

Thermalization-induced entropy-rate representation of internal energy in a damped quantum oscillator

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We study a harmonic oscillator weakly coupled to a thermal bath and evolving under the standard quantum-optical master equation. For zero-mean Gaussian states, entropy alone does not determine the internal energy because the latter also depends on nonequilibrium Gaussian structure. We show, however, that thermal relaxation yields the exact representation $E = E(S, \dot{S})$ for every mixed Gaussian state evolving under this thermal quantum-optical master equation. Explicitly, $E(S, \dot{S}) = \frac{\omega v(S)}{\nu_{\text{th}}} [\nu(S) + \frac{\dot{S}}{\gamma S'(v(S))}]$. Once the thermal Gorini–Kossakowski–Lindblad–Sudarshan equation is assumed, no further expansion in the damping rate, entropy rate, or distance from equilibrium is made. The entropy rate is not an alternative instantaneous state coordinate: It depends on the open-system dynamics and therefore carries information about the coupling to the bath. This is exposed by the isolated limit, where $\dot{S} = 0$ for all Gaussian states and the representation no longer determines the energy. The result thus identifies an exact thermalization-induced dynamical energetic relation, distinct from rate dependence generated by an external driving protocol. We also describe an operational test based on time-resolved purity and energy measurements. At fixed frequency, the bath only relaxes preexisting squeezing. We therefore also consider frequency modulation, which can drive an initially thermal state away from the instantaneous Gibbs family by generating off-Gibbs Gaussian structure. Within a local Gaussian continuation, the bath then relaxes this driven deformation, providing a dynamical realization of the entropy-rate dependence.

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I. INTRODUCTION

Equilibrium thermodynamics represents the internal energy in terms of instantaneous state variables, for example, $E = E(S, \lambda)$, with S the entropy and λ externally controlled parameters. Away from equilibrium, additional state variables are often introduced, while rate-dependent quantities also arise in finite-time and irreversible thermodynamics [1–6]. For open quantum systems, related questions are studied in quantum and stochastic thermodynamics [7–9]. Here, we ask a narrower question: Can ordinary thermal relaxation itself yield an exact reduced energetic representation involving the rate of a thermodynamic variable?

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We address this question in the simplest standard setting, a harmonic oscillator weakly coupled to a thermal bath. For a zero-mean Gaussian state, the entropy is determined by the symplectic eigenvalue of the covariance matrix, whereas the energy also depends on nonequilibrium Gaussian structure. Entropy alone therefore does not determine the energy away from the Gibbs family.

For the thermal quantum-optical master equation, the covariance dynamics provides an additional relation between the energy-carrying second moment and the entropy rate. Eliminating that moment gives the exact reduced relation

$$E = E(S, \dot{S}), \quad (1)$$

valid for every mixed Gaussian state and every nonzero damping rate compatible with the Born-Markov-secular regime. No expansion in the damping rate, in $\dot{S}$, or in the distance from equilibrium is required.

The meaning of Eq. (1) is dynamical rather than kinematic. The quantity $\dot{S}$ is not an additional instantaneous coordinate of the Gaussian state: Its value is fixed only after the open-system generator has been specified. Thus, the representation $E(S, \dot{S})$ contains information about thermalization that is absent from the instantaneous entropy alone. This distinction becomes sharp in the isolated limit. When the damping vanishes, $\dot{S} = 0$ for every Gaussian state although equal-entropy states can still have different energies, and the entropy-rate representation ceases to close. The derivative dependence is therefore induced by the bath coupling rather than by the speed of an externally imposed protocol.

The paper is organized as follows. Section II specifies the model and assumptions. We derive this relation directly from the exact second-moment dynamics in Sec. III, identify its physically accessible domain, and give an operational test based on time-resolved measurements of purity and mean energy, and a time-dependent bath temperature in Sec. IV. We then turn to a driven oscillator, where frequency modulation provides a physical mechanism for generating the off-Gibbs Gaussian structure whose subsequent relaxation is encoded in the entropy rate in Sec. V. The driven construction is formulated as a local Gaussian continuation of the exactly controlled fixed-frequency problem, with its additional assumptions and regime of validity stated explicitly. Technical details are deferred to the Appendixes.

II. MODEL, ASSUMPTIONS, AND NOTATION

We set $\hbar = k_B = m = 1$. For an oscillator of fixed frequency ω , we use the dimensionless quadratures

$$X = \sqrt{\omega} x, \quad P = \frac{p}{\sqrt{\omega}}, \quad [X, P] = i, \quad (2)$$

for which

$$H = \frac{\omega}{2}(X^2 + P^2). \quad (3)$$

We restrict attention to Gaussian states with vanishing first moments. Such a state is specified by the real symmetric covariance matrix

$$\sigma_{ij} = \frac{1}{2} \langle \{R_i, R_j\} \rangle, \quad R = (X, P). \quad (4)$$

Of its three independent components, two rotational invariants are sufficient for the energetic and entropic discussion of the isotropic generator used below [10, 11]:

$$s = \text{tr } \sigma, \quad v = \sqrt{\det \sigma}. \quad (5)$$

The remaining parameter is the orientation of the covariance ellipse. It enters neither the energy nor the entropy and, at constant ω , rotates independently; Appendix A gives the component evolution explicitly.

It is convenient to introduce the anisotropy, or squeezing magnitude,

$$r^2 = \frac{1}{4}(\sigma_{11} - \sigma_{22})^2 + \sigma_{12}^2 = \frac{s^2}{4} - \nu^2, \quad \nu \geq \frac{1}{2}. \quad (6)$$

The variable r is not independent of (s, ν) . It is useful because $r = 0$ is equivalent to $\sigma = \nu \mathbb{I}$, namely, an unsqueezed Gibbs state of the fixed oscillator Hamiltonian at some temperature.

The energy and von Neumann entropy are

$$E = \langle H \rangle = \frac{\omega s}{2}, \quad (7)$$

$$S(\nu) = \left(\nu + \frac{1}{2}\right) \ln \left(\nu + \frac{1}{2}\right) - \left(\nu - \frac{1}{2}\right) \ln \left(\nu - \frac{1}{2}\right), \quad (8)$$

with

$$S'(\nu) = \ln \frac{\nu + \frac{1}{2}}{\nu - \frac{1}{2}} > 0 \quad (\nu > 1/2). \quad (9)$$

At fixed ν ,

$$s = 2\sqrt{\nu^2 + r^2}, \quad E = \omega\sqrt{\nu^2 + r^2}, \quad (10)$$

so equal entropy does not imply equal energy on the full Gaussian state space.

The reference dynamics is the standard thermal quantum-optical master equation [12–15],

$$\dot{\rho} = -i[H, \rho] + \gamma(N + 1)\mathcal{D}[a]\rho + \gamma N\mathcal{D}[a^\dagger]\rho, \quad (11)$$

where

$$\mathcal{D}[c]\rho = c\rho c^\dagger - \frac{1}{2}\{c^\dagger c, \rho\}, \quad a = \frac{X + iP}{\sqrt{2}}, \quad (12)$$

and

$$N = \frac{1}{e^{\omega/T_{\text{bath}}} - 1}, \quad \nu_{\text{th}} = N + \frac{1}{2} = \frac{1}{2} \coth \frac{\omega}{2T_{\text{bath}}}. \quad (13)$$

We assume weak system-bath coupling, an initially factorized state, and the Born-Markov and secular approximations. In particular,

$$\gamma \ll \omega, \quad \tau_B \ll \gamma^{-1}, \quad (14)$$

where τ_B is the bath correlation time. Equation (11) preserves Gaussianity and vanishing first moments. The benchmark results through Sec. IV E assume constant ω , γ , and T_{bath} . Section IV F then relaxes the latter two assumptions while keeping ω fixed. This deliberate specialization removes any external protocol from the main result.

