

Single-Q and double-Q magnetic orders: A theoretical analysis of inelastic neutron scattering in a centrosymmetric structure

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Recent discoveries of multi-Q magnetic structures in centrosymmetric compounds have stimulated growing interest in their microscopic origin and observable properties. Here, we calculate the dynamical magnetic structure factor for a double-Q magnetic structure and compare it with that of a single-Q configuration. Our results demonstrate how inelastic neutron scattering can distinguish the double-Q state from competing single-Q states in this square-lattice setting.

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I. INTRODUCTION

In recent years, significant effort has been devoted to the study of noncollinear magnetic structures [1–4]. Among these, topological spin textures such as skyrmions have attracted particular attention due to their potential applications in next-generation computing technologies [5,6]. More recently, interest has grown following the demonstration that complex spin configurations, such as skyrmion lattices [7–11], can form spin-moiré superlattices [12], leading to uniquely modulated electronic behavior. These developments have established an active area of research in which the potential applications of topological spin structures are being explored.

The majority of the investigated systems that exhibit multi-Q magnetic structures are noncentrosymmetric. In these systems, the Dzyaloshinskii-Moriya interaction [13,14] is a key factor in the formation of various noncollinear spin structures. An external magnetic field can change a single-Q magnetic structure, turning it into a multi-Q one [15,16].

In centrosymmetric materials, frustrated interactions can result in various noncollinear structures [17–19]. In particular, several theoretical studies have predicted the formation of multi-Q magnetic structures in centrosymmetric systems under specific conditions [20–23]. Despite these theoretical advances, clear experimental evidence for the existence of topological multi-Q structures in centrosymmetric compounds has only emerged very recently.

In the past few years, a limited number of centrosymmetric materials have been reported to host skyrmion-lattice phases. These include Gd_2PdSi_3 [24–28], GdRu_2Si_2 [29–31], EuAl_4 [32–34], EuGa_2Al_2 [35,36], $\text{Sr}_3\text{Fe}_2\text{O}_7$ [37,38], and SrFeO_3 [39,40]. It was shown that despite the presence of inversion symmetry, they can host multi-Q magnetic ordering and topological phases such as skyrmion and hedgehog lattices. Experimental studies have revealed the most important signatures

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of the multi- $\mathbf{Q}$ magnetic structures such as the topological Hall effect and magnetoresistance anomalies associated with spin fluctuations.

Recently, the magnetic state of $\text{Sr}_3\text{Fe}_2\text{O}_7$ has been examined in inelastic neutron scattering experiments [37,38]. $\text{Sr}_3\text{Fe}_2\text{O}_7$ crystallizes in a layered Ruddlesden-Popper structure composed of double perovskite layers of corner-sharing FeO_6 octahedra. Upon cooling, it undergoes a continuous transition from a high-temperature metallic phase to a low-temperature insulating state at $T_{\text{MIT}} = 340$ K, followed by the onset of noncollinear magnetic order below $T_N = 115$ K [37,41]. The system exhibits pronounced two-dimensionality and strong magnetic frustration. Recent systematic investigations have revealed a complex magnetic phase diagram that may host both single- $\mathbf{Q}$ and multi- $\mathbf{Q}$ magnetic phases, based on the magnetic structure factor [38]. This observation motivates a detailed theoretical analysis of their neutron scattering signatures. However, directly confirming multi- $\mathbf{Q}$ magnetic structures is still lacking.

The distinction between multi- $\mathbf{Q}$ and single- $\mathbf{Q}$ magnetic orders by means of their spin-wave spectra is a well-known concept. Early spin-wave calculations for a triple- $\mathbf{Q}$ structure showed clearly different excitation branches compared with the corresponding single- $\mathbf{Q}$ case [42]. More recent work on powder inelastic-neutron-scattering data has demonstrated the practical usefulness of this approach for identifying multi- $\mathbf{Q}$ order, and a comprehensive theoretical analysis of triple- $\mathbf{Q}$ dynamics on a triangular lattice has further clarified the generic spectral fingerprints [43–45].

In the present study, we focus on a square-lattice model with anisotropic exchange interactions that is directly relevant to the layered compounds Sr_2FeO_4 and $\text{Sr}_3\text{Fe}_2\text{O}_7$. By constructing the complete phase diagram in the anisotropy parameter plane and performing linear-spin-wave calculations of the inelastic neutron cross section for the competing single- $\mathbf{Q}$ (helix, spiral) and double- $\mathbf{Q}$ configurations, we identify a set of robust, experimentally accessible signatures of double- $\mathbf{Q}$ order—most prominently, a rotonlike magnon mode, the suppression of the outer vertical branch, and a gapless phason at both propagation vectors. These features persist after realistic domain averaging and magnon damping are taken into account, providing a concrete guide for future inelastic neutron-scattering experiments on the iron-oxide materials of interest.

The paper is organized as follows. After a brief introduction, we construct the magnetic phase diagram as a function of the anisotropy parameters. We then present a detailed analysis of the dynamical structure factor for the magnetic phases identified in the phase diagram. Finally, we conclude with a discussion and summary of our key findings.

II. THE MODEL

We investigate the following Hamiltonian defined on a square lattice:

$$H = \sum_{ij} \sum_{\alpha\beta} J_{ij}^{\alpha\beta} S_i^\alpha S_j^\beta, \quad (1)$$

where i, j and $\alpha, \beta = x, y, z$ are the site and spin indices, correspondingly. The interaction is parametrized by sets of three parameters ($J_\ell, \Delta_\ell, \delta_\ell$) as was proposed for $\text{Sr}_3\text{Fe}_2\text{O}_7$ in Ref. [38]:

$$J_{ij}^{\alpha\beta} = J_\ell \delta^{\alpha\beta} + \Delta_\ell (\hat{r}_{ij}^\alpha \hat{r}_{ij}^\beta - \frac{1}{2} \delta^{\alpha\beta}), \quad \text{for } \alpha, \beta = x, y,$$

$$J_{ij}^{zz} = J_\ell + \delta_\ell,$$

where $\ell = 1, \dots, 3$ corresponds to the nearest-neighbor, next-nearest-neighbor, and third-nearest-neighbor interactions. Here, $\hat{r}_{ij} = (\mathbf{r}_i - \mathbf{r}_j)/|\mathbf{r}_i - \mathbf{r}_j|$, where $\mathbf{r}_i$ is the coordinate of the i th spin. For $\Delta_\ell = \delta_\ell = 0$, the model becomes the isotropic Heisenberg model.

