end-point heights h_{i-1}, h_i enter the sector integrals on the same footing as the slope.

This is the point at which a naive treatment fails. One is tempted to translate each sector to a local variable in which $F(x) = m_i x$ vanishes at an end point of the sector; this is correct *only* for a sector that ends on the axis (the rightmost one and, for even n , the leftmost one). Replacing $F \rightarrow m_i x$ in every sector discards the intercept, removes the dominant contribution of the strip width, and yields an expression that is not merely inaccurate but can be negative—hence unphysical. The correct reduction below keeps c_i (equivalently h_{i-1}, h_i) throughout.

By additivity of the integral over the partition of $[x_0, x_M]^2$ into sector rectangles, Eq. (8) becomes a finite sum,

$$\iint g dx dx' = \sum_{i=1}^M \sum_{j=1}^M \int_{x_{i-1}}^{x_i} dx \int_{x_{j-1}}^{x_j} dx' g(F(x), F(x'), |x - x'|) \equiv \sum_{i,j} J_{ij}. \quad (11)$$

B. Self-sector integral J_{ii}

Fix a sector and drop the index i . With $F(x) = mx + c$ on a strip of width Δ , pass to sum and difference variables $s = x + x'$ and $d = x - x'$, for which $a + b = ms + 2c \equiv w$ and $a - b = md$. Since g is even in d , the Jacobian and the two half planes combine to give, after $ds = dw/m$,

$$J_{ii} = \frac{1}{m} \int_0^\Delta dd \int_{2h_{i-1}+md}^{2h_i-md} dw \hat{g}(w, d), \quad (12)$$

where $\hat{g}(w, d) = 2[w \sinh^{-1} \frac{w}{d} - md \sinh^{-1} m - \sqrt{d^2 + w^2} + d\sqrt{1 + m^2}]$ is the kernel (6) expressed through $w = a + b$. Defining

$$P(w, d) = \left(\frac{w^2}{2} - \frac{d^2}{4} \right) \sinh^{-1} \frac{w}{d} - \frac{3w}{4} \sqrt{d^2 + w^2}, \quad (13)$$

the s integral is elementary and one obtains the exact result

$$J_{ii} = \frac{2}{m} [T(h_i, -1) - T(h_{i-1}, +1)] + \frac{2\Delta^3}{3} (\sqrt{1 + m^2} - m \sinh^{-1} m), \quad (14)$$

with

$$T(h, \sigma) = \int_0^\Delta P(2h + \sigma md, d) dd. \quad (15)$$

The remaining d integral in Eq. (15) is elementary: The substitution of $w = 2h + \sigma md$ into Eq. (13) produces terms of the form $\int d^k \sinh^{-1}(2h/d + \sigma m) dd$ ($k = 0, 1, 2$) and $\int (2h + \sigma md) \sqrt{d^2 + (2h + \sigma md)^2} dd$, each of which integrates to inverse-hyperbolic and algebraic functions of the end-point data. Equation (14) has been verified against direct two-dimensional quadrature to 25 significant digits, including sectors with $h_{i-1}, h_i \neq 0$. Setting $h_{i-1} = h_i = 0$ (the spurious “zero-intercept” choice) in Eq. (14) recovers the incorrect expression of a naive treatment and demonstrates explicitly how the end-point heights enter.

C. Cross-sector integral J_{ij}

For $i \neq j$, the two strips occupy disjoint x intervals, so $|x - x'|$ has a fixed sign and, writing $t = |x - x'|$, the inner integral over the right-hand strip gives $a + b = C_+ + m_j t$ and $a - b = C_- - m_j t$, with $C_\pm$ constant at fixed x . The same antiderivative family as in Eqs. (13)–(15) applies, so J_{ij} is again elementary. Adjacent sectors share a corner where $t \rightarrow 0$ while $a + b \rightarrow 2h > 0$; the resulting $\ln(1/t)$ behavior is integrable, so J_{ij} is finite for all pairs.

D. Summary of the method

In summary, once the transverse coordinate has been integrated out, the self-energy of any regular n -gon is the exact finite sum $\sum_{i,j} J_{ij}$ of sector-pair integrals [Eqs. (11)–(15)], each an elementary combination of inverse-hyperbolic functions and square roots of the sector data $\{m_i, \Delta_i, h_{i-1}, h_i\}$. The representation holds for every n and involves no approximation. Whether the sum collapses to a compact closed form is a separate, arithmetic question, governed by the algebraic relations among the geometric constants of the polygon; it is taken up in Secs. IV and V and Appendix C.

IV. CLOSED FORMS FOR $n = 3$ AND $n = 4$

A. Equilateral triangle ($n = 3$)

With a vertex on the axis, the upper boundary is the single segment

$$F(x) = \frac{R-x}{\sqrt{3}}, \quad x \in \left[-\frac{R}{2}, R\right], \quad (16)$$

so $m = -1/\sqrt{3}$, $h_0 = \sqrt{3}R/2$ at the left vertex, and $h_1 = 0$ at the apex on the axis. (Note the slope $-1/\sqrt{3}$, not $-\sqrt{3}$.) Evaluating Eq. (14) and inserting $A_3 = \frac{3\sqrt{3}}{4}R^2$ gives the closed form

$$C_3 = \frac{U_3 R}{k_e Q^2} = \frac{2 \ln 3}{\sqrt{3}} = 1.268\,568\,201\,195\,128\dots \quad (17)$$

In terms of the side length $L = \sqrt{3}R$, this reads $U_3 = 2 \ln 3 (k_e Q^2/L)$, consistent with Ref. [18].

B. Square ($n = 4$)

For the square oriented with a vertex on the axis,

$$F(x) = R - |x|, \quad x \in [-R, R], \quad (18)$$

which is two affine sectors of slopes ± 1 meeting at the top vertex $(0, R)$, where $F = R \neq 0$ —again an interior vertex with nonzero height. Evaluating Eq. (14) and the cross term with area $2R^2$ yields

$$C_4 = \frac{\sqrt{2}(1 + 3 \sinh^{-1} 1) - 2}{3} = 1.051\,188\,334\,404\,826\dots \quad (19)$$

In side-length units $L = \sqrt{2}R$,

$$U_4 = \left[2 \sinh^{-1} 1 - \frac{2}{3}(\sqrt{2} - 1) \right] \frac{k_e Q^2}{L} = 1.486\,606\dots \frac{k_e Q^2}{L}, \quad (20)$$

in agreement with Refs. [17,18].

Both closed forms were confirmed independently: The quadrature value of Eq. (17) matches $2 \ln 3/\sqrt{3}$ to 18 digits, and Eq. (19) was recovered from the high-precision value by integer-relation detection.

V. GENERAL n RESULT: EXACT REDUCTION AND BENCHMARKS

For general n , let $\alpha = 2\pi/n$. The ordered upper vertices are $(R \cos k\alpha, R \sin k\alpha)$ for the values of k with $\sin k\alpha \geq 0$, from which the sector data $\{m_i, \Delta_i, h_{i-1}, h_i\}$ follow directly, and C_n is assembled from Eqs. (8), (11), and (14).