III. EXACT THERMALIZATION OF GAUSSIAN SECOND MOMENTS

Equation (11) gives the Lyapunov equation

$$\dot{\sigma} = \omega(J\sigma - \sigma J) - \gamma(\sigma - \nu_{\text{th}}\mathbb{I}), \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (15)$$

The Hamiltonian term rotates the covariance ellipse, while the dissipator drives it toward $\nu_{\text{th}}\mathbb{I}$. Taking the trace gives

$$\dot{s} = -\gamma(s - 2\nu_{\text{th}}). \quad (16)$$

The Hamiltonian rotation preserves $\det \sigma$. Using $d(\det \sigma)/dt = \det \sigma \operatorname{tr}(\sigma^{-1} \dot{\sigma})$ and $\operatorname{tr} \sigma^{-1} = s/v^2$ for a 2×2 matrix, one obtains

$$\dot{v} = \gamma \left(\frac{v_{\text{th}} s}{2v} - v \right). \quad (17)$$

Finally, differentiating Eq. (6) gives

$$\frac{d}{dt} r^2 = -2\gamma r^2. \quad (18)$$

Hence,

$$s(t) = 2v_{\text{th}} + [s(0) - 2v_{\text{th}}]e^{-\gamma t}, \quad (19)$$

$$r^2(t) = r^2(0)e^{-2\gamma t}, \quad r(t) = r(0)e^{-\gamma t}. \quad (20)$$

The covariance orientation rotates at angular velocity ω (the two anisotropy components rotate at 2ω), while its magnitude r decays with the damping envelope. A component derivation and the exact matrix solution are given in Appendix A.

Equations (16) and (18) show the simple structure behind the result. In the variables (s, r^2) , the energy-carrying trace and the squared squeezing relax independently:

$$s - 2v_{\text{th}} \propto e^{-\gamma t}, \quad r^2 \propto e^{-2\gamma t}. \quad (21)$$

The entropy variable

$$v = \sqrt{\frac{s^2}{4} - r^2} \quad (22)$$

mixes these two sectors. This mixing prevents E from being a function of S alone. In the open system, however, the same covariance dynamics also determines the entropy rate. Once the thermal generator is specified, Eq. (17) therefore provides an additional dynamical relation that allows the energy-carrying trace s to be eliminated in favor of v and $\dot{v}$. This is the origin of the representation $E = E(S, \dot{S})$ derived below.

IV. EXACT RATE-DEPENDENT ENERGETIC CLOSURE

A. Derivation of $E(S, \dot{S})$

Solving Eq. (17) for s gives

$$s = \frac{2v}{v_{\text{th}}} \left(v + \frac{\dot{v}}{\gamma} \right), \quad \gamma > 0. \quad (23)$$

Consequently,

$$E(v, \dot{v}) = \frac{\omega v}{v_{\text{th}}} \left(v + \frac{\dot{v}}{\gamma} \right). \quad (24)$$

For mixed states, $v > 1/2$, the entropy is invertible and $\dot{S} = S'(v)\dot{v}$, so

$$E(S, \dot{S}) = \frac{\omega v(S)}{v_{\text{th}}} \left[v(S) + \frac{\dot{S}}{\gamma S'(v(S))} \right]. \quad (25)$$

No expansion in γ , in the rate, or in the distance from equilibrium has been made. Equation (25) is an exact identity on the mixed-state interior for Gaussian states evolving under Eq. (11). At fixed entropy, the energy is affine in the entropy rate,

$$\left(\frac{\partial E}{\partial \dot{S}} \right)_S = \frac{\omega v}{v_{\text{th}} \gamma S'(v)} > 0. \quad (26)$$

The pure-state boundary is better described by the regular variable $q = v^2$ introduced below (see Appendix B).

B. Physical domain and contour plot

The whole $(S, \dot{S})$ plane is not physically accessible. From $r^2 \geq 0$ and Eq. (6), one has $s \geq 2v$, which through Eq. (23) becomes

$$\dot{S} \geq \gamma S'(\nu)(\nu_{\text{th}} - \nu). \quad (27)$$

This is an exact lower boundary, not a compact bound. Equality is attained precisely for $r = 0$, i.e., by the one-parameter family

$$\mathcal{M}_G = \{\sigma = \nu \mathbb{I} \mid \nu \geq 1/2\}. \quad (28)$$

Every point of $\mathcal{M}_G$ is a Gibbs state of the same fixed Hamiltonian at some temperature. We refer to this one-parameter set as the Gibbs family. The bath-selected equilibrium state is the single point $\nu = \nu_{\text{th}}$ on this manifold.

The eliminated variable can be reconstructed explicitly. Combining Eqs. (23) and (6),

$$r^2 = v^2 \left[\left(\frac{\nu + \dot{S}/[\gamma S'(\nu)]}{\nu_{\text{th}}} \right)^2 - 1 \right]. \quad (29)$$

Equation (29) should be understood conditionally on the thermal generator. Once γ and ν_{th} are specified, a measured pair $(S, \dot{S})$ determines the value of r^2 that is compatible with that generator. Indeed, for fixed S and fixed generator, as shown in Appendix B,

$$\frac{\partial \dot{S}}{\partial (r^2)} = \frac{\gamma \nu_{\text{th}} S'(\nu)}{2\nu\sqrt{\nu^2 + r^2}} > 0 \quad (\gamma > 0, \nu > 1/2), \quad (30)$$

so this inference is unique.

This does not define an intrinsic change of coordinates on the Gaussian state space. The quantity r^2 is determined by the instantaneous state, whereas $\dot{S}$ depends additionally on the generator. Changing the bath coupling changes $\dot{S}$ while leaving the instantaneous covariance matrix unchanged. Equation (29) is therefore a generator-dependent dynamical inference, not a state-space reparametrization.

The corresponding static description in the (S, r^2) plane is straightforward. At fixed entropy, the squeezing raises the energy according to $E = \omega\sqrt{\nu(S)^2 + r^2}$ with $r^2 \geq 0$. Thus, the familiar Gaussian-state description uses the entropy together with an explicit internal coordinate, r^2 .

Figure 1 compares these two descriptions directly. Panel (a) shows the static (S, r^2) representation, where the physical boundary is simply $r^2 = 0$. Panel (b) should not be interpreted as a second coordinate chart on the same instantaneous state space. It shows instead the entropy rates assigned to Gaussian states by the specified thermal generator. The dependence on γ is essential: The same instantaneous state would occupy a different point in the $(S, \dot{S})$ plane under a different generator.

Under the relaxation map, the straight boundary $r^2 = 0$ is carried into

$$\dot{S} = \gamma S'(\nu)(\nu_{\text{th}} - \nu), \quad (31)$$

which is the curved lower boundary of the physically accessible region in the $(S, \dot{S})$ plane. The two marked states in Fig. 1 make the correspondence explicit. They have the same symplectic eigenvalue, and hence the same entropy, but different squeezing and therefore different energies. Panel (a) shows that the states differ in their instantaneous anisotropy, whereas panel (b) shows that the specified thermal generator assigns different entropy rates to those states.

The horizontal line $\dot{S} = 0$ is not an equilibrium manifold; equilibrium is the single intersection of $\dot{S} = 0$ with the Gibbs-family boundary at $\nu = \nu_{\text{th}}$. The figure also includes a region with $\dot{S} < 0$.

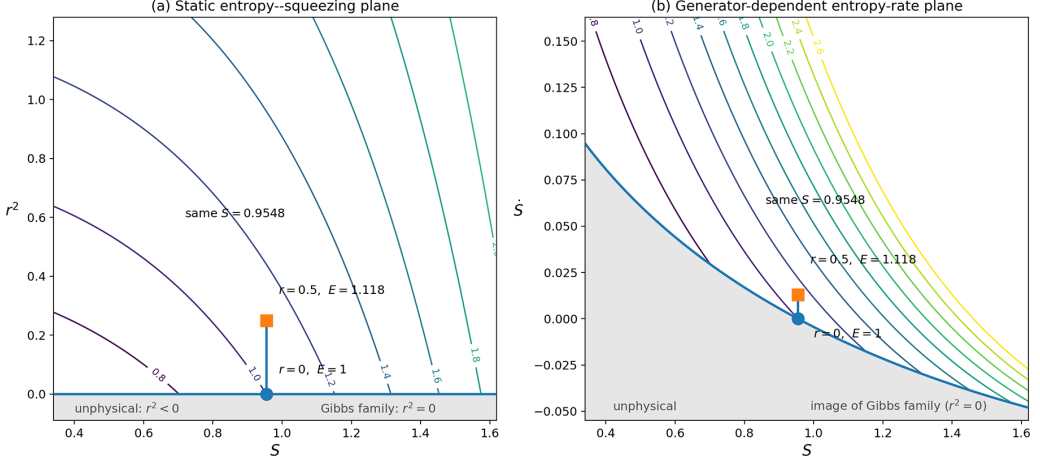


FIG. 1. Static state variables and the generator-dependent entropy-rate representation for $\omega = 1$, $\gamma = 0.1$, and $v_{\text{th}} = 1$. (a) In the instantaneous Gaussian-state description, the physical domain is $r^2 \geq 0$, with the Gibbs family at $r^2 = 0$. (b) For the specified thermal GKLS generator, each Gaussian state acquires an entropy rate according to Eq. (17), producing the allowed region in the $(S, \dot{S})$ plane. Its lower boundary corresponds to unsqueezed states and is $\dot{S} = \gamma S'(\nu)(v_{\text{th}} - \nu)$. The marked initial states have the same entropy, $S = 0.9548$, but different squeezing. Under the specified bath dynamics, they consequently have different initial entropy rates and different energies.