III. MAGNETIC PHASES

First, we discuss magnetic structures within the isotropic Heisenberg model Hamiltonian ($\Delta_\ell = \delta_\ell = 0$, $\ell = 1, \dots, 3$). Spin configurations that minimize the classical energy can be deter-

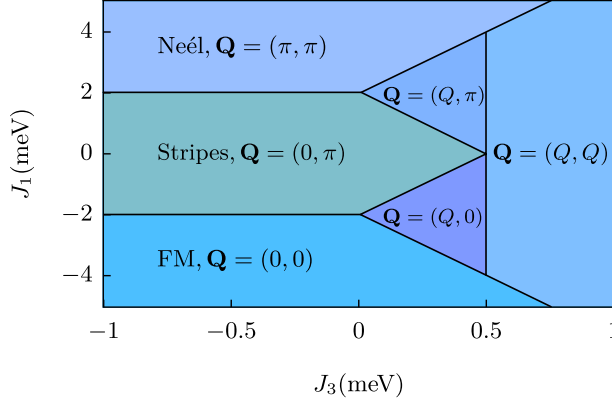


FIG. 1. Ground state of the isotropic J_1 - J_2 - J_3 Heisenberg model of a classical spin system. The states are described as helices with the corresponding propagation vectors $\mathbf{Q}$. Parameters: $J_2 = 1$ meV.

mined analytically, e.g., using the Luttinger-Tisza method. In this regime, the system stabilizes in a single- $\mathbf{Q}$ magnetic structure (helices), which is characterized by a propagation vector $\mathbf{Q} = (Q, 0)$ (helix along x axis or y axis) and $\mathbf{Q} = (Q, Q)$ (helix along the diagonal) [46]. Here, $4(J_2/2 + J_3) \cos(Q) + J_1 = 0$ and we assume here and below that the lattice constant $a = 1$. Because the Hamiltonian does not incorporate spin-orbit interaction, it remains invariant under rotations of the spin plane.

The results of the calculations are summarized in Fig. 1. As can be seen from the phase diagram, noncollinear structures are realized in the presence of frustrated interactions, specifically when the nearest-neighbor interaction is ferromagnetic ($J_1 < 0$) and the next-nearest-neighbor and third-nearest-neighbor interactions are antiferromagnetic ($J_{2,3} > 0$).

However, to explore more complex states with multiple $\mathbf{Q}$ vectors in the parameter space of δ_1 and Δ_2 , where the Luttinger-Tisza method is not applicable, the classical ground state must be determined using numerical approaches such as Monte Carlo simulations [47,48]. We employ the Metropolis Monte Carlo algorithm on spin lattices typically consisting of 21×21 sites [49]. Starting from a random spin configuration at high temperature, the system is gradually cooled to a low-temperature state, allowing it to relax into a candidate low-energy configuration. To ensure that the resulting spin configurations, in particular, for multi- $\mathbf{Q}$ states, are not artifacts of finite system size or boundary conditions, we perform additional simulations for varying lattice sizes (see Fig. S1 in the Supplemental Material [49]).

The spin configurations obtained from the Monte Carlo simulations are subsequently refined by local energy minimization. For this step, we employ a finer parameter grid with a spacing of approximately 0.005 in both Δ_2 and δ_1 . This procedure enables a more accurate determination of the stability regions of the various magnetic phases and yields the final phase diagram shown in Fig. 2. The resulting magnetic orders include helical, spiral, and double- $\mathbf{Q}$ phases. Representative spin textures are shown in Fig. 3, where the S^x and S^y components are represented by arrows and the S^z component is encoded by the color scale. The cycloidal spiral state shown in Fig. 3(a) can, to leading order, be described as follows:

$$S_i^x = S \cos(\mathbf{Q}\mathbf{R}_i), \quad S_i^y = S \sin(\mathbf{Q}\mathbf{R}_i), \quad S_i^z = 0,$$

where the propagation vector $\mathbf{Q} = (2\pi/7, 2\pi/7)$.

The helix shown in Fig. 3(b) can be described as

$$S_i^{x'} = S \cos(\mathbf{Q}\mathbf{R}_i), \quad S_i^{z'} = S \sin(\mathbf{Q}\mathbf{R}_i), \quad S_i^{y'} = 0.$$

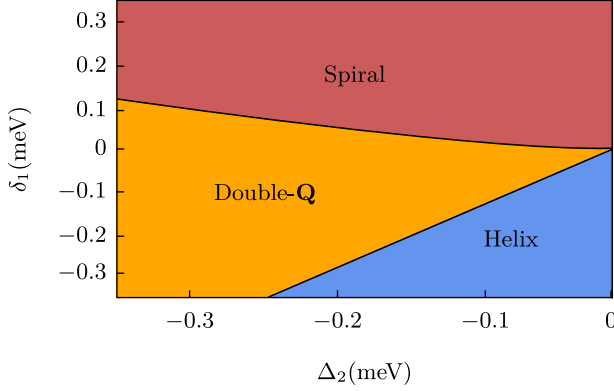


FIG. 2. Ground state of the anisotropic Heisenberg model. The diagram reveals several magnetic phases: spiral—a spiral phase with spin rotation in XY plane; helix—a “proper screw” rotation perpendicular to $(1, 1)$ and double- $\mathbf{Q}$ configurations. The exchange parameters are $J_1 = -7.2$ meV, $J_2 = 1.05$ meV, and $J_3 = 2.1$ meV.