Because every J_{ij} is elementary, an exact closed form for C_n exists for every n ; the question is only whether it collapses to a compact expression. Whether it does turns out to depend on the arithmetic of the vertex coordinates rather than on n alone. The hexagon has its field of constants in

$\mathbb{Q}(\sqrt{3})$, with clean self-sector contributions (in units of $R = 1$)

$$J_{\text{slant}} = -\frac{1}{4} \ln 3 + \left(\frac{1}{2} + \frac{\sqrt{3}}{6}\right) \ln(2 + \sqrt{3}), \quad (21)$$

$$J_{\text{flat}} = 2\sqrt{3} - \frac{14}{3} + 3 \ln 3 + 2\sqrt{3} \ln(2 + \sqrt{3}). \quad (22)$$

The two cross-sector integrals are elementary as well. Their inner integration returns the explicit primitive of Sec. III C; the outer integration reduces, after one integration by parts, to integrals of the type $\int P(x) dx / [(\ell_1 x + \ell_0) \sqrt{q(x)}]$ with q a quadratic over $\mathbb{Q}(\sqrt{3})$, which are elementary—no dilogarithm appears. Both cross-sector integrals are evaluated this way: The slant-slant integral closes directly under computer algebra, and the slant-flat integral is obtained from the same explicit primitive (verified by differentiation), the removable limit at the shared vertex taken explicitly. Integer-relation detection on the separately computed high-precision values returns the identical closed forms (Appendix A), an independent confirmation. The results are

$$J_{01} = \frac{1}{24}(8 + 75 \ln 3 + 27\sqrt{3} \ln 3 - 30 \ln(2 + \sqrt{3}) - 34\sqrt{3} \ln(2 + \sqrt{3})), \quad (23)$$

$$J_{02} = \frac{1}{12}(-52 + 24\sqrt{3} + 9 \ln 3 - 27\sqrt{3} \ln 3 + 6 \ln(2 + \sqrt{3}) + 20\sqrt{3} \ln(2 + \sqrt{3})), \quad (24)$$

for the slant-flat (J_{01}) and slant-slant (J_{02}) pairs. Every constant lies in $K = \mathbb{Q}(\sqrt{3}) \oplus \mathbb{Q}(\sqrt{3}) \ln 3 \oplus \mathbb{Q}(\sqrt{3}) \ln(2 + \sqrt{3})$; in particular, *no* $\ln 2$ occurs. Assembling, with $A_6 = \frac{3\sqrt{3}}{2}$,

$$C_6 = \frac{2J_{\text{slant}} + J_{\text{flat}} + 4J_{01} + 2J_{02}}{2A_6^2} \quad (25)$$

and simplifying in exact arithmetic collapses every term, yielding the compact closed form

$$C_6 = \frac{4\sqrt{3} - 8 + 11 \ln 3 - 2 \ln(2 + \sqrt{3})}{9}. \quad (26)$$

This is an exact term-by-term evaluation, not a numerical fit; as an independent check, integer-relation detection applied to the separately computed 90-digit value of C_6 returns the same expression with residual below 10^{-80} (numerically 0.931 002 512 419 ..., matching the benchmark to all digits computed). To our knowledge, this hexagon value has not appeared before; together with C_3 and C_4 it shows that compact closed forms persist past the square.

The pentagon is the most intricate of the elementary cases. Its vertex heights are the nested radicals $\sin 36^\circ, \sin 72^\circ \propto \sqrt{10 \mp 2\sqrt{5}}$, so the field of constants is the quartic $K_4 = \mathbb{Q}(\sqrt{5}, \sqrt{10 - 2\sqrt{5}})$ rather than $\mathbb{Q}(\sqrt{5})$. The transcendental content, however, is remarkably small: Every asinh that arises—the slope constants $\text{asinh}(\tan 18^\circ) = \frac{3}{2} \ln \varphi - \frac{1}{4} \ln 5$ and $\text{asinh}(\tan 54^\circ) = \frac{3}{2} \ln \varphi + \frac{1}{4} \ln 5$ with $\varphi = (1 + \sqrt{5})/2$, and likewise the end-point constants from the self- and cross-sectors—reduces to $\ln \varphi$ and $\ln 5$, while every algebraic prefactor (slopes, heights, and the end-point radicals $\sqrt{\Delta^2 + w^2}$) lies in K_4 . The two sectors give (in units of $R = 1$), with $s_2 = \sqrt{10 - 2\sqrt{5}}$ and $s_{10} = \sqrt{50 - 10\sqrt{5}} = \sqrt{5} s_2$,

$$J_{00} = \frac{1}{48}(10s_2 - 10s_{10} + (285s_2 + 105s_{10}) \ln \varphi - (30s_2 + 15s_{10}) \ln 5), \quad (27)$$

$$J_{11} = \frac{1}{96}((30s_{10} - 30s_2) \ln \varphi + (15s_2 + 5s_{10}) \ln 5), \quad (28)$$

$$J_{01} = \frac{1}{96}(10s_{10} - 10s_2 + (555s_2 + 255s_{10}) \ln \varphi - (140s_2 + 75s_{10}) \ln 5). \quad (29)$$

TABLE I. Dimensionless self-energy coefficient $C_n = U_n R / (k_e Q^2)$ of a uniformly charged regular n -gon of circumradius R . Values for $n = 3, 4, 5, 6$ are the closed forms (17), (19), (30), and (26); the rest are from high-precision quadrature of Eq. (8). The last row is the disk limit $8/(3\pi)$. The values are generated by the scheme of Appendix A and are available from the authors on request at higher precision.

n	C_n
3	1.268 568 201 195 128
4	1.051 188 334 404 826
5	0.970 766 373 674 870
6	0.931 002 512 419 454
7	0.908 177 021 093 881
8	0.893 786 112 507 559
9	0.884 100 930 176 800
10	0.877 259 474 376 232
11	0.872 242 198 882 440
12	0.868 450 779 575 236
∞	0.848 826 363 156 775 ($=8/3\pi$)

Assembling $C_5 = (J_{00} + J_{11} + 2J_{01})/(2A_5^2)$, with $A_5 = \frac{5}{2} \sin 72^\circ$, the log-free part cancels identically and the result is purely logarithmic:

$$C_5 = \sqrt{10 - 2\sqrt{5}} \left[\frac{15 + 7\sqrt{5}}{10} \ln \varphi - \frac{15 + 11\sqrt{5}}{60} \ln 5 \right]. \quad (30)$$

Numerically $C_5 = 0.970\,766\,373\,674\,870\dots$, matching the benchmark to all digits computed. Thus, the pentagon is *not* the first noncompact case. Compact closed forms persist through both the pentagon and the hexagon. The pentagon merely raises the field of constants from quadratic to quartic, the price of its odd vertex count; earlier work had reported closed forms only through the square [17,18]. High-precision values of $C_n = U_n R / (k_e Q^2)$ for general n are collected in Table I; these are exact to all digits shown and are intended as benchmarks.

The sequence is strictly decreasing in n : At fixed circumradius, the polygon grows toward the disk, and the coefficient relaxes monotonically toward $8/(3\pi)$ *from above*.

The benchmarks quantify how strongly the shape of the confining region matters. Relative to the disk of the same circumradius, the triangle stores 49.5% more energy, the square 23.8%, the pentagon 14.4%, the hexagon 9.7%, the heptagon 7.0%, and the octagon 5.3%; beyond $n \simeq 12$, the excess is below 2.5% and is reproduced to better than 10^{-6} by the expansion of Sec. VI. In simulations of confined Coulomb systems with a uniform neutralizing background in a polygonal cell [18,19], C_n is the exact background-background term that must be subtracted to isolate the correlation energy; the closed forms and Table I supply this constant to any required precision. Because the equilibrium distribution of a conductor minimizes the energy at fixed charge, C_n is also a rigorous upper bound for the dimensionless energy $UR/(k_e Q^2) = R/(2k_e C)$ of a conducting polygonal plate of capacitance C (see Sec. VIII).

VI. DISK LIMIT AND ASYMPTOTIC EXPANSION

A. Disk limit

As $n \rightarrow \infty$ at fixed R , the polygon converges to the disk of radius R , both pointwise ($F(x) \rightarrow \sqrt{R^2 - x^2}$) and in area ($A_n \rightarrow \pi R^2$). The kernel (6) is bounded by an integrable majorant uniformly in n , so by dominated convergence the integral in Eq. (8) tends to its value for the disk. For the

uniformly charged disk, the self-energy is the classical result

$$U_\infty = \frac{8}{3\pi} \frac{k_e Q^2}{R}, \quad (31)$$

implying that

$$C_\infty = \frac{8}{3\pi} = 0.848\,826\,363\dots, \quad (32)$$

which is reproduced by Table I. This result establishes the circular disk as the asymptotic limit of the regular-polygon sequence. The remaining task is to quantify the finite- n corrections and determine how the geometric signatures of the polygonal boundary persist as the disk limit is approached.