This does not violate the second law: $\dot{S}$ is the oscillator entropy rate, not the total entropy production. Appendix C makes this distinction explicit using Spohn's inequality.

For the parameters shown, the two marked states have $\nu = 1$ with $r = 0$ and $r = 0.5$; they share the entropy $S = 0.9548$ and correspond, respectively, to $\dot{S} = 0$ and $\dot{S} \simeq 1.30 \times 10^{-2}$. The point is that, once the thermal generator is specified, the entropy rate provides the dynamical relation needed to determine the energy without independently measuring the squeezing.

C. Why thermalization is essential

The role of thermalization can be seen most directly by comparing the open and isolated oscillators. For the thermal Gorini–Kossakowski–Lindblad–Sudarshan (GKLS) dynamics, Eq. (17) gives

$$\dot{\nu} = \gamma \left(\frac{\nu_{\text{th}} s}{2\nu} - \nu \right), \quad (32)$$

and therefore, for every specified generator with $\gamma > 0$, the energy-carrying variable s can be eliminated algebraically to get Eq. (23). This is the dynamical input behind Eq. (25).

It is important that $\dot{S}$ in this relation is not an additional instantaneous coordinate of the Gaussian state. Its value depends on both the state and the open-system generator. Using $s = 2\sqrt{\nu^2 + r^2}$, for example,

$$\dot{S} = \gamma S'(\nu) \left[\frac{\nu_{\text{th}}}{\nu} \sqrt{\nu^2 + r^2} - \nu \right]. \quad (33)$$

Thus, the same instantaneous Gaussian state would in general have a different entropy rate if the damping rate or the bath temperature were changed. The representation $E(S, \dot{S})$ is therefore not a reparametrization of the instantaneous Gaussian state space. Rather, it is a relation between the energy and trajectory-level data generated by a specified thermalizing dynamics.

The distinction becomes sharp in the isolated limit. Setting $\gamma = 0$ in Eqs. (16)–(18) gives $\dot{s} = \dot{r} = \dot{v} = 0$ and hence

$$\dot{S} = 0 \quad (34)$$

for every Gaussian state. Nevertheless, states with the same v , and therefore the same entropy, can have different squeezing r and hence different energies,

$$E = \omega\sqrt{v^2 + r^2}. \quad (35)$$

Consequently, the pair $(S, \dot{S})$ cannot determine the energy of the isolated oscillator: All equal-entropy states have the same $\dot{S} = 0$ regardless of their energy.

For any nonzero damping rate within the regime of validity of the thermal master equation, Eq. (32) supplies precisely the additional dynamical relation needed to eliminate s and obtain Eq. (25). The limit $\gamma \rightarrow 0^+$ should therefore be distinguished from the exactly isolated case. Along a thermalizing trajectory $\dot{v} = O(\gamma)$, so the ratio $\dot{v}/\gamma$ in Eq. (23) may remain finite for arbitrarily weak but nonzero coupling. At $\gamma = 0$ itself, however, the dynamical relation disappears and the entropy-rate closure is lost.

In this precise sense, the derivative dependence in

$$E = E(S, \dot{S}) \quad (36)$$

is induced by thermalization. It does not arise from the speed of an externally imposed protocol: In the benchmark considered here the Hamiltonian frequency is constant and no external driving is present. Instead, the bath provides a dynamical relation between the instantaneous state and its entropy rate, and it is this relation that makes the reduced energetic representation possible.

D. Closed trajectory equation for an entropy-related variable

For a specified thermal generator, the trajectory-level closure admits a particularly simple form in terms of $q = v^2$. The variable q is a monotonic reparametrization of the entropy in the mixed-state interior and remains regular at the pure-state boundary. Since $\dot{q} = 2v\dot{v}$, Eq. (24) becomes

$$E(q, \dot{q}) = \frac{\omega}{v_{\text{th}}} \left(q + \frac{\dot{q}}{2\gamma} \right). \quad (37)$$

At the same time, Eq. (16) implies

$$\dot{E} = -\gamma(E - \omega v_{\text{th}}). \quad (38)$$

Eliminating E between Eqs. (37) and (38) gives

$$\ddot{q} + 3\gamma\dot{q} + 2\gamma^2(q - v_{\text{th}}^2) = 0. \quad (39)$$

Thus, the original first-order reduced dynamics of two Gaussian state variables has been rewritten as a second-order equation for one entropy-related coordinate.

The origin of Eq. (39) is transparent once the two relaxation channels are separated. Equations (19) and (20) give $s - 2v_{\text{th}} \propto e^{-\gamma t}$ and $r^2 \propto e^{-2\gamma t}$, and $q = s^2/4 - r^2$ is quadratic in the first and linear in the second. Squaring therefore produces no exponent other than those already present, so that

$$q(t) - v_{\text{th}}^2 = C_1 e^{-\gamma t} + C_2 e^{-2\gamma t}, \quad (40)$$

with

$$C_1 = v_{\text{th}}[s(0) - 2v_{\text{th}}], \quad C_2 = \frac{1}{4}[s(0) - 2v_{\text{th}}]^2 - r^2(0). \quad (41)$$

Equation (39) is thus nothing but $(d_t + \gamma)(d_t + 2\gamma)[q - v_{\text{th}}^2] = 0$. What is not automatic is that a quantity built from a square root of a determinant should be an exact affine combination of the two

relaxation modes; this is what makes the second-order equation linear and closed. For a specified thermal generator, admissible initial data $(q(0), \dot{q}(0))$ determine a unique trajectory of Eq. (39). These data should not be interpreted as intrinsic coordinates of the instantaneous Gaussian state: $\dot{q}(0)$ depends on the generator as well as on that state. The physical-domain condition derived in Appendix B specifies which initial position-velocity pairs are compatible with Gaussian states under the chosen generator.

Since $q = q(S)$, Eq. (39) can also be written directly as an entropy equation,

$$q'(S)\ddot{S} + q''(S)\dot{S}^2 + 3\gamma q'(S)\dot{S} + 2\gamma^2[q(S) - v_{\text{th}}^2] = 0. \quad (42)$$

In this precise sense, the entropy trajectory can be described by a closed second-order equation once the generator is fixed: S specifies the coordinate and $\dot{S}$ supplies the second datum needed to determine its trajectory. We note, however, that the coordinate in which the reduced dynamics is linear is $q = v^2$ and not S itself; Eq. (42) is nonlinear because $q(S)$ is. The variable in which the closed trajectory equation becomes linear is therefore $q = v^2$, rather than S itself.

There is a second reason for preferring the derivative representation. The squeezing variable r^2 is specific to the single-mode Gaussian oscillator, and a different open system may require a different internal variable to complete its instantaneous energetic description. The elimination mechanism itself is more general. Suppose that a state coordinate Y and an additional internal variable Z determine the energy according to

$$E = \Phi(Y, Z), \quad (43)$$

while the relaxation dynamics gives

$$\dot{Y} = \mathcal{R}(Y, Z). \quad (44)$$

Whenever this relation can be inverted locally for Z , the internal variable may be eliminated,

$$Z = Z(Y, \dot{Y}), \quad E = \tilde{E}(Y, \dot{Y}). \quad (45)$$

For the oscillator studied here, Y may be taken as the entropy-related coordinate and Z as the squeezing information. Thus the significance of the $(S, \dot{S})$ representation is not that $\dot{S}$ is universally preferable to an internal variable, but that a specified relaxation law can make the internal variable inferable from state-and-rate data.

E. Operational content

The generator dependence of $\dot{S}$ gives Eq. (25) direct operational content. The test does not require reconstructing the squeezing variable r^2 . Instead, the mean energy and the state purity can be measured independently as functions of time and their trajectories compared with the relation predicted by the thermal GKLS generator.