We use a rotated coordinate system $x'y'z'$ in which the y' axis lies along the propagation vector $\mathbf{Q} = (2\pi/7, 2\pi/7)$, and the $x'z'$ plane is perpendicular to y' . The propagation vector is close to the one observed experimentally.

Figure 3(c) shows the double- $\mathbf{Q}$ phase. Its magnetic structure factor is given in Fig. 3(f). It consists of two pairs of peaks at the propagation vectors $\mathbf{Q} = (2\pi/7, 2\pi/7)$ and $\mathbf{Q}' = (-2\pi/7, 2\pi/7)$ with different magnitudes. In a leading approximation, the spin configuration can be described as a superposition of a helix and modulation in (π, π) direction, characterized by propagation vectors $\mathbf{Q}$ and $\mathbf{Q}'$:

$$\begin{aligned} S_i^{x'} &= A_1 \sin(\mathbf{Q}\mathbf{R}_i), \\ S_i^{y'} &= A_2 \sin(\mathbf{Q}'\mathbf{R}_i), \\ S_i^{z'} &= A_1 \cos(\mathbf{Q}\mathbf{R}_i). \end{aligned} \quad (2)$$

Here, the parameters A_1 and A_2 determine the relative amplitudes of the spin components along each direction. We found that $A_2 \approx A_1/\sqrt{3}$; the corresponding amplitude-squared ratio of 3 matches the intensity ratio of the two dominant peaks in Fig. 3(f), confirming that the out-of-plane $S^{y'}$ modulation is subdominant. The ratio corresponds to Phase I in Ref. [38].

IV. MAGNETIC STRUCTURE FACTOR

Now we calculate the magnetic structure factor

$$|\mathcal{F}_{\mathbf{q}}|^2 = \frac{1}{N} \sum_{\alpha=x,y,z} \left| \sum_i S_i^{\alpha} e^{-i\mathbf{q}\mathbf{R}_i} \right|^2, \quad (3)$$

where $\mathbf{R}_i$ is the coordinate of the i th spin. For a perfectly ordered state, the magnetic static structure factor exhibits sharp δ -function Bragg peaks at the ordering wave vectors precisely as shown in Figs. 3(d)–3(f). Panels (d) and (e) present the magnetic static structure factor for single- $\mathbf{Q}$ structures, while panel (f) shows the double- $\mathbf{Q}$ structure. In the latter case, the structure is characterized by two dominant peaks corresponding to the leading modulations discussed in the previous section, along with weaker subleading peaks of negligible intensity.

A single- $\mathbf{Q}$ helix is characterized by a single pair of Bragg peaks at $\pm\mathbf{Q}$. On the square lattice, however, the C_4 symmetry renders the two diagonal propagation vectors $\mathbf{Q} = (Q, Q)$ and $\mathbf{Q}' = (-Q, Q)$ equivalent. Consequently, a macroscopic sample generically fragments into domains

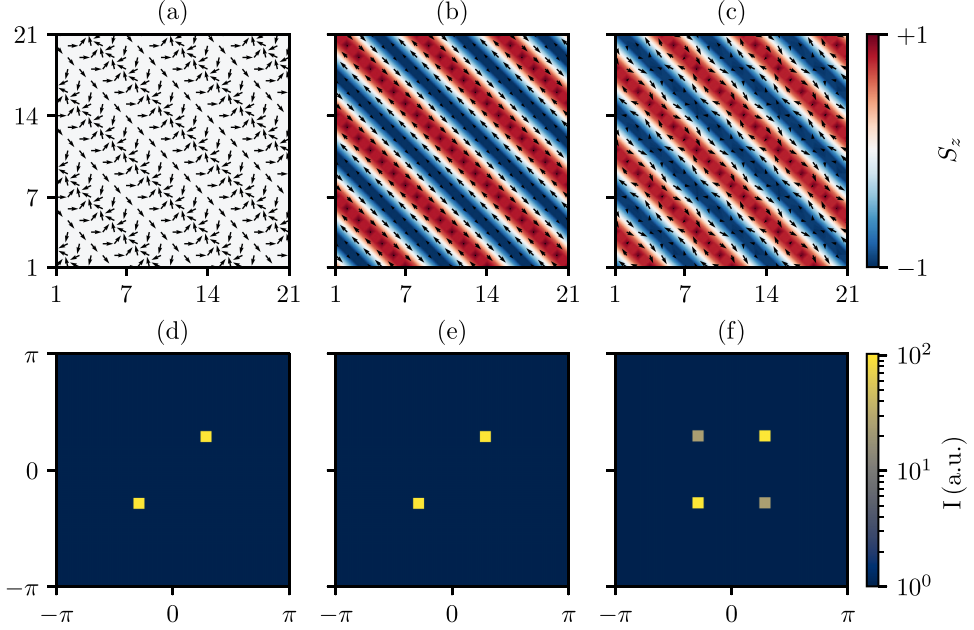


FIG. 3. Spin structures [panels (a)–(c)]. The in-plane components S^x and S^y are depicted as arrows and the out-of-plane component S^z is presented by color: (a) spiral, (b) helix, (c) double- $\mathbf{Q}$, and (d)–(f) corresponding magnetic structure factors. Parameters for the double- $\mathbf{Q}$ state: $J_1 = -7.2$ meV, $J_2 = 1.05$ meV, $J_3 = 2.1$ meV, $\Delta_2 = -0.36$ meV, and $\delta_1 = 0.072$ meV. Parameters for the spiral and helix states are given by $J_1 = -2J_2 \cos \theta - 4J_3 \cos \theta$, with $\theta = 2\pi/7$, $J_2 = 1.05$ meV, $J_3 = 2.1$ meV, $\Delta_{1,2,3} = 0$, and $\delta_{1,2,3} = 0$.