B. Leading coefficient: A boundary-perturbation derivation

The $1/n^2$ coefficient can be obtained exactly. Let us write the dimensionless coefficient of the self-energy as $C_n = \frac{R}{2A_n^2} \mathcal{I}_n$ with the purely geometric quadruple integral

$$\mathcal{I}_n = \int_{\Omega_n} d^2r \int_{\Omega_n} \frac{d^2r'}{|\mathbf{r} - \mathbf{r}'|}, \quad \mathcal{I}_\infty = \frac{16\pi}{3} R^3, \quad (33)$$

The disk value corresponds to $C_\infty = 8/3\pi$.

In polar form, the regular n -gon of circumradius R has boundary $r(\varphi) = R[1 + \delta(\varphi)]$, where within each edge $\delta = \cos \varepsilon / \cos \psi - 1$, with $\varepsilon = \pi/n$ and $\psi \in [-\varepsilon, \varepsilon]$ the angle measured from the edge midpoint; $\delta = O(\varepsilon^2)$ and $\delta \leq 0$ (the polygon lies inside its circumcircle).

To first order in the boundary displacement,

$$\delta \mathcal{I} = 2 \int_{\delta\Omega} g(\mathbf{r}) d^2r, \quad g(\mathbf{r}) = \int_{D_R} \frac{d^2r'}{|\mathbf{r} - \mathbf{r}'|}, \quad (34)$$

where g is the potential of a uniform unit-density disk D_R and $\delta\Omega$ is the signed boundary layer. On the rim, this potential is *constant*, $g(R) = 4R$, so only the mean displacement survives:

$$\delta \mathcal{I}^{(1)} = 2R^2 g(R) \oint \delta d\varphi = 8R^3 \oint \delta d\varphi. \quad (35)$$

Using $\int \sec \psi d\psi = \sinh^{-1}(\tan \psi)$,

$$\oint \delta d\varphi = 2n[\cos \varepsilon \sinh^{-1}(\tan \varepsilon) - \varepsilon] = -\frac{2\pi^3}{3n^2} + O(n^{-4}), \quad (36)$$

so $\delta \mathcal{I}^{(1)} = -16\pi^3 R^3/(3n^2)$. The same layer fixes the area defect $\delta A = R^2 \oint \delta d\varphi = -2\pi^3 R^2/(3n^2)$, consistent with $A_n = \pi R^2(1 - \frac{2\pi^2}{3n^2} + \dots)$. Hence,

$$\frac{\delta C}{C_\infty} = \frac{\delta \mathcal{I}}{\mathcal{I}_\infty} - \frac{2\delta A}{A_\infty} = -\frac{\pi^2}{n^2} + \frac{4\pi^2}{3n^2} = +\frac{\pi^2}{3n^2}, \quad (37)$$

which gives

$$a_2 = \frac{8}{3\pi} \times \frac{\pi^2}{3} = \frac{8\pi}{9} = 2.792\,526\,8\dots \quad (38)$$

Because the rim potential is independent of φ , the oscillatory part of δ (the corrugation, with angular harmonics at multiples of n) does not contribute at first order; it enters at second order and controls the next term.

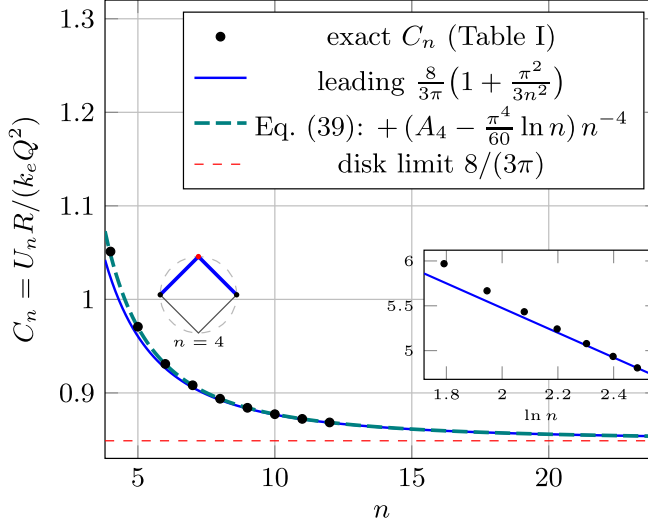


FIG. 2. Self-energy coefficient C_n vs n . Filled circles are the exact values of Table I. The blue curve is the leading $1/n^2$ law from Eq. (38); the teal dashed curve adds the fourth-order terms of Eq. (39), including the corner logarithm $-\frac{\pi^4}{60}n^{-4}\ln n$, which visibly improves the match already at $n = 5, 6$ (the asymptotic series is not expected to capture $n = 3, 4$). The red dashed line is the disk limit. *Upper left*: The $n = 4$ square in the same vertex-on-axis orientation; its upper boundary (blue) is a single tent with one interior vertex of height $h_1 = R$, the case whose closed form is $C_4 = [\sqrt{2}(1 + 3 \sinh^{-1} 1) - 2]/3$. *Lower right*: The rescaled fourth-order remainder $(C_n - C_\infty - a_2 n^{-2})n^4$, for $6 \leq n \leq 12$, plotted against $\ln n$; the straight line is Eq. (39) with the constants of Appendix B and no fitted parameter. Its slope $-2\pi^3/45$ is the corner logarithm.

C. Fourth-order term: A corner logarithm

The next correction is *not* a pure power of $1/n$. Its structure follows from the second variation of the energy about the disk, which is evaluated in closed form in Appendix B; here, we state the result and then show how it is confirmed numerically. The n -fold symmetry confines the boundary corrugation to the harmonics $m = kn$, and the boundary Hessian of the Coulomb energy grows logarithmically with m [Eq. (B7)], so the fourth-order term acquires a $\ln n$:

$$C_n = \frac{8}{3\pi} \left[1 + \frac{\pi^2}{3n^2} - \frac{\pi^4}{60} \frac{\ln n}{n^4} + \frac{A_4}{n^4} + O\left(\frac{\ln n}{n^6}\right) \right]. \quad (39)$$

Equivalently, the absolute coefficient of $n^{-4} \ln n$ is $-2\pi^3/45 = -1.378\,056\,74\dots$. Numerically, the rescaled remainder $(C_n - C_\infty - a_2/n^2)n^4$, computed for $12 \leq n \leq 84$ to more than 20 digits, does not approach a constant but drifts linearly in $\ln n$; a pure power-law fit is ill-conditioned, whereas a regression with free $\ln n$ amplitudes is stable and returns the $\ln n$ coefficient equal to $-2\pi^3/45$ to 11 significant figures. The same drift is already apparent in the values of Table I for $6 \leq n \leq 12$, which approach the line predicted by Eqs. (39) and (40) with no fitted parameter (inset of Fig. 2). Note that $-\pi^4/60 = -\frac{3}{2}\zeta(4)$, where $\zeta(x)$ is the Riemann zeta function. The accompanying constant is [Appendix B, Eq. (B9)]

$$A_4 = \frac{3}{2}[\zeta'(4) + (8 - \gamma - 2 \ln 2) \zeta(4)] = 9.696\,783\,124\,508\dots, \quad (40)$$

with γ the Euler-Mascheroni constant (so $a_4 = \frac{8}{3\pi}A_4 = 8.230\,89\dots$). The regression of Appendix A 5, with A_4 left free, reproduces Eq. (40) to 13 significant figures, the residual being set by the neglected $n^{-6} \ln n$ tower rather than by data precision.

Accuracy at small n . The four-term truncation of Eq. (39) represents the exact polygon energies well beyond its nominal range of validity. Against the closed forms of Table I the relative error of the expansion is -2.1×10^{-2} at $n = 3$, -3.0×10^{-3} at $n = 4$, -6.4×10^{-4} at $n = 5$, and only -1.7×10^{-4} at $n = 6$ —so even the hexagon is reproduced to better than two parts in 10^4 by an expansion derived in the $n \rightarrow \infty$ limit. The agreement sharpens rapidly beyond the cases treated in closed form here: Using the high-precision quadrature values of Sec. VII, the truncated Eq. (39) matches the true C_n to 5.4×10^{-5} at $n = 7$ and 1.9×10^{-5} at $n = 8$, is 2.3×10^{-6} at $n = 10$, falls below 10^{-6} for $n = 11$ (7.1×10^{-7}), and is 8.3×10^{-8} at $n = 12$, the residual decreasing in step with the neglected $O(n^{-6} \ln n)$ term. Thus, for every $n \geq 7$ the polygon self-energy is already given by the disk value and three correction terms to better than one part in 10^4 , and the exact closed forms of Secs. IV and V are needed only for the most strongly faceted polygons $n \leq 6$.