For a one-mode Gaussian state, the purity is

$$\mathcal{P} := \text{tr } \rho^2 = \frac{1}{2\nu}, \quad (46)$$

so that a direct purity measurement determines ν , and hence the entropy, without resolving the orientation or anisotropy of the covariance matrix. The energy is independently given by

$$E = \omega \left(\bar{n} + \frac{1}{2} \right) = \frac{\omega S}{2}. \quad (47)$$

Thus the test requires scalar energetic and purity information rather than full covariance tomography. The price is that the purity must be measured as a time-resolved quantity.

A particularly useful form of the prediction avoids numerical differentiation altogether. Introducing

$$q := v^2, \quad (48)$$

Equation (17) gives

$$\dot{q} + 2\gamma q = \gamma v_{\text{th}} s = \frac{2\gamma v_{\text{th}}}{\omega} E(t). \quad (49)$$

Hence, for a specified thermal generator,

$$q(t) = e^{-2\gamma(t-t_0)} q(t_0) + \frac{2\gamma v_{\text{th}}}{\omega} \int_{t_0}^t e^{-2\gamma(t-\tau)} E(\tau) d\tau. \quad (50)$$

Since $q = 1/(4\mathcal{P}^2)$, the measured energy trajectory together with the single initial purity $\mathcal{P}(t_0)$ predicts the complete purity, and therefore entropy, trajectory.

In practice, γ and v_{th} may be determined independently from the bath temperature and an energy-relaxation calibration. For a nonstationary energy trace, they may also be obtained from a fit to

$$E(t) = \omega v_{\text{th}} + [E(t_0) - \omega v_{\text{th}}] e^{-\gamma(t-t_0)}. \quad (51)$$

Once these generator parameters and the initial purity are fixed, Eq. (50) contains no additional dynamical fit parameters. The measured purity trajectory therefore provides an independent test of the thermal Gaussian relaxation law.

The importance of this test is that the energy relaxation alone does not determine the full dissipative dynamics. To see this, replace the isotropic thermal diffusion matrix by

$$D = \gamma \text{diag}(d_1, d_2), \quad \bar{d} = \frac{d_1 + d_2}{2}. \quad (52)$$

For two reservoirs with the same damping rate and the same trace of D , the trace equation is identical,

$$\dot{s} = -\gamma(s - 2\bar{d}), \quad (53)$$

and therefore the same initial state produces the same energy trajectory. The symplectic eigenvalue, however, obeys

$$\frac{dq}{dt} + 2\gamma q - \gamma \bar{d} s = \gamma(d_2 - d_1)u, \quad u = \frac{\sigma_{11} - \sigma_{22}}{2}. \quad (54)$$

Thus, reservoirs that are indistinguishable through $E(t)$ can be distinguished through their purity or entropy trajectories.

For the thermal benchmark, one has $d_1 = d_2 = v_{\text{th}}$, so the right-hand side of Eq. (54) vanishes identically and Eq. (50) follows. An anisotropic diffusion matrix instead sources the anisotropy component u and produces a nonzero residual. For generic transient initial conditions, this residual contains a damped component at the orientation frequency 2ω ; at finite anisotropy it can also contain a stationary contribution. The detailed evolution is given in Appendix B.

Operationally, Eq. (25) is therefore not merely a way of rewriting the internal energy. Once the thermal generator has been calibrated, it gives an independent consistency relation between two scalar time series, energy and purity. A failure of the measured purity trajectory to satisfy Eq. (50) signals a failure of the assumed thermal isotropic relaxation law, with anisotropic diffusion providing one explicit example.

F. Time-dependent bath temperature

We first retain a fixed oscillator frequency but allow the bath temperature and, if desired, the damping rate to vary in time. Within a time-local thermal GKLS description,

$$v_{\text{th}} \rightarrow v_{\text{th}}(t) = \frac{1}{2} \coth \frac{\omega}{2T_{\text{bath}}(t)}, \quad \gamma \rightarrow \gamma(t), \quad (55)$$

and Eq. (15) retains the same local form. The instantaneous bath target is

$$\sigma_{\text{target}}(t) = \nu_{\text{th}}(t)\mathbb{I}. \quad (56)$$

Changing $T_{\text{bath}}(t)$ moves the target along the same fixed Gibbs family $\mathcal{M}_G$; it does not change that family itself.

The exact local equations are

$$\dot{s} = -\gamma(t)[s - 2\nu_{\text{th}}(t)], \quad (57)$$

$$\dot{v} = \gamma(t) \left[\frac{\nu_{\text{th}}(t)s}{2v} - v \right], \quad (58)$$

$$\frac{d}{dt}r^2 = -2\gamma(t)r^2. \quad (59)$$

Hence,

$$r^2(t) = r^2(t_0) \exp \left[-2 \int_{t_0}^t \gamma(u) du \right]. \quad (60)$$

If the state starts unsqueezed, it remains unsqueezed. Bath-temperature driving at fixed ω therefore produces motion along the Gibbs family, not transverse squeezing.

Whenever $\gamma(t) > 0$, the same elimination can be performed pointwise:

$$E(S, \dot{S}; t) = \frac{\omega v(S)}{\nu_{\text{th}}(t)} \left[v(S) + \frac{\dot{S}}{\gamma(t)S'(v(S))} \right]. \quad (61)$$

No explicit $\dot{T}_{\text{bath}}$ term appears. The time-dependent bath enters through the instantaneous target and relaxation rate. The explicit t dependence of Eq. (61) is of the same character as the control-parameter dependence λ in the equilibrium form $E = E(S, \lambda)$: $\nu_{\text{th}}(t)$ and $\gamma(t)$ are externally prescribed parameters, not additional state variables.

V. DRIVEN OSCILLATOR: GENERATION AND RELAXATION OF OFF-GIBBS STRUCTURE

The fixed-frequency analysis establishes the exact entropy-rate representation of the internal energy under standard thermal relaxation. It does not, however, provide a mechanism for generating the off-Gibbs Gaussian structure from an initially Gibbs state. Indeed, at fixed oscillator frequency, Eq. (18) gives $\frac{d}{dt}r^2 = -2\gamma r^2$, so that

$$r(t_0) = 0 \implies r(t) = 0. \quad (62)$$

The same conclusion holds when only the bath temperature changes: The thermal target moves along the fixed Gibbs family, but no transverse Gaussian deformation is generated. Thus, thermal relaxation can remove preexisting squeezing, but at fixed ω it cannot create it for the isotropic thermal generator considered here.

A different ingredient is required if the off-Gibbs structure is to arise dynamically. We now show that frequency modulation supplies such a mechanism. Consider

$$H(t) = \frac{1}{2}[p^2 + \omega^2(t)x^2]. \quad (63)$$

The driven construction has a different logical status from the fixed-frequency benchmark. Equation (25) follows exactly from the standard thermal GKLS equation, whereas the construction below is a local Gaussian continuation whose validity requires additional assumptions on the bath and driving timescales. Its purpose is to identify how Hamiltonian driving can generate the Gaussian structure on which thermal relaxation subsequently acts. Technical details are collected in Appendix D.

A. Invariant description of the driven Gaussian state

Write the zero-mean Gaussian state as

$$\rho(t) = Z^{-1} \exp[-\mathcal{I}(t)/T_0], \quad (64)$$

where T_0 is a fixed reference scale and

$$\mathcal{I}(t) = \frac{1}{2}[g_-(t)p^2 + g_0(t)\{x, p\} + g_+(t)x^2]. \quad (65)$$

For isolated evolution, the quadratic kernel satisfies

$$D_t \mathcal{I} \equiv \partial_t \mathcal{I} + i[H(t), \mathcal{I}] = 0. \quad (66)$$

The symplectic kernel scale

$$\omega_{\mathcal{I}} = \sqrt{g_- g_+ - g_0^2} \quad (67)$$

is therefore exactly constant under the unitary dynamics, even when $\omega(t)$ varies [16–20]. The drive changes the shape and orientation of the quadratic kernel, but not this invariant scale.