in which the helix propagates along either diagonal. Since elastic neutron scattering adds these domains incoherently, the static structure factor of such a multidomain single- $\mathbf{Q}$ state is the superposition of individual single-domain patterns and exhibits peaks at both $\mathbf{Q}$ and $\mathbf{Q}'$ at identical positions to those of the genuine double- $\mathbf{Q}$ order in Fig. 3(f). Moreover, the relative peak intensities are determined by the (uncontrolled) domain populations, rendering the intensity imbalance an unreliable fingerprint. The static structure factor alone thus cannot distinguish a single-domain double- $\mathbf{Q}$ state from a multidomain ensemble of single- $\mathbf{Q}$ domains. This ambiguity motivates the dynamical response analysis in the following sections.

V. MAGNON SPECTRUM

To find the excitations, the Holstein-Primakoff transformation is applied [50,51]. For this purpose, we introduce a local coordinate system such that the quantization axis is parallel to the classical spin $\mathbf{S}_i$. Correspondingly, the spin operator transforms $\hat{\mathbf{S}} \rightarrow \hat{\hat{\mathbf{S}}}$. For notational simplicity, we omit the operator symbol.

Then, we change the enumeration of the spins, introducing an internal index m , labeling the position of the spin inside the unit cell and an external index n , which runs over the unit cells. Then, the Holstein-Primakoff yields

$$\begin{aligned}\tilde{S}_{nm}^x &\rightarrow \frac{\sqrt{2S}}{2}(\hat{b}_{nm} + \hat{b}_{nm}^\dagger), \\ \tilde{S}_{nm}^y &\rightarrow \frac{\sqrt{2S}}{2i}(\hat{b}_{nm} - \hat{b}_{nm}^\dagger), \\ \tilde{S}_{nm}^z &\rightarrow S - \hat{b}_{nm}^\dagger \hat{b}_{nm}.\end{aligned}\tag{4}$$

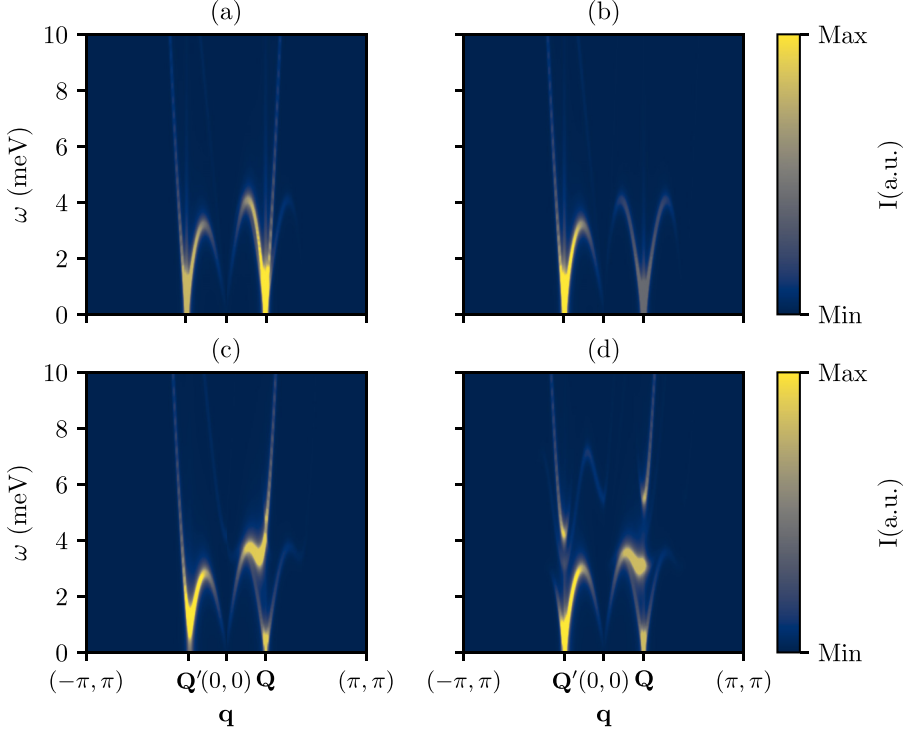


FIG. 4. Contour maps of the calculated inelastic neutron scattering cross section $I(\mathbf{q}, \omega)$ for the following magnetic structures: (a) helix, (b) spiral, (c) helix with anisotropy, and (d) double- $\mathbf{Q}$. The parameters are the same as in Fig. 3. For the helix with anisotropy, we use the same exchange parameters as for the double- $\mathbf{Q}$ state, while the anisotropy parameters are $\delta_1 = -0.16$ meV and $\Delta_2 = -0.16$ meV.

Noticing the translation invariance of the Hamiltonian, we apply the Bloch theorem, followed by the Bogoliubov transformation: $\hat{b}_{\mathbf{q}m} = \sum_{m'} U_{\mathbf{q},mm'} \hat{a}_{\mathbf{q}m'} + V_{\mathbf{q},mm'} \hat{a}_{-\mathbf{q}m'}^\dagger$ [49]. The matrices $U_{\mathbf{q},mm'}$ and $V_{\mathbf{q},mm'}$ are chosen so that the new operators $a_{\mathbf{q}m'}$ satisfy commutation relations for Bose particles.