Structural origin. Neither the $\zeta(4)$ amplitude nor the form of A_4 is accidental; both follow from the second variation of $\mathcal{I}$ about the disk. Writing the corrugation as a Fourier series in φ , the n -fold periodicity confines its spectrum to harmonics $m = kn$, and the parabolic tooth $\delta(\psi) \simeq \frac{1}{2}(\psi^2 - \varepsilon^2)$ gives, as $n \rightarrow \infty$,

$$a_{kn} = \frac{2(-1)^k}{k^2 n^2} + O(n^{-4}), \quad k = 1, 2, \dots \quad (41)$$

Restricted to the rim, the Coulomb kernel $|\mathbf{r} - \mathbf{r}'|^{-1} \sim (R|\theta|)^{-1}$ is logarithmically singular, so the second-shape-derivative (Hessian) eigenvalues grow as $\hat{\lambda}_m = \mu \ln m + \nu + O(m^{-1})$. With $a_{kn}^2 = 4/(k^4 n^4)$, the oscillatory second-order contribution to the bracket is

$$\sum_{k \geq 1} \hat{\lambda}_{kn} a_{kn}^2 = \frac{4}{n^4} [\mu \zeta(4) \ln n - \mu \zeta'(4) + \nu \zeta(4)] + \dots, \quad (42)$$

using $\sum_k k^{-4} = \zeta(4) = \pi^4/90$ and $\sum_k k^{-4} \ln k = -\zeta'(4)$. Thus, the logarithm and its $\zeta'(4)$ companion share the prefactor μ , while the regular part ν of the Hessian—a renormalized corner energy set by the regularization of the singular rim kernel, the source of the γ and $\ln 2$ in Eq. (40)—sets the remaining $\zeta(4)$ piece. This is why A_4 is transcendental rather than algebraic. The closed-form Hessian of Appendix B gives $-\frac{3}{2}\zeta(4) = 4\mu \zeta(4)$, fixing the universal Hessian coefficient at the clean rational

$$\mu = -\frac{3}{8}, \quad \nu = \frac{3}{8}(8 - \gamma - 2 \ln 2), \quad (43)$$

the latter read off by matching the $\zeta(4)$ part of Eq. (42) to Eq. (40). Appendix B carries out this evaluation in full: The boundary Hessian is obtained in closed form [Eq. (B7)], and the assembly of Eqs. (B2)–(B9) yields both the amplitude $-\pi^4/60$ and Eq. (40) without any fitted input; the regression of Appendix A 5 is an independent confirmation. Equivalently, each vertex is a wedge of interior angle $\pi - 2\pi/n$, and logarithmic corner terms of exactly this type are familiar from heat-kernel, Casimir, and Laplacian-determinant expansions on domains with corners; they also explain why a naive Richardson extrapolation of C_n in powers of $1/n^2$ is ill-conditioned.

The leading correction is *positive*: The polygon energy lies above the disk value and relaxes to it from above as n^{-2} , a curvature-controlled approach. Figure 2 compares the tabulated values with Eq. (39).

VII. NUMERICAL VERIFICATION

All analytical results presented in this work were subjected to extensive and independent numerical verification. First, the reduced interaction kernel was validated against direct numerical evaluation of the original Coulomb integral, yielding agreement to essentially machine precision. For example, the kernel in Eq. (6) reproduces the brute-force double integral $\iint dy dy' / \sqrt{d^2 + (y - y')^2}$ to 18 digits. This confirms that the dimensional reduction employed throughout the analysis preserves the exact electrostatic interaction.

Second, the expressions derived for individual sector contributions were compared with direct numerical quadrature over the corresponding 2D domains. The self-sector formula in Eq. (14) matches direct 2D quadrature of g over a sector to 25 digits, for sectors with nonzero end-point heights. Overall, for representative sectors with nonvanishing end-point heights, the agreement extended to more than 20 decimal places, providing a stringent test of the sector decomposition and the associated integral reductions.

Third, the benchmark values reported for regular polygons were computed using high-precision $\tanh$ -sinh quadrature applied to the reduced integral representation. Particular care was taken in treating the integrable diagonal singularity, which was isolated analytically before numerical evaluation. The coefficients in Table I were obtained by $\tanh$ -sinh quadrature of Eq. (8) with the diagonal singularity isolated at an open end point, and cross-checked against the closed forms for $n = 3-6$, and were further verified by comparison with the asymptotic approach to the circular-disk limit. The consistency of these independent calculations establishes the correctness of both the analytical formulas and the numerical implementation.

Finally, the asymptotic expansion for large numbers of polygon sides was examined in detail. The leading correction to the disk limit was derived analytically using boundary-perturbation methods and subsequently confirmed by numerical calculations over a wide range of polygon orders. The fourth-order coefficients, derived in closed form in Appendix B, were checked against exact and high-precision numerical data: A regression with free $\ln n$ amplitudes recovers the logarithmic coefficient to 11 and the constant A_4 to 13 significant digits, whereas a pure power-law model is numerically singular. This behavior reflects the persistent influence of the polygon's corners even in the nearly circular regime and provides an additional, highly nontrivial check on the asymptotic theory.

Taken together, these validations establish the accuracy and internal consistency of the entire framework. The benchmark energies reported here therefore constitute reliable reference values for Monte Carlo, molecular-dynamics, and related numerical studies of finite Coulomb systems confined to polygonal geometries.

VIII. CONCLUSIONS

We have reduced the electrostatic self-energy of a uniformly charged regular n -gon to an exact 2D integral with a closed-form interaction kernel, and subsequently to a finite sum of affine-sector contributions. The reduction exploits the piecewise-linear structure of the boundary and leads to a representation in which each sector contributes an elementary integral. A key point of the construction is the presence of finite intercept terms associated with the nonzero heights of the interior vertices. These terms enter each pairwise interaction explicitly and are required for consistency of the formulation; neglecting them leads to incorrect scaling already at low n .

Within this framework, exact closed-form expressions are obtained for several low- n cases, including the triangle, square, pentagon, and hexagon. The resulting expressions reproduce known results where available and extend them to new cases in a unified setting. The pentagon is algebraically the most intricate case, involving intermediate quantities that reside in a quartic number field, but the final result simplifies to a compact expression involving logarithmic constants. Despite the increasing algebraic complexity with decreasing symmetry, all cases follow a consistent trend: The self-energy decreases monotonically with increasing n and converges smoothly toward the circular disk limit.

The asymptotic behavior for large n is obtained via a boundary-perturbation expansion about the disk geometry. The leading correction scales as n^{-2} and has a universal coefficient independent of the polygonal details beyond symmetry. Higher-order corrections reveal a more subtle structure. In addition to analytic powers of $1/n$, a nonanalytic contribution proportional to $n^{-4} \ln n$ appears. This term originates from corner-induced singularities in the second variation of the electrostatic energy and therefore encodes the persistent effect of polygonal vertices even in the near-continuum limit. The associated coefficients, both the logarithmic amplitude $-\pi^4/60$ and the constant A_4 , are

obtained in closed form from the boundary Hessian of the disk and confirmed numerically to 11 and 13 significant digits, respectively.

The affine-sector reduction is not tied to regular polygons. Any polygon symmetric about the x axis is covered as written, since only the piecewise linearity of $F(x)$ was used. For a general simple polygon with upper and lower boundaries $F_+(x) \geq F_-(x)$ that are not mirror images, the transverse integration over $y \in [F_-(x), F_+(x)]$, and $y' \in [F_-(x'), F_+(x')]$, with $\Phi(u) = u \sinh^{-1}(u/d) - \sqrt{d^2 + u^2}$, gives

$$g = \Phi(F_+ - F'_-) - \Phi(F_+ - F'_+) - \Phi(F_- - F'_-) + \Phi(F_- - F'_+), \quad (44)$$

which reduces to Eq. (6) when $F_{\pm} = \pm F$ and $F'_{\pm} = \pm F'$; the sector integrals then involve the same antiderivative families as Eqs. (13)–(15). Nonconvex polygons are handled by cutting the x range at every vertex so that each slice is a union of trapezoidal strips. Two further generalizations follow at little cost: A piecewise polynomial rather than uniform density multiplies the integrand by polynomials in the coordinates and leaves the transverse integrals elementary, and the mutual energy of two disjoint polygons is a sum of cross-sector terms J_{ij} alone. What is lost for irregular shapes is not exactness but compactness: The vertex coordinates no longer lie in a small number field, so the collapse described in Appendix C does not occur.