It is also useful to define

$$\omega_{\text{eff}} = \frac{g_+ + \omega^2 g_-}{2\omega_{\mathcal{I}}}, \quad (68)$$

which determines the physical energy. The exact Gaussian dictionary is

$$\nu = \frac{1}{2} \coth \frac{\omega_{\mathcal{I}}}{2T_0}, \quad E = \nu \omega_{\text{eff}}. \quad (69)$$

Thus, $\omega_{\mathcal{I}}$ determines the mixedness of the Gaussian state, whereas ω_{eff} measures its energetic deformation relative to the instantaneous oscillator Hamiltonian. On the instantaneous Gibbs family,

$$\mathcal{I} = \lambda H(t), \quad (70)$$

one has

$$\omega_{\text{eff}} = \omega(t), \quad (71)$$

while nonadiabatic deformation generically produces $\omega_{\text{eff}} \neq \omega$ [21].

B. Local relaxation toward a moving Gibbs target

To describe weak thermal relaxation in the driven problem, we adopt the local Gaussian continuation

$$\boxed{D_t \mathcal{I} = -\alpha(t)[\mathcal{I} - g(t)H(t)],} \quad g(t) = \frac{T_0}{T_{\text{bath}}(t)}. \quad (72)$$

Extensions of invariant methods to open systems have been considered previously [22]; here, Eq. (72) is used specifically as a local relaxation model anchored to the fixed-frequency thermal problem. It is not assumed to be a universal microscopic GKLS generator for arbitrary driving.

The origin of the off-Gibbs structure is most transparent if we define the deviation from the instantaneous thermal target,

$$\delta \mathcal{I} := \mathcal{I} - gH. \quad (73)$$

Equation (72) then gives

$$\boxed{D_t \delta \mathcal{I} + \alpha \delta \mathcal{I} = -\dot{g}H - g\dot{H}.} \quad (74)$$

The two source terms have different roles. At fixed ω , the term $\dot{g}H$ moves the target only along the Gibbs family. Frequency modulation is qualitatively different because

$$\dot{H} = \omega \dot{\omega} x^2. \quad (75)$$

In terms of the instantaneous ladder operator

$$a_t = \sqrt{\frac{\omega}{2}} x + \frac{i}{\sqrt{2\omega}} p, \quad (76)$$

this becomes

$$\dot{H} = \frac{\dot{\omega}}{2} (a_t^2 + a_t^{\dagger 2} + 2a_t^\dagger a_t + 1). \quad (77)$$

The terms a_t^2 and $a_t^{\dagger 2}$ are transverse to the instantaneous Gibbs family and generate pair coherence, or equivalently, squeezing. Consequently, even if the state initially coincides with the instantaneous thermal target,

$$\delta\mathcal{I}(t_0) = 0, \quad (78)$$

a nonzero $\dot{\omega}(t_0)$ generically drives it away from that family.

The physical division of labor is therefore

- frequency driving
- off-Gibbs Gaussian deformation, thermal relaxation
- decay of that deformation.

This is the mechanism absent from the fixed-frequency benchmark.

C. Relaxation of the invariant scale and energetic representation

The local relaxation law also determines the evolution of the invariant kernel scale. The component equations following from Eq. (72) imply

$$\dot{\omega}_{\mathcal{I}} = \alpha(g\omega_{\text{eff}} - \omega_{\mathcal{I}}). \quad (79)$$

This equation cleanly separates the unitary and dissipative roles. For $\alpha = 0$,

$$\dot{\omega}_{\mathcal{I}} = 0 \quad (80)$$

even under nonadiabatic frequency driving. The drive can therefore generate squeezing while leaving the entropy unchanged in the isolated system. Only the coupling to the bath makes $\omega_{\mathcal{I}}$, and hence the entropy, evolve.

Solving Eq. (79) for the energetic scale gives

$$\omega_{\text{eff}} = \frac{1}{g} \left(\omega_{\mathcal{I}} + \frac{\dot{\omega}_{\mathcal{I}}}{\alpha} \right). \quad (81)$$

For $\alpha(t) > 0$, solving Eq. (79) for the energetic scale gives

$$E = \frac{1}{2g} \left(\omega_{\mathcal{I}} + \frac{\dot{\omega}_{\mathcal{I}}}{\alpha} \right) \coth \frac{\omega_{\mathcal{I}}}{2T_0}. \quad (82)$$

Since the Gaussian entropy is determined by $\omega_{\mathcal{I}}$, Eq. (82) is the driven analog of the generator-conditioned entropy-rate representation found in the fixed-frequency problem.

This statement should not be interpreted as an intrinsic change of coordinates on Gaussian state space. The quantity $\dot{\omega}_{\mathcal{I}}$, like $\dot{S}$, depends on the specified dissipative dynamics. Equation (82) is therefore a trajectory-level energetic relation conditioned on the local relaxation model (72).

D. Relation to the fixed-frequency benchmark

The fixed-frequency and driven relations have different levels of control. At fixed ω , Eq. (25) is the appropriate result because it follows directly from the standard thermal master equation (11). Equation (82), in contrast, follows from the additional modeling assumption (72).

The two become quantitatively equivalent at fixed frequency only when the rate α is chosen so that the invariant-kernel evolution reproduces the GKLS entropy trajectory. The corresponding matching condition is derived in Appendix D. In particular, α should not in general be identified with the benchmark damping rate γ away from the near-target regime.

The two calculations therefore answer complementary questions. The fixed-frequency benchmark establishes that thermalization gives an exact relation between the internal energy and the entropy rate. The driven construction shows how the nonequilibrium Gaussian structure entering that relation can itself be generated dynamically:

frequency modulation
 $\longrightarrow$ squeezing
 $\longrightarrow$ thermal relaxation
 $\longrightarrow$ nontrivial entropy dynamics.

A driven but isolated oscillator realizes the first two steps but has $\dot{S} = 0$; an undriven thermal oscillator realizes relaxation but cannot generate squeezing from a Gibbs initial state. Both ingredients are required for the full dynamical process.

It is useful to distinguish two independently controlled facts from their combination in the driven open system. First, nonadiabatic unitary frequency modulation can generically generate off-Gibbs Gaussian structure while leaving the von Neumann entropy unchanged; this follows from the isolated dynamics and does not rely on Eq. (72). Second, in the exactly controlled fixed-frequency thermal GKLS benchmark, off-Gibbs structure affects the entropy rate through Eq. (17). Combining these two mechanisms into a single driven open-system trajectory, and obtaining the quantitative relation (82), requires the local Gaussian continuation (72). The qualitative division of labor—unitary driving generates off-Gibbs structure, while thermal relaxation makes its subsequent evolution contribute to entropy change—is established by the two independently controlled limits; only their quantitative synthesis in a driven open system depends on the local continuation.

E. Regime of validity

The local construction requires the bath to lose memory before the resonant data entering the system-bath coupling vary appreciably. Let

$$\tau_I = \left| \frac{\omega_I}{\dot{\omega}_I} \right|, \quad \tau_A = \left| \frac{A}{\dot{A}} \right|, \quad (83)$$

where $A(t)$ denotes the matrix-element prefactor in the decomposition of the system coupling operator in the invariant basis. A compact Markov hierarchy is

$$\tau_B \ll \min(\alpha^{-1}, \tau_I, \tau_A). \quad (84)$$

If a secular coarse-graining time Δt is introduced, one further requires

$$\max(\tau_B, \omega_I^{-1}) \ll \Delta t \ll \min(\alpha^{-1}, \tau_I, \tau_A). \quad (85)$$

These conditions delimit the status of the driven construction. The fixed-frequency entropy-rate relation remains the exactly controlled benchmark; the driven model supplies a local realization of how frequency modulation generates transverse Gaussian structure and how subsequent thermalization converts that structure into nontrivial entropy dynamics. The full covariance-kernel dictionary,

component evolution equations, matching to the fixed-frequency GKLS benchmark, and derivation of the timescale conditions are given in Appendix D.

VI. DISCUSSION

The analysis above separates two logically distinct ingredients of the problem. The first is thermal relaxation itself. For the standard fixed-frequency thermal GKLS oscillator, the covariance equations imply the exact mixed-state relation

$$E = E(S, \dot{S}), \quad (86)$$

with the explicit form given in Eq. (25). No external driving is required for this result. The entropy rate is not an instantaneous state coordinate: It is determined jointly by the Gaussian state and the specified open-system generator. The entropy-rate representation is therefore a trajectory-level energetic relation conditioned on thermalizing dynamics, not a reparametrization of the instantaneous Gaussian state space.