Keeping only quadratic terms, we get the quadratic-in-operators Hamiltonian

$$H = \sum_{\mathbf{q}} \sum_m \varepsilon_{\mathbf{q}m} \hat{a}_{\mathbf{q}m}^\dagger \hat{a}_{\mathbf{q}m}.$$

VI. DYNAMICAL STRUCTURE FACTOR

Now, we address the central question of our work: the dynamical response of the magnetic structures. At $T = 0$, it can be obtained via the fluctuation-dissipation relation,

$$S_{\alpha\beta}(\mathbf{q}, \omega) = \frac{1}{\pi} \chi''_{\alpha\beta}(\mathbf{q}, \omega),$$

for $\omega > 0$. So, only the imaginary part of the susceptibility $\chi''_{\alpha\beta}(\mathbf{q}, \omega)$ enters the inelastic neutron cross section, which takes the form [52]

$$I(\mathbf{q}, \omega) \sim - \sum_{\alpha\beta} \left(\delta_{\alpha\beta} - \frac{q_\alpha q_\beta}{q^2} \right) \chi''_{\alpha\beta}(\mathbf{q}, \omega). \quad (5)$$

The results for $I(\mathbf{q}, \omega)$ in the unfolded Brillouin zone are presented in Fig. 4.

Figures 4(a) and 4(b) show the dynamical structure factor for single- $\mathbf{Q}$ states, the helix and the cycloidal spiral state. In both cases, the magnon dispersion $\omega(\mathbf{q})$ vanishes at Γ point and at the

propagation vector $\mathbf{Q}$. This behavior is consistent with the Goldstone theorem [53]. In addition, it vanishes at $\mathbf{Q}'$ due to the approximation of the linear spin waves of the $1/S$ expansion, used in these calculations. However, it is known that the gap at this point opens in higher orders of the $1/S$ expansion [54,55].

Figure 4(c) shows the helix state in the presence of anisotropy. Compared with the isotropic helix in Fig. 4(a), anisotropy lifts the degeneracy at $\mathbf{Q}$ and resolves the low-energy response into two branches. At $\mathbf{Q}'$, the overall structure remains similar to the isotropic case, including the pronounced outer vertical branch. But in contrast to the isotropic case, the anisotropy opens a finite gap of order ~ 1 meV. The gap also reflects the absence of the second ordering vector, which in the double- $\mathbf{Q}$ state protects the gapless phason mode. The spectrum therefore retains the qualitative single- $\mathbf{Q}$ character, but with anisotropy-induced mode splitting and gap opening.

Figure 4(d) displays the calculated inelastic neutron-scattering cross section of the double- $\mathbf{Q}$ structure. At the Γ point, one finds a Goldstone mode whose spectral weight in the inelastic neutron-scattering cross section vanishes, similarly to the helix case. In contrast to the isotropic helix, however, an additional rotonlike mode appears at the $\mathbf{Q}$ point beside the Goldstone mode. By analyzing the contributions to $\chi_{\alpha\beta}$, we find that the phason mode, the mode corresponding to in-plane phase shift of the magnetic helix, preserves its Goldstone nature and remains gapless. At the same time, the spin-tilting mode, corresponding to tilting of the spins away from the helix plane, becomes gapped. In addition, a similar gapped mode appears in the anisotropic helix state. However, in that case the pronounced outer vertical branch remains clearly visible. At the $\mathbf{Q}'$ point, the phason mode vanishes similar to helix. However, the spectral weight of the outer branch is reduced in comparison with the helix. The rotonlike mode is shifted to higher energies and becomes smeared out in comparison with the $\mathbf{Q}$ point. Because Figs. 4(c) and 4(d) use identical exchange constants and differ only in anisotropy and in the magnetic structure, the comparison isolates the structural effect: The loss of the outer vertical branch and the gapless phason are generic fingerprints of double- $\mathbf{Q}$ order, whereas the gap magnitude and branch splitting are specific to the chosen parameters. Overall, the calculated cross section looks much more similar to the experimental one for $\text{Sr}_3\text{Fe}_2\text{O}_7$ in the relevant low-energy regime.

VII. DISCUSSION AND CONCLUSION

In this paper, we study a double- $\mathbf{Q}$ structure on a square lattice, which represents the simplest model allowing single- $\mathbf{Q}$ and double- $\mathbf{Q}$ structures. The model corresponds to Sr_2FeO_4 , a system with a single magnetic layer and a noncollinear magnetic structure [56]. To date, however, only elastic neutron-scattering experiments on polycrystalline samples have been performed for this material. At the same time, the most important outcome of our analysis, namely, the prediction of a rotonlike mode in a double- $\mathbf{Q}$ state, suppression of outer vertical branch at $\mathbf{Q}'$, and Goldstone mode at $\mathbf{Q}'$, is also relevant to the bilayer compound $\text{Sr}_3\text{Fe}_2\text{O}_7$. However, because this system is bilayered, the q_z dependence of the dispersion becomes important. In particular, at point $(0, 0, \pi)$ the dispersion has a gap, in contrast to point $(0, 0, 0)$. Indeed, such a gap is observed experimentally [38], since the measurements were performed at momentum transfer (q, q, π) [57].

Overall, the calculated double- $\mathbf{Q}$ spectrum is in qualitative agreement with several characteristic features of the measured response of $\text{Sr}_3\text{Fe}_2\text{O}_7$ in the range from Γ to the $\mathbf{Q}'$ point (cf. Fig. S1 of Ref. [38]). A fully quantitative comparison would require a bilayer model including the q_z -dependent dispersion and the experimental resolution function, which is beyond the scope of this work. Since real crystals generically form multidomain configurations, we have further verified that these features persist after domain averaging. As shown in Fig. S2 of the Supplemental Material [49], calculations for two-domain magnetic structures demonstrate that the characteristic signatures remain clearly resolved. The main characteristic features of the single- $\mathbf{Q}$ and double- $\mathbf{Q}$ structures are summarized in Tables S1 and S2 of the Supplemental Material [49].