The uniform-density problem is the natural counterpart of the equipotential (conductor) problem recalled in the Introduction, and the two are quantitatively related. Since the equilibrium distribution minimizes the energy at fixed total charge, the uniform-density energy is a rigorous upper bound on the conductor energy, $U_{\text{cond}} = Q^2/(2C) \leq U_{\text{uniform}}$, for every shape. For the disk, $C = 8\epsilon_0 R$ gives $U_{\text{cond}}R/(k_e Q^2) = \pi/4 = 0.7854$, that is, $3\pi^2/32 = 92.5\%$ of the uniform value $8/(3\pi)$; the difference measures the energy released when the charge relaxes toward the edge. For polygonal plates no closed form for C is known: The equilibrium density diverges as the inverse square root of the distance to the edge and carries additional corner singularities whose exponents depend on the interior angle [25], so that the problem is an open-plate integral equation treated so far only numerically [26–28]. The present results bear on it in two ways. First, Table I bounds and calibrates such numerical conductor energies. Second, since the sector reduction extends to piecewise-polynomial densities on the same partition (previous paragraph), it supplies exactly integrated matrix elements for a Ritz treatment of the conductor problem, in which the density is expanded in polynomials on the sectors and the energy is minimized at fixed charge. The corner mechanism found here, with its $n^{-4} \ln n$ signature, suggests that the capacitance of a regular polygonal plate approaches that of the disk with nonanalytic corrections of the same type. Extending the analysis to equipotential polygons along these lines is a natural continuation of this work.

Overall, these results provide a unified and systematically controlled description of the electrostatic self-energy of uniformly charged regular polygons. They clarify the role of boundary geometry in determining finite-size corrections and establish how the smooth circular limit emerges from discrete polygonal domains while retaining identifiable signatures of corner singularities in higher-order terms.

ACKNOWLEDGMENTS

J.B. acknowledges fruitful discussions with Joana Rosselló, Maria del Mar Batle, and Regina Batle. J.B. received no funding for the present work. The research of O.C. was supported in part by the PRISE and PIER grants from Prairie View A&M University.

DATA AVAILABILITY

All results reported in this work are analytic, or are high-precision numbers generated from the analytic expressions given in the text; the algorithms used to produce them are described in full in Appendix A. The numerical values and the arbitrary-precision routines that generate them are available from the authors on reasonable request.

APPENDIX A: NUMERICAL METHODS

All values quoted in this work were produced with arbitrary-precision arithmetic (typically 30–65 significant digits) using the `mpmath` library [29] so that quadrature and round-off errors lie far below the digits reported. The description below is intended to be complete enough for an independent reimplementaion: It specifies the analytic preintegration that removes the diagonal ridge, the quadrature rule and its working precision, the exact algebraic end-point data entering each sector, and the constant basis used in the integer-relation searches. Three ingredients enter.

1. Closed-form inner integration

For each ordered pair of affine sectors, the inner integral over x' of the kernel (6) is performed analytically. Writing the kernel through $w = a \pm b$ and $t = |x - x'|$, every term reduces to one of the elementary antiderivatives

$$\int \sinh^{-1} \frac{A}{t} dt, \quad \int t \sinh^{-1} \frac{A}{t} dt, \quad \int \sqrt{t^2 + A^2} dt, \quad (\text{A1})$$

(with A an affine function of the fixed variable), each expressible in $\sinh^{-1}$ and algebraic functions. The diagonal block $i = j$ contains the integrable logarithmic ridge at $x' = x$; it is isolated analytically by splitting the inner integral at the diagonal, so the antiderivative is evaluated at an interior limit $t \rightarrow 0^+$ where it is finite, leaving no singular quadrature.

2. Outer integration

The remaining one-dimensional integral over x is smooth (the inner step has removed the ridge) and is evaluated by adaptive $\tanh$ -sinh (double-exponential) quadrature [30]. The resulting C_n agree with the closed forms (17), (19), (30), and (26) and with direct two-dimensional quadrature of Eq. (8) to all digits reported in Table I. For even n , the reflection symmetry $g(-x) = g(x)$ of the integrand halves the cost, which made n up to 84 accessible.

3. Identification of analytic constants

The triangle, square, and hexagon closed forms are obtained by symbolic integration. For the hexagon this means all four sector integrals: The self-sectors [Eqs. (21) and (22)] and the cross-sectors [Eqs. (23) and (24)], whose explicit antiderivatives we verified by differentiation; their exact assembly gives Eq. (26). As an independent confirmation, integer-relation detection by the PSLQ algorithm [31] applied to the separately computed 90-digit value of C_6 against a basis of candidate constants returns the same relation, exact to a residual below 10^{-80} .

4. The pentagon

The self-sectors are obtained in closed form by direct integration. The substitution $w = 2h + \sigma md$ in Eq. (15) reduces $T(h, \sigma)$ to the elementary integrals $\int_0^\Delta d^k [(1 + m^2)d^2 + 4\sigma mhd + 4h^2]^{-1/2} dd$ ($k = 0, 1, 2$) and $\int_0^\Delta (2h + \sigma md)\sqrt{\cdot} dd$, whose primitives are asinh-plus-algebraic; evaluating them at the exact algebraic end points gives J_{00}, J_{11} as elements of $\mathbb{Q}(\sqrt{5}, \sqrt{10 - 2\sqrt{5}}) \otimes \{1, \ln \varphi, \ln 5\}$. The only transcendental inputs are the closed forms $\operatorname{asinh} \tan 18^\circ = \frac{3}{2} \ln \varphi - \frac{1}{4} \ln 5$ and $\operatorname{asinh} \tan 54^\circ = \frac{3}{2} \ln \varphi + \frac{1}{4} \ln 5$, which follow from the identities $(\tan 18^\circ + \sec 18^\circ)^2 = \varphi^3/\sqrt{5}$ and $(\tan 54^\circ + \sec 54^\circ)^2 = \varphi^3\sqrt{5}$; every asinh that arises in T is verified to reduce to one of these (its argument equals $\pm \tan 18^\circ$ or $\pm \tan 54^\circ$ exactly), and every algebraic prefactor—slopes, intercepts, heights $\sin 36^\circ$, $\sin 72^\circ$, and the end-point radicals $\sqrt{\Delta^2 + w^2}$ —lies in K_4 . The cross-sector J_{01} is obtained in closed form by the same route carried to second order. Its inner primitive (Sec. III C) leaves an outer integrand that is a sum of terms $x^k \operatorname{asinh}[(rx + s)/(ux + v)]$ with coefficients in K_4 ; one integration by parts sends each

to $x^{k+1} \operatorname{asinh}(\cdot)/(k+1)$ plus $\int x^{k+1} dx/[(ux+v)\sqrt{R(x)}]$ with $R = (ux+v)^2 + (rx+s)^2$, and the Lagrange identity fixes the constant of the inner radical integral, $\Lambda = (rv - su)^2$, so the latter is a single asinh —no dilogarithm appears. The resulting elementary antiderivative, which we verified by differentiation to 50 digits, is evaluated between the sector limits; the integrand is finite at the shared vertex, where two asinh arguments degenerate to constants and R becomes a perfect square, and the removable limit there is taken explicitly. This reproduces Eq. (29) to the full working precision. As an independent check, J_{01} is *a priori* an element of the explicit 12-dimensional space $K_4 \otimes \{1, \ln \varphi, \ln 5\}$, and integer-relation detection on its 90-digit value returns the same Eq. (29) with small integer coefficients (residual below 10^{-80}). Assembling Eqs. (27)–(29) gives Eq. (30), in which the algebraic part cancels.