The isolated limit shows why the bath is essential to this representation. When $\gamma = 0$, the symplectic eigenvalue and hence the entropy are constants of motion, so that $\dot{S} = 0$ for every Gaussian state. Equal-entropy states can nevertheless have different energies because they may carry different squeezing. The entropy rate therefore contains no additional energetic information in the isolated oscillator. For nonzero thermal coupling, Eq. (17) supplies the additional dynamical relation that allows the energy-carrying second moment to be eliminated in favor of S and $\dot{S}$. Within the fixed-frequency benchmark, this is the precise sense in which the rate dependence is induced by thermalization.

Thermalization, however, does not by itself generate the off-Gibbs structure that makes this relation nontrivial. At fixed oscillator frequency,

$$\frac{d}{dt}r^2 = -2\gamma r^2, \quad (87)$$

so an initially unsqueezed state remains unsqueezed. The same remains true when only the bath temperature varies: The bath target moves along the fixed Gibbs family, while no transverse squeezing is produced. Thus the fixed-frequency problem establishes the exact energetic relation and describes the relaxation of preexisting nonequilibrium Gaussian structure, but it does not provide a mechanism for creating that structure from an initially Gibbs state.

This is the physical role of the driven construction. When the oscillator frequency varies, the instantaneous Gibbs family itself moves. The tracking equation

$$D_t \delta \mathcal{I} + \alpha \delta \mathcal{I} = -\dot{g} H - g \dot{H} \quad (88)$$

shows that the two sources have different geometric effects. The term $\dot{g} H$ moves the target tangentially along the Gibbs family, whereas $\dot{H}$ contains pair terms in the instantaneous oscillator basis and generically drives the state transversely away from that family. Frequency modulation therefore provides a physical mechanism for generating squeezing, while the bath subsequently relaxes the generated deformation.

The roles of driving and thermalization are consequently complementary. A driven but isolated oscillator can develop squeezing and an energetic departure from the instantaneous Gibbs family, but its von Neumann entropy remains constant and no entropy-rate closure follows. Conversely, an undriven thermal oscillator supplies the exact relation $E(S, \dot{S})$ but cannot generate squeezing from an initially Gibbs state. In the driven open oscillator, the two mechanisms operate together:

- frequency driving
- off-Gibbs Gaussian deformation
- thermal relaxation and nontrivial $\dot{S}$.

This combined process is the dynamical realization of the entropy-rate energetic structure identified by the fixed-frequency benchmark.

This separation also clarifies the relation to finite-time thermodynamics. The appearance of $\dot{S}$ in the energetic representation is not caused by the speed of the driving protocol: Eq. (25) already holds for a time-independent Hamiltonian. External driving has a different role here. It generates the transverse Gaussian structure on which the bath subsequently acts. The derivative dependence is supplied by the relaxation law, whereas the drive supplies a mechanism for producing the nonequilibrium structure whose relaxation makes that dependence physically active.

The two parts of the analysis have different levels of generality. The fixed-frequency result follows exactly from the standard weak-coupling Born-Markov-secular master equation. The driven construction is instead a local Gaussian continuation based on Eq. (72). Its purpose is not to claim a universal microscopic GKLS equation for an arbitrarily driven bath, but to exhibit a controlled realization in which the unitary drive generates transverse Gaussian structure while the invariant scale remains suitable for a local relaxation description. Its validity therefore requires the hierarchy summarized in Eqs. (84) and (85), and the rate α should not in general be identified with the fixed-frequency rate γ away from the matching regime.

Because the entropy rate is fixed jointly by the state and the generator, the relation is falsifiable without any reconstruction of the squeezing variable. The protocol is given in Sec. IV E: Two scalar time series, the purity and the mean energy, are measured independently, and Eq. (25) asserts that they are not independent of one another. What such a test probes is accordingly the assumed structure of the dissipator rather than any new energetic effect of squeezing. An anisotropic reservoir illustrates the distinction, since it can reproduce the energy trajectory exactly while failing the entropy-rate relation.

One point of interpretation should be kept separate from this. The physically accessible region of Fig. 1 contains states with $\dot{S} < 0$, so the oscillator entropy may decrease along a thermalizing trajectory. Only the total entropy production, which also counts the flow to the bath, is constrained,

$$\dot{\Sigma} = \dot{S} - \frac{\dot{Q}}{T_{\text{bath}}} \geq 0, \quad (89)$$

established for the same GKLS generator in Appendix C.

Nothing in this construction changes the usual definition $E = \text{tr}(\rho H)$ as an instantaneous state function on the full Gaussian state space. The new statement concerns a reduced, generator-conditioned representation of that energy. The fixed-frequency oscillator shows that thermal relaxation permits this representation to close exactly on $(S, \dot{S})$, while the driven oscillator shows how the off-Gibbs Gaussian structure entering that relation can be generated dynamically rather than inserted through the initial condition.

The same logic suggests extensions to multimode Gaussian systems, but with an important qualification. If independent modes each obey an appropriate thermal Gaussian relaxation law, one may obtain mode-resolved relations

$$E_k = E_k(S_k, \dot{S}_k). \quad (90)$$

This does not imply a closed scalar relation $E_{\text{tot}} = E(S_{\text{tot}}, \dot{S}_{\text{tot}})$, because the distribution of entropy and entropy rate among the modes generally carries additional information. Determining when an analogous generator-dependent elimination closes in multimode or non-Gaussian systems remains an open problem.

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX A: COMPONENT COVARIANCE DYNAMICS

This Appendix derives the invariant equations used in Sec. III directly from the covariance components and makes the decoupling of the covariance orientation explicit.

Write

$$\sigma = \begin{pmatrix} a & c \\ c & b \end{pmatrix}. \quad (\text{A1})$$

Substituting this into Eq. (15) gives

$$\dot{a} = 2\omega c - \gamma(a - v_{\text{th}}), \quad (\text{A2})$$

$$\dot{b} = -2\omega c - \gamma(b - v_{\text{th}}), \quad (\text{A3})$$

$$\dot{c} = \omega(b - a) - \gamma c. \quad (\text{A4})$$

The trace equation follows immediately:

$$\dot{s} = \dot{a} + \dot{b} = -\gamma(s - 2v_{\text{th}}). \quad (\text{A5})$$

Introduce the two anisotropy components $u = (a - b)/2$ and $v = c$. They obey

$$\dot{u} = 2\omega v - \gamma u, \quad (\text{A6})$$

$$\dot{v} = -2\omega u - \gamma v. \quad (\text{A7})$$

Thus, the anisotropy vector (u, v) rotates at angular frequency 2ω , while its magnitude decays with envelope $e^{-\gamma t}$. Since $r^2 = u^2 + v^2$, one obtains

$$\frac{d}{dt}r^2 = 2u\dot{u} + 2v\dot{v} = -2\gamma r^2. \quad (\text{A8})$$

This also shows why the orientation is irrelevant for the energetic closure: The phase of $u + iv$ does not enter either s or r^2 .

The full covariance solution can be written compactly as

$$\sigma(t) = v_{\text{th}}\mathbb{I} + e^{-\gamma t}R(t)[\sigma(0) - v_{\text{th}}\mathbb{I}]R^T(t), \quad (\text{A9})$$

where

$$R(t) = e^{\omega J t} = \begin{pmatrix} \cos \omega t & \sin \omega t \\ -\sin \omega t & \cos \omega t \end{pmatrix}. \quad (\text{A10})$$

Equation (A9) displays the isotropic relaxation and the Hamiltonian rotation separately.

Finally, since $v^2 = \det \sigma$,

$$\frac{d}{dt}v^2 = \det \sigma \operatorname{tr}(\sigma^{-1}\dot{\sigma}) = -2\gamma v^2 + \gamma v_{\text{th}}s, \quad (\text{A11})$$

where the Hamiltonian commutator contributes zero. Therefore,

$$\dot{v} = \gamma \left(\frac{v_{\text{th}}s}{2v} - v \right), \quad (\text{A12})$$

which is Eq. (17).

APPENDIX B: GENERATOR-DEPENDENT ENTROPY-RATE RELATION, ADMISSIBLE DOMAIN, AND DISSIPATOR TEST

This Appendix collects three technical results used in Sec. IV. First, it characterizes the relation between the instantaneous squeezing and the entropy rate for a specified thermal GKLS generator. Second, it gives the admissible state-and-rate domain and discusses the pure-state boundary. Third, it derives the scalar consistency relation used in Sec. IV E and shows explicitly how it fails for an anisotropic dissipator.

The relation between r^2 and $\dot{S}$ should throughout be understood conditionally on the generator parameters γ and ν_{th} . It is not an intrinsic coordinate transformation on Gaussian state space.