In summary, we have investigated inelastic neutron scattering cross section for a set of magnetic configurations that can arise in a two-dimensional square-lattice system. By constructing the phase

diagram as a function of the anisotropy parameters, we identified the regions of parameter space in which the different magnetic structures are stabilized. We then analyzed the corresponding dynamical structure factors. We have demonstrated that the characteristic feature of the double- $\mathbf{Q}$ structure in inelastic neutron scattering is a rotonlike magnon mode, suppression of outer vertical branch at $\mathbf{Q}'$, and Goldstone mode at $\mathbf{Q}'$. Recent findings by M. Mostovoy with collaborators [58] show similar results.

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

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- [1] Y. Tokura and S. Seki, Multiferroics with spiral spin orders, *Adv. Mater.* **22**, 1554 (2010).
 - [2] T. Kimura, Magnetoelectric hexaferrites, *Annu. Rev. Condens. Matter Phys.* **3**, 93 (2012).
 - [3] N. Nagaosa and Y. Tokura, Topological properties and dynamics of magnetic skyrmions, *Nat. Nanotechnol.* **8**, 899 (2013).
 - [4] W. Jiang, G. Chen, K. Liu, J. Zang, S. G. te Velthuis, and A. Hoffmann, Skyrmions in magnetic multilayers, *Phys. Rep.* **704**, 1 (2017).
 - [5] Y. Tokura and N. Kanazawa, Magnetic skyrmion materials, *Chem. Rev.* **121**, 2857 (2021).
 - [6] K. K. Mishra, A. H. Lone, S. Srinivasan, H. Fariborzi, and G. Setti, Magnetic skyrmion: From fundamental physics to pioneering applications, *Appl. Phys. Rev.* **12**, 011315 (2025).
 - [7] K. Hamamoto, M. Ezawa, and N. Nagaosa, Quantized topological Hall effect in skyrmion crystal, *Phys. Rev. B* **92**, 115417 (2015).
 - [8] B. Göbel, A. Mook, J. Henk, and I. Mertig, Unconventional topological Hall effect in skyrmion crystals caused by the topology of the lattice, *Phys. Rev. B* **95**, 094413 (2017).
 - [9] Z. Wang, Y. Su, S.-Z. Lin, and C. D. Batista, Skyrmion crystal from RKKY interaction mediated by 2D electron gas, *Phys. Rev. Lett.* **124**, 207201 (2020).
 - [10] T. Kurumaji, N. Paul, S. Fang, P. M. Neves, M. Kang, J. S. White, T. Nakajima, D. Graf, L. Ye, M. K. Chan, *et al.*, Electronic commensuration of a spin moiré superlattice in a layered magnetic semimetal, *Sci. Adv.* **11**, eadu6686 (2025).
 - [11] C. Back, V. Cros, H. Ebert, K. Everschor-Sitte, A. Fert, M. Garst, T. Ma, S. Mankovsky, T. L. Monchesky, M. Mostovoy, N. Nagaosa, S. S. P. Parkin, C. Pfleiderer, N. Reyren, A. Rosch, Y. Taguchi, Y. Tokura, K. von Bergmann, and J. Zang, The 2020 skyrmionics roadmap, *J. Phys. D: Appl. Phys.* **53**, 363001 (2020).
 - [12] K. Shimizu, S. Okumura, Y. Kato, and Y. Motome, Spin moiré engineering of topological magnetism and emergent electromagnetic fields, *Phys. Rev. B* **103**, 184421 (2021).
 - [13] I. Dzyaloshinsky, A thermodynamic theory of “weak” ferromagnetism of antiferromagnetics, *J. Phys. Chem. Solids* **4**, 241 (1958).
 - [14] T. Moriya, Anisotropic superexchange interaction and weak ferromagnetism, *Phys. Rev.* **120**, 91 (1960).
 - [15] U. K. Rößler, A. N. Bogdanov, and C. Pfleiderer, Spontaneous skyrmion ground states in magnetic metals, *Nature (London)* **442**, 797 (2006).
 - [16] S. Mühlbauer, B. Binz, F. Jonietz, C. Pfleiderer, A. Rosch, A. Neubauer, R. Georgii, and P. Böni, Skyrmion lattice in a chiral magnet, *Science* **323**, 915 (2009).
 - [17] J. Villain, La structure des substances magnetiques, *J. Phys. Chem. Solids* **11**, 303 (1959).