5. Asymptotic constants

The leading coefficient $a_2 = 8\pi/9$ is derived analytically (Sec. VI). The fourth-order $\ln n$ amplitude was extracted by least-squares regression of $(C_n - C_\infty - a_2/n^2)$ on the model $\sum_{k \geq 2} (a_{2k} + b_{2k} \ln n) n^{-2k}$ over $12 \leq n \leq 84$; a pure power-law model ($b_{2k} \equiv 0$) is numerically singular, whereas leaving the $\ln n$ amplitudes free yields a stable fit whose order-4 amplitude equals $-2\pi^3/45$ to 11 significant figures. The companion constant $A_4 = 9.696\,783\,124\,508\dots$ was then identified as $\frac{3}{2}[\zeta'(4) + (8 - \gamma - 2 \ln 2)\zeta(4)]$ [Eq. (40)]. The relation is stable: Recomputing the C_n at 60-digit working precision and cross-validating the regression over different node subsets pins A_4 to ~ 13 significant figures, the ceiling being set by the asymptotic extrapolation (the $n^{-6} \ln n$ tower), not by data precision. Neither coefficient is fitted: both are derived in Appendix B. The regression described here is retained as the independent numerical confirmation, and the three-term relation $(A_4 - \frac{3}{2}\zeta'(4))/(\frac{3}{2}\zeta(4)) = 8 - \gamma - 2 \ln 2$, confirmed to 12 digits, provides a further check.

We emphasize that no result reported in this paper is obtained by fitting. The closed forms of Secs. IV and V are term-by-term symbolic evaluations, the leading asymptotic coefficient is derived in Sec. VI, and the fourth-order coefficients are derived in Appendix B; the high-precision computations described here serve exclusively as independent verification.

APPENDIX B: ANALYTIC DERIVATION OF THE FOURTH-ORDER CONSTANT

Here, we evaluate the boundary Hessian of Sec. VI in closed form, turning the identification of A_4 [Eq. (40)] and of the logarithmic amplitude $-\pi^4/60$ into derived results. We set $R = 1$ and write $H_p = \sum_{j=1}^p 1/j$ for the harmonic number.

The energy as a shape functional. Write the boundary in polar form $r(\psi) = 1 + \rho(\psi)$. Since $I = \iint_{\Omega \times \Omega} |\mathbf{r} - \mathbf{r}'|^{-1}$ scales as $(\text{length})^3$, while the area A scales as $(\text{length})^2$ and the normalization length R as $(\text{length})^1$, the ratio $C = RI/(2A^2)$ is invariant under dilation: It is a pure *shape* functional. The disk maximizes the Coulomb energy at fixed area (by the Riesz rearrangement inequality), so it is a critical point of C and the first variation in each nonradial harmonic $\cos m\psi$ ($m \geq 1$) vanishes.

Decompose $\rho = \rho_0 + \sum_{m \geq 1} a_m \cos m\psi$ into mean and oscillatory parts. Two facts fix the expansion of $C = I/(2A^2)$ to quadratic order. (1) I is homogeneous of degree 3 in the boundary, so the pure-mean part is exact: $I[\rho_0] = (1 + \rho_0)^3 I_0 = I_0(1 + 3\rho_0 + 3\rho_0^2 + \dots)$. (2) The disk is a critical point of C , so the first variation in every $\cos m\psi$, $m \geq 1$, vanishes, and by orthogonality of the harmonics the mean-oscillatory cross terms vanish at second order as well. Writing $I[a_m \cos m\psi]/I_0 = 1 + \frac{1}{2}\beta_m a_m^2 + O(a_m^4)$ for a single harmonic, and using the exact area $A = \frac{1}{2} \oint (1 + \rho)^2 d\psi = \pi(1 + 2\rho_0 + \rho_0^2 + \frac{1}{2} \sum_m a_m^2)$, one has

$$\frac{C}{C_\infty} = \frac{I/I_0}{(A/\pi)^2} = \left(1 + 3\rho_0 + 3\rho_0^2 + \frac{1}{2} \sum_m \beta_m a_m^2\right) \left(1 - 4\rho_0 + 10\rho_0^2 - \sum_m a_m^2\right) + O(\rho^3), \quad (\text{B1})$$

and multiplying out gives

$$\frac{C_n}{C_\infty} = 1 - \rho_0 + \rho_0^2 + \frac{1}{2} \sum_{m \geq 1} \hat{\Lambda}_m a_m^2 + O(\rho^3), \quad (\text{B2})$$

where $\hat{\Lambda}_m = \beta_m - 2$ is the scale-invariant (area-normalized) shape-Hessian eigenvalue and β_m is defined by $I[a_m \cos m\psi]/I_0 = 1 + \frac{1}{2}\beta_m a_m^2 + O(a_m^4)$; the -2 is the contribution $-\sum a_m^2$ of the area to the Jacobian. For the polygon, the corrugation amplitude is $O(n^{-2})$, so the omitted cubic terms are $O(n^{-6})$.

The corrugation spectrum. With $t = \pi/n$,

$$\rho_0 = \frac{\cos t}{t} \ln(\sec t + \tan t) - 1 = -\frac{\pi^2}{3n^2} + O(n^{-6}), \quad (\text{B3})$$

the order- t^4 term canceling identically; the parabolic tooth has the Bernoulli-Fourier coefficients $a_{kn} = 2/(n^2 k^2)$ [on one edge $\delta(\psi) = \cos \varepsilon / \cos \psi - 1 = \frac{1}{2}(\psi^2 - \varepsilon^2) + O(\varepsilon^4)$ with $\varepsilon = \pi/n$, and the Fourier series of ψ^2 on $[-\varepsilon, \varepsilon]$, extended with period 2ε , is $\varepsilon^2/3 + \sum_{k \geq 1} 4\varepsilon^2(-1)^k (k\pi)^{-2} \cos(kn\psi)$, whence $\rho_0 = -\varepsilon^2/3 + O(\varepsilon^4)$ and $a_{kn} = 2(-1)^k \varepsilon^2 / (k\pi)^2 = 2(-1)^k / (k^2 n^2)$], agreeing with Eq. (41) up to the sign fixed by the choice of angular origin (immaterial for the energy, which depends on a_{kn}^2). Equation (B3) already furnishes the leading term, $C_n/C_\infty = 1 - \rho_0 + \dots = 1 + \pi^2/(3n^2) + \dots$, recovering $a_2 = 8\pi/9$.

The shape Hessian of the disk. Writing $I = \iint d\psi d\psi' H(r(\psi), r(\psi'); \psi - \psi')$ with $H(b, b'; \phi) = \int_0^b \int_0^{b'} rr' [r^2 + r'^2 - 2rr' \cos \phi]^{-1/2} dr dr'$, a Taylor expansion of the upper limits about $b = b' = 1$ gives, for $\rho = a \cos m\psi$,

$$\delta^2 I = \frac{1}{2} \iint [H_{bb} \rho^2 + 2H_{bb'} \rho \rho' + H_{b'b'} \rho'^2] d\psi d\psi' = \pi a^2 [\hat{H}_{bb}^{(0)} + \hat{H}_{bb'}^{(m)}], \quad (\text{B4})$$

because $\int_0^{2\pi} \cos^2 m\psi d\psi = \pi$ and $\iint f(\psi - \psi') \cos m\psi \cos m\psi' d\psi d\psi' = \pi \hat{f}^{(m)}$ for any even 2π -periodic f . Hence, $\beta_m = (2\pi/I_0) [\hat{H}_{bb}^{(0)} + \hat{H}_{bb'}^{(m)}]$, where the superscript denotes the angular Fourier coefficient $\hat{f}^{(k)} = \int_0^{2\pi} f(\phi) \cos k\phi d\phi$ and $H_{bb}, H_{bb'}$ are second derivatives of H at $b = b' = 1$. Three inputs evaluate in closed form. (1) The disk Coulomb integral is $I_0 = \iint_{\text{disk}} |\mathbf{r} - \mathbf{r}'|^{-1} = 16\pi/3$, the classical mean reciprocal distance $16/(3\pi)$ of the unit disk. (2) The mixed term is $H_{bb'}(\phi) = \frac{1}{2} \csc(\phi/2)$ (directly from the definition, $\partial_b \partial_{b'} H$ at $b = b' = 1$ is the integrand evaluated at $r = r' = 1$, namely, $[2 - 2 \cos \phi]^{-1/2}$), so the whole m dependence is