1. Conditional relation and admissible domain

For fixed ν , Eq. (10) gives

$$s = 2\sqrt{\nu^2 + r^2}. \quad (\text{B1})$$

Substitution into Eq. (17) yields

$$\dot{S} = \gamma S'(\nu) \left[\frac{\nu_{\text{th}}}{\nu} \sqrt{\nu^2 + r^2} - \nu \right]. \quad (\text{B2})$$

For a specified thermal generator with $\gamma > 0$ and for $\nu > 1/2$,

$$\frac{\partial \dot{S}}{\partial (r^2)} = \frac{\gamma \nu_{\text{th}} S'(\nu)}{2\nu \sqrt{\nu^2 + r^2}} > 0. \quad (\text{B3})$$

Thus, at fixed entropy and for fixed generator parameters, the entropy rate determines uniquely the value of r^2 compatible with that dynamics. Solving Eq. (B2) for r^2 reproduces Eq. (29). This is a generator-dependent dynamical inference, not an identification of r^2 and $\dot{S}$ as intrinsic state coordinates.

The variable

$$q := \nu^2 \quad (\text{B4})$$

provides a convenient description of the admissible state-and-rate data. From Eq. (37),

$$s = \frac{2}{\nu_{\text{th}}} \left(q + \frac{\dot{q}}{2\gamma} \right), \quad (\text{B5})$$

and therefore

$$r^2 = \frac{1}{\nu_{\text{th}}^2} \left(q + \frac{\dot{q}}{2\gamma} \right)^2 - q. \quad (\text{B6})$$

A physical Gaussian covariance must satisfy the uncertainty condition

$$q \geq \frac{1}{4} \quad (\text{B7})$$

together with $r^2 \geq 0$. Since $s > 0$, the physical branch of the latter condition is

$$\boxed{\dot{q} \geq 2\gamma(\nu_{\text{th}}\sqrt{q} - q)}. \quad (\text{B8})$$

Equality corresponds to $r = 0$. Equation (39) therefore defines a closed trajectory equation only together with the admissibility conditions (B7) and (B8); arbitrary mathematical initial data $(q, \dot{q})$ need not correspond to a physical Gaussian state evolving under the specified generator.

For completeness, the constants in Eq. (40) are fixed by $q_0 = q(0)$ and $\dot{q}_0 = \dot{q}(0)$ as

$$C_1 = 2(q_0 - v_{\text{th}}^2) + \frac{\dot{q}_0}{\gamma}, \quad (\text{B9})$$

$$C_2 = -(q_0 - v_{\text{th}}^2) - \frac{\dot{q}_0}{\gamma}. \quad (\text{B10})$$

2. Pure-state boundary

The entropy representation requires one additional qualification. At the pure-state boundary $v = 1/2$,

$$S'(v) = \ln \frac{v + \frac{1}{2}}{v - \frac{1}{2}} \longrightarrow \infty. \quad (\text{B11})$$

Consequently, formulas written directly in terms of $(S, \dot{S})$ become singular at this boundary for generic thermalizing trajectories. This is why Eq. (25) was stated for mixed states.

The variable $q = v^2$, in contrast, remains regular at $q = 1/4$, and Eq. (37) continues to provide the appropriate state-and-rate representation there. The zero-temperature vacuum fixed point is exceptional: There $v = 1/2$ and $\dot{v} = 0$, and the corresponding expressions should be understood through the limiting fixed trajectory.

3. Failure for an anisotropic dissipator

Using $q = v^2$, Eq. (17) gives Eq. (49) and its integrated solution (50) of the main text. Thus, once the thermal generator is specified, an energy trajectory and a single initial value of q determine the symplectic-eigenvalue trajectory predicted by the model.

The special role of the isotropic thermal diffusion matrix can be seen by replacing the diffusion term in Eq. (15) by a general symmetric matrix D . The covariance equation then takes the form

$$\dot{\sigma} = \omega(J\sigma - \sigma J) - \gamma\sigma + D. \quad (\text{B12})$$

Using $\frac{d}{dt} \det \sigma = \det \sigma \operatorname{tr}(\sigma^{-1} \dot{\sigma})$, together with $\det \sigma \sigma^{-1} = \operatorname{adj} \sigma$ for a 2×2 matrix, one obtains

$$\frac{dq}{dt} = -2\gamma q + \operatorname{tr}(\operatorname{adj} \sigma D). \quad (\text{B13})$$

The Hamiltonian rotation again drops out of the determinant.

For the thermal isotropic choice $D = \gamma v_{\text{th}} \mathbb{I}$, Eq. (B13) reduces to Eq. (49). Now, consider instead

$$D = \gamma \operatorname{diag}(d_1, d_2), \quad \bar{d} := \frac{d_1 + d_2}{2}. \quad (\text{B14})$$

We restrict $d_1 > 0$ and $d_2 > 0$ to values compatible with complete positivity of the Gaussian Markovian dynamics; in the present normalization, this requires $d_1 d_2 \geq 1/4$. Writing $\sigma_{11} = \frac{s}{2} + u$, $\sigma_{22} = \frac{s}{2} - u$, and $v := \sigma_{12}$ gives Eq. (54).

The same anisotropic diffusion also modifies the evolution of the orientation variables themselves. The covariance components obey

$$\dot{\sigma}_{11} = 2\omega\sigma_{12} - \gamma\sigma_{11} + \gamma d_1, \quad (\text{B15})$$

$$\dot{\sigma}_{22} = -2\omega\sigma_{12} - \gamma\sigma_{22} + \gamma d_2, \quad (\text{B16})$$

$$\dot{\sigma}_{12} = \omega(\sigma_{22} - \sigma_{11}) - \gamma\sigma_{12}, \quad (\text{B17})$$

and hence

$$\dot{u} = 2\omega v - \gamma u + \frac{\gamma}{2}(d_1 - d_2), \quad (\text{B18})$$

$$\dot{v} = -2\omega u - \gamma v. \quad (\text{B19})$$

Unlike the isotropic equations (A6) and (A7), Eq. (B18) contains a constant source. The anisotropy therefore relaxes not to zero but to

$$u_{\text{ss}} = \frac{\gamma^2(d_1 - d_2)}{2(4\omega^2 + \gamma^2)}, \quad (\text{B20})$$

$$v_{\text{ss}} = -\frac{\gamma\omega(d_1 - d_2)}{4\omega^2 + \gamma^2}. \quad (\text{B21})$$

Writing $\Delta t = t - t_0$, the exact transient is

$$u(t) = u_{\text{ss}} + e^{-\gamma\Delta t}[(u_0 - u_{\text{ss}})\cos(2\omega\Delta t) + (v_0 - v_{\text{ss}})\sin(2\omega\Delta t)], \quad (\text{B22})$$

$$v(t) = v_{\text{ss}} + e^{-\gamma\Delta t}[-(u_0 - u_{\text{ss}})\sin(2\omega\Delta t) + (v_0 - v_{\text{ss}})\cos(2\omega\Delta t)]. \quad (\text{B23})$$

Equation (54) therefore contains, in general, both a stationary contribution and a damped transient at the orientation frequency 2ω . The 2ω component is a useful signature of anisotropic diffusion during the transient, but it is not by itself universal: for example, a state prepared directly in the anisotropic stationary covariance has only the stationary contribution.

The distinction from the isotropic thermal generator is especially transparent when the two dissipators have the same diffusion trace, $\vec{d} = v_{\text{th}}$. Nontrivial same-trace anisotropic choices satisfying this condition exist for $v_{\text{th}} > 1/2$; at the zero-temperature limit $v_{\text{th}} = 1/2$, complete positivity together with the same-trace condition forces $d_1 = d_2 = 1/2$. Their trace equations are then identical to Eq. (16), so that the same initial value of s produces the same energy trajectory. The q dynamics, however, satisfies

$$\dot{q} + 2\gamma q = \frac{2\gamma v_{\text{th}}}{\omega} E(t) + \gamma(d_2 - d_1)u(t). \quad (\text{B24})$$

Integrating gives

$$q(t) = e^{-2\gamma(t-t_0)}q(t_0) + \frac{2\gamma v_{\text{th}}}{\omega} \int_{t_0}^t e^{-2\gamma(t-\tau)} E(\tau) d\tau + \gamma(d_2 - d_1) \int_{t_0}^t e^{-2\gamma(t-\tau)} u(\tau) d\tau. \quad (\text{B25})$$

The last term is absent for the isotropic thermal generator. Hence, two dissipators can produce exactly the same energy relaxation while predicting different q and entropy trajectories.