- [18] T. A. Kaplan, Classical spin-configuration stability in the presence of competing exchange forces, *Phys. Rev.* **116**, 888 (1959).
- [19] A. Yoshimori, A new type of antiferromagnetic structure in the rutile type crystal, *J. Phys. Soc. Jpn.* **14**, 807 (1959).
- [20] T. Okubo, S. Chung, and H. Kawamura, Multiple- q states and the skyrmion lattice of the triangular-lattice Heisenberg antiferromagnet under magnetic fields, *Phys. Rev. Lett.* **108**, 017206 (2012).
- [21] S. Hayami, R. Ozawa, and Y. Motome, Effective bilinear-biquadratic model for noncoplanar ordering in itinerant magnets, *Phys. Rev. B* **95**, 224424 (2017).
- [22] R. Ozawa, S. Hayami, K. Barros, G.-W. Chern, Y. Motome, and C. D. Batista, Vortex crystals with chiral stripes in itinerant magnets, *J. Phys. Soc. Jpn.* **85**, 103703 (2016).
- [23] A. O. Leonov and M. Mostovoy, Multiply periodic states and isolated skyrmions in an anisotropic frustrated magnet, *Nat. Commun.* **6**, 8275 (2015).
- [24] T. Kurumaji, T. Nakajima, M. Hirschberger, A. Kikkawa, Y. Yamasaki, H. Sagayama, H. Nakao, Y. Taguchi, T.-H. Arima, and Y. Tokura, Skyrmion lattice with a giant topological Hall effect in a frustrated triangular-lattice magnet, *Science* **365**, 914 (2019).
- [25] H. Zhang, Q. Huang, L. Hao, J. Yang, K. Noordhoek, S. Pandey, H. Zhou, and J. Liu, Anomalous magnetoresistance in centrosymmetric skyrmion-lattice magnet Gd_2PdSi_3 , *New J. Phys.* **22**, 083056 (2020).
- [26] S. Spachmann, A. Elghandour, M. Frontzek, W. Löser, and R. Klingeler, Magnetoelastic coupling and phases in the skyrmion lattice magnet Gd_2PdSi_3 discovered by high-resolution dilatometry, *Phys. Rev. B* **103**, 184424 (2021).
- [27] J. A. M. Paddison, B. K. Rai, A. F. May, S. Calder, M. B. Stone, M. D. Frontzek, and A. D. Christianson, Magnetic interactions of the centrosymmetric skyrmion material Gd_2PdSi_3 , *Phys. Rev. Lett.* **129**, 137202 (2022).
- [28] M. Gomišek, T. J. Hicken, M. N. Wilson, K. J. A. Franke, B. M. Huddart, A. Štefančič, S. J. R. Holt, G. Balakrishnan, D. A. Mayoh, M. T. Birch, S. H. Moody, H. Luetkens, Z. Guguchia, M. T. F. Telling, P. J. Baker, S. J. Clark, and T. Lancaster, Anisotropic skyrmion and multi- q spin dynamics in centrosymmetric Gd_2PdSi_3 , *Phys. Rev. Lett.* **134**, 046702 (2025).
- [29] N. D. Khanh, T. Nakajima, S. Hayami, S. Gao, Y. Yamasaki, H. Sagayama, H. Nakao, R. Takagi, Y. Motome, Y. Tokura, T.-H. Arima, and S. Seki, Zoology of multiple- Q spin textures in a centrosymmetric tetragonal magnet with itinerant electrons, *Adv. Sci.* **9**, 2105452 (2022).
- [30] G. D. A. Wood, D. D. Khalyavin, D. A. Mayoh, J. Bouaziz, A. E. Hall, S. J. R. Holt, F. Orlandi, P. Manuel, S. Blügel, J. B. Staunton, O. A. Petrenko, M. R. Lees, and G. Balakrishnan, Double- Q ground state with topological charge stripes in the centrosymmetric skyrmion candidate GdRu_2Si_2 , *Phys. Rev. B* **107**, L180402 (2023).
- [31] J. A. Paddison, J. Bouaziz, A. F. May, Q. Zhang, S. Calder, D. Abernathy, J. B. Staunton, S. Blügel, and A. D. Christianson, Spin dynamics of the centrosymmetric skyrmion material GdRu_2Si_2 , *Cell Rep. Phys. Sci.* **5**, 102280 (2024).
- [32] R. Takagi, N. Matsuyama, V. Ukleev, L. Yu, J. S. White, S. Francoual, J. R. L. Mardegan, S. Hayami, H. Saito, K. Kaneko, K. Ohishi, Y. Ōnuki, T.-H. Arima, Y. Tokura, T. Nakajima, and S. Seki, Square and rhombic lattices of magnetic skyrmions in a centrosymmetric binary compound, *Nat. Commun.* **13**, 1472 (2022).
- [33] M. Gen, R. Takagi, Y. Watanabe, S. Kitou, H. Sagayama, N. Matsuyama, Y. Kohama, A. Ikeda, Y. Ōnuki, T. Kurumaji, T.-H. Arima, and S. Seki, Rhombic skyrmion lattice coupled with orthorhombic structural distortion in EuAl_4 , *Phys. Rev. B* **107**, L020410 (2023).
- [34] A. N. Korshunov, A. S. Sukhanov, S. Gebel, M. S. Pavlovskii, N. D. Andriushin, Y. Gao, J. M. Moya, E. Morosan, and M. C. Rahn, Phonon softening and atomic modulations in EuAl_4 , *Phys. Rev. B* **110**, 045102 (2024).
- [35] J. M. Moya, S. Lei, E. M. Clements, C. S. Kengle, S. Sun, K. Allen, Q. Li, Y. Y. Peng, A. A. Husain, M. Mitrano, M. J. Krogstad, R. Osborn, A. B. Puthirath, S. Chi, L. Debeer-Schmitt, J. Gaudet, P. Abbamonte, J. W. Lynn, and E. Morosan, Incommensurate magnetic orders and topological Hall effect in the square-net centrosymmetric EuGa_2Al_2 system, *Phys. Rev. Mater.* **6**, 074201 (2022).