$$\hat{H}_{bb'}^{(m)} - \hat{H}_{bb'}^{(0)} = \int_0^{2\pi} \frac{\cos m\phi - 1}{2 \sin(\phi/2)} d\phi = -4 \left(H_{2m} - \frac{1}{2} H_m \right), \quad (\text{B5})$$

obtained by setting $\phi = 2\theta$ and using $(1 - \cos 2m\theta)/\sin \theta = 2 \sum_{k=1}^m \sin((2k-1)\theta)$ together with $\int_0^\pi \sin((2k-1)\theta) d\theta = 2/(2k-1)$ and $\sum_{k=1}^m (2k-1)^{-1} = H_{2m} - \frac{1}{2} H_m$. (3) The remaining constant $\Xi \equiv \hat{H}_{bb}^{(0)} + \hat{H}_{bb'}^{(0)}$ is finite although each piece diverges at $\phi \rightarrow 0$. It is fixed without any regularization by the homogeneity of H : Since $H(\lambda b, \lambda b'; \phi) = \lambda^3 H(b, b'; \phi)$, Euler's relation $bH_b + b'H_{b'} = 3H$ differentiated once more with respect to b gives, at $b = b' = 1$, $H_{bb} + H_{bb'} = 2H_b$, whence

$$\Xi = 2 \int_0^{2\pi} H_b d\phi = 2 \int_0^{2\pi} d\phi \int_0^1 \frac{r' dr'}{\sqrt{1 + r'^2 - 2r' \cos \phi}} = 2 g(R=1) = 8, \quad (\text{B6})$$

because the last double integral is the potential of the unit-density unit disk at a point of its rim, $g(R) = 4R$. The same value follows from integrating $\hat{H}_{bb}^{(0)}$ by parts, $\Xi = 16 - 4\pi + 8 \int_0^{\pi/2} \cos^2 \theta (1 + \sin \theta)^{-1} d\theta = 16 - 4\pi + 8(\pi/2 - 1) = 8$. Substituting $I_0 = 16\pi/3$, $\Xi = 8$, and

Eq. (B5) yields the eigenvalue in exact closed form, with no fitting,

$$\hat{\Lambda}_m = \beta_m - 2 = 1 - \frac{3}{2} \left(H_{2m} - \frac{1}{2} H_m \right), \quad (\text{B7})$$

whose large-order expansion, from $H_{2m} - \frac{1}{2} H_m = \frac{1}{2} \ln m + \ln 2 + \frac{\gamma}{2} + O(m^{-2})$, is $\hat{\Lambda}_m = -\frac{3}{4} \ln m + (1 - \frac{3}{4} \gamma - \frac{3}{2} \ln 2) + O(m^{-2})$. The absence of an m^{-1} term is not accidental: The $1/(2p)$ corrections of $H_p = \ln p + \gamma + 1/(2p) + O(p^{-2})$ cancel between H_{2m} and $\frac{1}{2} H_m$. This is why the first correction beyond Eq. (39) is $O(n^{-6} \ln n)$ rather than $O(n^{-5})$, in agreement with the residual scaling quoted earlier.

Assembly. With $\hat{\Lambda}_{kn} = -\frac{3}{4} \ln(kn) + \nu + O((kn)^{-2})$ and $a_{kn}^2 = 4/(k^4 n^4)$,

$$\frac{1}{2} \sum_{k \geq 1} \hat{\Lambda}_{kn} a_{kn}^2 = \frac{2}{n^4} \sum_{k \geq 1} \frac{-\frac{3}{4} \ln n - \frac{3}{4} \ln k + \nu}{k^4} = \frac{2}{n^4} \left[-\frac{3}{4} \zeta(4) \ln n + \frac{3}{4} \zeta'(4) + \nu \zeta(4) \right] + O(n^{-6}). \quad (\text{B8})$$

Inserting $a_{kn} = 2/(n^2 k^2)$ and Eq. (B7) into Eq. (B2) and using $\sum_k k^{-4} = \zeta(4)$, $\sum_k k^{-4} \ln k = -\zeta'(4)$, the oscillatory sum contributes $\frac{2}{n^4} [\mu \zeta(4) \ln n - \mu \zeta'(4) + \nu \zeta(4)]$, with $(\mu, \nu) = (-\frac{3}{4}, 1 - \frac{3}{4} \gamma - \frac{3}{2} \ln 2)$. Together with the area term $\rho_0^2 = \pi^4/(9n^4) = 10 \zeta(4)/n^4$ from Eq. (B3), bracket is $2\mu \zeta(4) \ln n + [\frac{\pi^4}{9} - 2\mu \zeta'(4) + 2\nu \zeta(4)]$. The logarithmic amplitude is $2\mu \zeta(4) = -\frac{3}{2} \zeta(4) = -\pi^4/60$, and the constant is

$$A_4 = \frac{\pi^4}{9} + \frac{3}{2} \zeta'(4) + 2 \left(1 - \frac{3}{4} \gamma - \frac{3}{2} \ln 2 \right) \zeta(4) = \frac{3}{2} [\zeta'(4) + (8 - \gamma - 2 \ln 2) \zeta(4)], \quad (\text{B9})$$

identical to Eq. (40) (verified to 30 digits). The $\zeta'(4)$ companion, the shared prefactor $\frac{3}{2}$, and the $\gamma, \ln 2$ content thus all follow from Eq. (B7). In the bookkeeping of Sec. VI, where the area term ρ_0^2 is absorbed into the regular part of the Hessian, Eq. (B7) reproduces exactly the coefficients $\mu = -\frac{3}{8}$, $\nu = \frac{3}{8}(8 - \gamma - 2 \ln 2)$ quoted in Eq. (43) [the factor $\frac{1}{2}$ relative to $\mu = -\frac{3}{4}$ being the convention of Eq. (42), which omits the explicit $\frac{1}{2}$ of Eq. (B2)]. *Remarks.* Equation (B2) is exact through quadratic order, and the eigenvalue (B7) rests only on the two elementary evaluations (B5) and (B6) and the classical $I_0 = 16\pi/3$. As an independent finite- n test, the assembly (B2) with the exact (B7) reproduces the directly computed C_n with a residual decreasing rapidly with n (e.g., 9×10^{-8} at $n = 20$, 1.5×10^{-8} at $n = 30$), confirming the construction beyond the asymptotic coefficient itself.

APPENDIX C: ARITHMETIC STRUCTURE OF THE CLOSED FORMS

The closed forms of Secs. IV and V raise two questions that only the arithmetic of the polygon can answer: which transcendental constants can appear in C_n at all, and why compact expressions are found through $n = 6$ but not beyond. This Appendix shows that both are governed by the unit group of the real cyclotomic field generated by $\cos(\pi/n)$: The only logarithms that can survive are logarithms of cyclotomic units and of the primes ramifying in that field, and their number grows with its unit rank, which first exceeds one at $n = 7$.

Every $\sinh^{-1}$ produced by the sector integration has an argument equal to $\pm \tan(j\pi/2n)$, j an integer (the edge slopes), or to an end-point radical, and $\sinh^{-1}(\tan \theta) = \ln[(1 + \sin \theta)/\cos \theta]$, a logarithm of a real algebraic number. Hence, each closed form is a field-linear combination of 1 and logarithms of algebraic numbers—no dilogarithm, and no transcendental beyond logarithms, ever enters. The natural home of the coefficients is the maximal real subfield of the cyclotomic field, $K_n^+ = \mathbb{Q}(\cos \frac{\pi}{n}) = \mathbb{Q}(\zeta_{2n})^+$, of degree $[K_n^+ : \mathbb{Q}] = \frac{1}{2} \varphi(2n)$, extended where necessary by the radicals $\sin(k\pi/n)$ carried by the polygon heights.