- [36] A. M. Vibhakar, D. D. Khalyavin, J. M. Moya, P. Manuel, F. Orlandi, S. Lei, E. Morosan, and A. Bombardi, Competing charge and magnetic order in the candidate centrosymmetric skyrmion host EuGa_2Al_2 , *Phys. Rev. B* **108**, L100404 (2023).
- [37] J.-H. Kim, A. Jain, M. Reehuis, G. Khaliullin, D. C. Peets, C. Ulrich, J. T. Park, E. Faulhaber, A. Hoser, H. C. Walker, D. T. Adroja, A. C. Walters, D. S. Inosov, A. Maljuk, and B. Keimer, Competing exchange interactions on the verge of a metal-insulator transition in the two-dimensional spiral magnet $\text{Sr}_3\text{Fe}_2\text{O}_7$, *Phys. Rev. Lett.* **113**, 147206 (2014).
- [38] N. D. Andriushin, J. Muller, N. S. Pavlovskii, J. Grumbach, S. Granovsky, Y. V. Tymoshenko, O. Zaharko, A. Ivanov, J. Ollivier, M. Doerr, B. Keimer, M. Mostovoy, D. S. Inosov, and D. C. Peets, Reentrant multiple- $\mathbf{Q}$ magnetic order and a “spin meta-cholesteric” phase in $\text{Sr}_3\text{Fe}_2\text{O}_7$, *npj Quantum Mater.* **9**, 84 (2024).
- [39] S. Ishiwata, T. Nakajima, J.-H. Kim, D. S. Inosov, N. Kanazawa, J. S. White, J. L. Gavilano, R. Georgii, K. M. Seemann, G. Brandl, *et al.*, Emergent topological spin structures in the centrosymmetric cubic perovskite SrFeO_3 , *Phys. Rev. B* **101**, 134406 (2020).
- [40] N. D. Andriushin, J. Grumbach, A. A. Kulbakov, Y. V. Tymoshenko, Y. A. Onykienko, R. Firouzmandi, E. Cheng, S. Granovsky, Y. Skourski, J. Ollivier, H. C. Walker, V. Kocsis, B. Büchner, B. Keimer, M. Doerr, D. S. Inosov, and D. C. Peets, Anomalous quasielastic scattering contribution in the centrosymmetric multi- $\mathbf{Q}$ helimagnet SrFeO_3 , *Phys. Rev. X* **15**, 011038 (2025).
- [41] D. C. Peets, J.-H. Kim, P. Dosanjh, M. Reehuis, A. Maljuk, N. Aliouane, C. Ulrich, and B. Keimer, Magnetic phase diagram of $\text{Sr}_3\text{Fe}_2\text{O}_{7-\delta}$, *Phys. Rev. B* **87**, 214410 (2013).
- [42] J. Jensen and P. Bak, Spin waves in triple- $\vec{q}$ structures. Application to USb, *Phys. Rev. B* **23**, 6180(R) (1981).
- [43] J. A. M. Paddison, H. Zhang, J. Yan, M. J. Cliffe, M. A. McGuire, S.-H. Do, S. Gao, M. B. Stone, D. Dahlbom, K. Barros, C. D. Batista, and A. D. Christianson, Cubic double perovskites host noncoplanar spin textures, *npj Quantum Mater.* **9**, 48 (2024).
- [44] Y. Kato, S. Hayami, and Y. Motome, Spin excitation spectra in helimagnetic states: Proper-screw, cycloid, vortex-crystal, and hedgehog lattices, *Phys. Rev. B* **104**, 224405 (2021).
- [45] P. Park, W. Cho, C. Kim, Y. An, K. Iida, R. Kajimoto, S. Matin, S.-S. Zhang, C. D. Batista, and J.-G. Park, Spin dynamics of triple- $\mathbf{Q}$ magnetic orderings in a triangular lattice: Implications for multi- $\mathbf{Q}$ orderings in general two-dimensional lattices, *Phys. Rev. X* **15**, 031032 (2025).
- [46] The point $J_1 = J_3 = 0$ is a singular limit of the model. In this case, the system decomposes into two independent sublattices coupled only through J_2 , and the classical ground state is correspondingly degenerate, as the referee correctly notes. The stripe region in the phase diagram should therefore not be interpreted as representing a uniquely defined ordered state at this isolated point.
- [47] W. Janke, Monte Carlo simulations in statistical physics—from basic principles to advanced applications, in *Order, Disorder and Criticality: Advanced Problems of Phase Transition Theory*, edited by Y. Holovatch (World Scientific Publishing Co. Pte. Ltd., Singapore, 2012), Vol. 3, pp. 93–166.
- [48] J. Nocedal and S. J. Wright, *Numerical Optimization* (Springer, New York, 2006), pp. 164–192.
- [49] See Supplemental Material at <http://link.aps.org/supplemental/10.1103/bdxw-7tdf> for finite-size tests, details of the spin-wave and unfolding procedures, and the effects of magnon damping and multidomain configurations.
- [50] T. Holstein and H. Primakoff, Field dependence of the intrinsic domain magnetization of a ferromagnet, *Phys. Rev.* **58**, 1098 (1940).
- [51] D. C. Mattis, *The Theory of Magnetism I: Statics and Dynamics* (Springer, Berlin, 1981), pp. 138–206.
- [52] G. L. Squires, *Introduction to the Theory of Thermal Neutron Scattering* (Cambridge University Press, Cambridge, UK, 2012), pp. 129–145.
- [53] J. Goldstone, Field theories with superconductor solutions, *Nuovo Cimento* **19**, 154 (1961).
- [54] A. V. Chubukov, S. Sachdev, and T. Senthil, Large- S expansion for quantum antiferromagnets on a triangular lattice, *J. Phys.: Condens. Matter* **6**, 8891 (1994).
- [55] Y. V. Tymoshenko, Y. A. Onykienko, T. Müller, R. Thomale, S. Rachel, A. S. Cameron, P. Y. Portnichenko, D. V. Efremov, V. Tsurkan, D. L. Abernathy, J. Ollivier, A. Schneidewind, A. Piovano,

- V. Felea, A. Loidl, and D. S. Inosov, Pseudo-Goldstone magnons in the frustrated $S = 3/2$ Heisenberg helimagnet ZnCr_2Se_4 with a pyrochlore magnetic sublattice, [Phys. Rev. X **7**, 041049 \(2017\)](#).
- [56] P. Adler, M. Reehuis, N. Stüßer, S. A. Medvedev, M. Nicklas, D. C. Peets, J. Bertinshaw, C. K. Christensen, M. Etter, A. Hoser, L. Schröder, P. Merz, W. Schnelle, A. Schulz, Q. Mu, D. Bessas, A. Chumakov, M. Jansen, and C. Felser, Spiral magnetism, spin flop, and pressure-induced ferromagnetism in the negative charge-transfer-gap insulator Sr_2FeO_4 , [Phys. Rev. B **105**, 054417 \(2022\)](#).
- [57] Note that in the experiment in Ref. [38], the direction of the main propagation vector $\mathbf{Q}$ is along $(-\xi, \xi, 5)$ in contrast to our single-layer convention $\mathbf{Q} \parallel (\xi, \xi)$.
- [58] M. Mostovoy (private communication).