The numbers $\alpha_{n,k} = \tan(k\pi/n) + \sec(k\pi/n) = \exp[\sinh^{-1} \tan(k\pi/n)]$, and more generally $\tan(\theta) + \sec(\theta)$ for the slopes that occur, are either *units* ($1 + \sqrt{2}$ for the square, $2 + \sqrt{3}$ for the hexagon, powers of the golden unit φ for the pentagon, $\alpha_{7,1}$ and $\alpha_{7,2}$ below) or units multiplied by a power of a ramified prime, as $\tan 30^\circ + \sec 30^\circ = \sqrt{3}$ for the triangle and $(\tan 18^\circ + \sec 18^\circ)^2 = \varphi^3/\sqrt{5}$ for the pentagon. The logarithms of these non-unit factors account for $\ln 3$ and $\ln 5$, the logarithms of the rational primes ramifying in the fields involved. The number of *independent* unit logarithms available is the unit rank $(1/2)\varphi(2n) - 1$.

This explains the empirical dichotomy between $n \leq 6$ and $n \geq 7$. The degrees are

n	3	4	5	6	7	8,	(C1)
$\frac{1}{2}\varphi(2n)$	1	2	2	2	3	4,	
rank	0	1	1	1	2	3.	

For $n \leq 6$, the field K_n^+ is at most real quadratic, of unit rank 0 or 1: There is at most one fundamental unit, hence at most one unit logarithm, and the closed form is forced to be compact. Indeed the square has $\varepsilon = 1 + \sqrt{2}$ (the fundamental unit of $\mathbb{Q}(\sqrt{2})$), the hexagon $\varepsilon = 2 + \sqrt{3}$ of $\mathbb{Q}(\sqrt{3})$ together with $\ln 3$, and the pentagon the golden unit φ of $\mathbb{Q}(\sqrt{5})$ together with $\ln 5$ —exactly the transcendental content of Eqs. (19), (26), and (30). At $n = 7$, the field $\mathbb{Q}(\cos \frac{\pi}{7})$ becomes real *cubic* of unit rank 2: The generators $\alpha_{7,1}, \alpha_{7,2}$ are multiplicatively independent units (each with a reciprocal sextic minimal polynomial), so no collapse to a two-logarithm form is possible and the coefficients themselves are cubic irrationalities. Accordingly, $C_7 = 0.908\,177\,021\,1\dots$ and $C_8 = 0.893\,786\,112\,5\dots$ resist any small-basis integer relation, in contrast to $C_3, \dots, C_6$. These observations suggest that for every n the self-energy is a K_n^+ -linear combination of 1, the logarithms of a basis of cyclotomic units of K_n^+ , and the logarithms of the primes ramifying in K_n^+ , the number of logarithmic terms growing with the unit rank $\frac{1}{2}\varphi(2n) - 1$; the explicit determination of C_7 in this basis—a regulator computation in the cubic field $\mathbb{Q}(\cos \frac{\pi}{7})$ —is a natural next step.

-
- [1] J. C. Maxwell, *A Treatise on Electricity and Magnetism* (Clarendon Press, Oxford, 1873), Vol. 1.
 - [2] J. D. Jackson, *Classical Electrodynamics*, 3rd ed. (John Wiley and Sons, New York, 1999).
 - [3] W. R. Smythe, *Static and Dynamic Electricity*, 3rd ed. (McGraw-Hill, New York, 1968).
 - [4] D. J. Griffiths, *Introduction to Electrodynamics*, 3rd ed. (Prentice Hall, New Jersey, 1999).
 - [5] F. R. Zypman, Off-axis electric field of a ring of charge, *Am. J. Phys.* **74**, 295 (2006).
 - [6] H. R. Noh, Electrostatic potential of a charged ring: Applications to elliptic integral identities, *J. Korean Phys. Soc.* **71**, 37 (2017).
 - [7] V. Bochko and Z. K. Silagadze, On the electrostatic potential and electric field of a uniformly charged disk, *Eur. J. Phys.* **41**, 045201 (2020).
 - [8] J. Lekner, Electrostatics of two charged conducting spheres, *Proc. R. Soc. A* **468**, 2829 (2012).
 - [9] Y.-K. Yu, Electrostatics of charged dielectric spheres with application to biological systems. III. Rigorous ionic screening at the Debye-Hückel level, *Phys. Rev. E* **102**, 052404 (2020).
 - [10] T. LaFave, Jr., Correspondences between the classical electrostatic Thomson problem and atomic electronic structure, *J. Electrostat.* **71**, 1029 (2013).
 - [11] W. J. Atkinson, J. H. Young, and I. A. Brezovich, An analytic solution for the potential due to a circular parallel plate capacitor, *J. Phys. A: Math. Gen.* **16**, 2837 (1983).
 - [12] D. F. Bartlett and T. R. Corle, The circular parallel plate capacitor: A numerical solution for the potential, *J. Phys. A: Math. Gen.* **18**, 1337 (1985).
 - [13] G. Hummer, Electrostatic potential of a homogeneously charged square and cube in two and three dimensions, *J. Electrostat.* **36**, 285 (1996).
 - [14] G. Paffuti, Numerical and analytical results for the two discs capacitor problem, *Proc. R. Soc. A* **473**, 20160792 (2017).
 - [15] R. Beringer, Coulomb self-energy of axial figures, *Phys. Rev.* **131**, 1402 (1963).

- [16] V. Jadhao, Z. Yao, C. K. Thomas, and M. O. de la Cruz, Coulomb energy of uniformly charged spheroidal shell systems, *Phys. Rev. E* **91**, 032305 (2015).
- [17] O. Ciftja, Coulomb self-energy and electrostatic potential of a uniformly charged square in two dimensions, *Phys. Lett. A* **374**, 981 (2010).
- [18] J. Batle, O. Ciftja, M. Naseri, M. Ghoranneviss, A. Farouk, and M. Elhoseny, Equilibrium and uniform charge distribution of a classical two-dimensional system of point charges with hard-wall confinement, *Phys. Scr.* **92**, 055801 (2017).
- [19] V. M. Bedanov and F. M. Peeters, Ordering and phase transitions of charged particles in a classical finite two-dimensional system, *Phys. Rev. B* **49**, 2667 (1994).
- [20] O. Ciftja, Coulomb self-energy of a uniformly charged three-dimensional cube, *Phys. Lett. A* **375**, 766 (2011).
- [21] O. Ciftja, Quantitative analysis of shape-sensitive interaction of a charged nanoplate and a charged nanowire, *Nano* **10**, 1550114 (2015).
- [22] O. Ciftja, Stored Coulomb self-energy of a uniformly charged rectangular plate, *Adv. Math. Phys.* **2016**, 7207536 (2016).
- [23] O. Ciftja, Understanding electronic systems in semiconductor quantum dots, *Phys. Scr.* **88**, 058302 (2013).
- [24] R. Bubeck, P. Leiderer, and C. Bechinger, Fluctuation-induced order in two-dimensional colloidal clusters, *Europhys. Lett.* **60**, 474 (2002).
- [25] J. A. Morrison and J. A. Lewis, Charge singularity at the corner of a flat plate, *SIAM J. Appl. Math.* **31**, 233 (1976).
- [26] D. K. Reitan and T. J. Higgins, Accurate determination of the capacitance of a thin rectangular plate, *Trans. Am. Inst. Electr. Eng. Part I* **75**, 761 (1957).
- [27] F. H. Read, Improved extrapolation technique in the boundary element method to find the capacitances of the unit square and cube, *J. Comput. Phys.* **133**, 1 (1997).
- [28] F. H. Read, Capacitances and singularities of the unit triangle, square, tetrahedron and cube, *COMPEL* **23**, 572 (2004).
- [29] The mpmath development team, Mpmath: A Python library for arbitrary-precision floating-point arithmetic, version 1.3, 2023, <https://mpmath.org>.
- [30] H. Takahasi and M. Mori, Double exponential formulas for numerical integration, *Publ. Res. Inst. Math. Sci.* **9**, 721 (1974).
- [31] H. R. P. Ferguson, D. H. Bailey, and S. Arno, Analysis of PSLQ, an integer relation finding algorithm, *Math. Comp.* **68**, 351 (1999).