APPENDIX C: OSCILLATOR ENTROPY RATE AND TOTAL ENTROPY PRODUCTION

The appearance of negative values of $\dot{S}$ in Fig. 1 is consistent with the second law because $\dot{S}$ is the entropy rate of the oscillator alone. The total entropy production includes the entropy flow to the thermal bath.

Let

$$\rho_{\text{th}} = \frac{e^{-H/T_{\text{bath}}}}{\text{tr}(e^{-H/T_{\text{bath}}})} \quad (\text{C1})$$

be the stationary Gibbs state of the generator (11). Define

$$\mathcal{L}_{\text{th}}\rho = \gamma(N+1)\mathcal{D}[a]\rho + \gamma N\mathcal{D}[a^\dagger]\rho.$$

For a thermal GKLS dissipator satisfying detailed balance, Spohn's inequality [23] gives

$$\dot{\Sigma} = -\text{tr}[(\mathcal{L}_{\text{th}}\rho)(\ln \rho - \ln \rho_{\text{th}})] \geq 0. \quad (\text{C2})$$

The unitary part does not contribute to the von Neumann entropy rate, so

$$\dot{S} = -\text{tr}[(\mathcal{L}_{\text{th}}\rho) \ln \rho]. \quad (\text{C3})$$

At fixed Hamiltonian, the heat current from the bath is

$$\dot{Q} = \text{tr}[H\mathcal{L}_{\text{th}}\rho] = \dot{E} = -\gamma(E - \omega\nu_{\text{th}}). \quad (\text{C4})$$

Using $\ln \rho_{\text{th}} = -H/T_{\text{bath}} - \ln Z$, Eq. (C2) becomes

$$\dot{\Sigma} = \dot{S} - \frac{\dot{Q}}{T_{\text{bath}}} \geq 0. \quad (\text{C5})$$

Thus, $\dot{S}$ itself need not be non-negative. For example, an unsqueezed state with $\nu > \nu_{\text{th}}$ cools toward the bath and has

$$\dot{S} = \gamma S'(\nu)(\nu_{\text{th}} - \nu) < 0, \quad (\text{C6})$$

while Eq. (C5) still guarantees non-negative total entropy production. The same distinction applies to squeezed states in the negative- $\dot{S}$ portion of the physical region.

APPENDIX D: TECHNICAL DETAILS OF THE DRIVEN INVARIANT CONTINUATION

This Appendix collects the technical material behind the concise driven extension of Sec. V. None of the exact constant-frequency results in Secs. III and IV depends on this construction.

1. Kernel normalization and Gaussian dictionary

The Gaussian state is written as

$$\rho = Z^{-1} \exp[-\mathcal{I}/T_0], \quad (\text{D1})$$

where only the ratio $\mathcal{I}/T_0$ is physical. Following the convention of the main text, we fix the normalization by choosing T_0 as the temperature of the thermal state of $H(0)$ with the same entropy as the initial Gaussian state:

$$\nu(0) = \frac{1}{2} \coth \frac{\omega(0)}{2T_0}. \quad (\text{D2})$$

The driven kernel construction is stated for mixed initial Gaussian states with $T_0 > 0$; pure states may be approached as limiting cases. For an initially unsqueezed state, this is its ordinary initial temperature. With this convention, $\omega_{\mathcal{I}}$ has the same dimension as ω .

Williamson reduction of the quadratic kernel (65) gives

$$\langle x^2 \rangle = \frac{\nu g_-}{\omega_{\mathcal{I}}}, \quad \langle xp \rangle_s = -\frac{\nu g_0}{\omega_{\mathcal{I}}}, \quad \langle p^2 \rangle = \frac{\nu g_+}{\omega_{\mathcal{I}}}, \quad \nu = \frac{1}{2} \coth \frac{\omega_{\mathcal{I}}}{2T_0}. \quad (\text{D3})$$

Differentiating the last relation,

$$\dot{\nu} = -\frac{1}{T_0} \left(\nu^2 - \frac{1}{4} \right) \dot{\omega}_{\mathcal{I}}. \quad (\text{D4})$$

Equivalently,

$$\omega_{\mathcal{I}} = T_0 S'(\nu). \quad (\text{D5})$$

Using the instantaneous dimensionless quadratures $X = \sqrt{\omega}x$ and $P = p/\sqrt{\omega}$, one obtains

$$s = \frac{2\nu\omega_{\text{eff}}}{\omega}, \quad E = \nu\omega_{\text{eff}}, \quad (\text{D6})$$

$$r^2 = \nu^2 \left[\left(\frac{\omega_{\text{eff}}}{\omega} \right)^2 - 1 \right]. \quad (\text{D7})$$

Thus, $\omega_{\text{eff}} = E/\nu$ is the energetic scale associated with the Gaussian deformation, and $\omega_{\text{eff}} = \omega$ exactly for unsqueezed states.

For fixed ω , the covariance and kernel descriptions of the Gibbs family are equivalent. With ρ in Eq. (64), a Gibbs state (28) of the oscillator at temperature T may be written either as $\sigma = \nu(T)\mathbb{I}$ or as $\mathcal{I} = \lambda H$, with $\lambda = \frac{T_0}{T}$. Equivalently,

$$\boxed{\begin{aligned} \sigma = \nu\mathbb{I} &\iff \mathcal{I} = \lambda H \\ &\iff g_0 = 0, \quad g_- = \lambda, \quad g_+ = \lambda\omega^2 \\ &\iff \omega_{\text{eff}} = \omega \iff r = 0, \end{aligned}} \quad (\text{D8})$$

where

$$\omega_{\mathcal{I}} = \lambda\omega, \quad \nu(T) = \frac{1}{2} \coth \frac{\omega_{\mathcal{I}}}{2T_0} = \frac{1}{2} \coth \frac{\omega}{2T}. \quad (\text{D9})$$

The bath selects one particular member of this family. For a bath temperature T_{bath} ,

$$\lambda = g := \frac{T_0}{T_{\text{bath}}}, \quad \mathcal{I}_{\text{target}} = gH, \quad (\text{D10})$$

and therefore

$$\omega_{\mathcal{I}}^{\text{target}} = g\omega, \quad \nu_{\text{th}} = \frac{1}{2} \coth \frac{g\omega}{2T_0} = \frac{1}{2} \coth \frac{\omega}{2T_{\text{bath}}}. \quad (\text{D11})$$

2. Component equations and invariant-scale relaxation

Substituting Eq. (65) into the local relaxation law (72) gives

$$\dot{g}_- = -2g_0 - \alpha(g_- - g), \quad (\text{D12})$$

$$\dot{g}_0 = \omega^2 g_- - g_+ - \alpha g_0, \quad (\text{D13})$$

$$\dot{g}_+ = 2\omega^2 g_0 - \alpha(g_+ - g\omega^2). \quad (\text{D14})$$

Differentiating $\omega_{\mathcal{I}}^2 = g_- g_+ - g_0^2$ and using these equations gives

$$\dot{\omega}_{\mathcal{I}} = \alpha(g\omega_{\text{eff}} - \omega_{\mathcal{I}}), \quad (\text{D15})$$

which is Eq. (79). In particular, $\alpha = 0$ implies $\dot{\omega}_{\mathcal{I}} = 0$ exactly, irrespective of the Hamiltonian driving.

3. Matching to the fixed-frequency GKLS benchmark

The exact covariance dictionary does not imply that Eq. (72) is an exact rewriting of the benchmark master equation. At fixed frequency, Eq. (17) becomes

$$\dot{v} = \gamma \left(\frac{\nu_{\text{th}}\omega_{\text{eff}}}{\omega} - v \right). \quad (\text{D16})$$

On the other hand, combining Eqs. (D4) and (79) gives

$$\dot{v} = -\frac{\alpha}{T_0} \left(v^2 - \frac{1}{4} \right) (g\omega_{\text{eff}} - \omega_{\mathcal{I}}). \quad (\text{D17})$$

Equality of the entropy trajectories therefore requires

$$\alpha(g\omega_{\text{eff}} - \omega_{\mathcal{I}}) = \frac{T_0\gamma}{v^2 - \frac{1}{4}} \left(v - \frac{v_{\text{th}}\omega_{\text{eff}}}{\omega} \right). \quad (\text{D18})$$

Hence, α should not in general be identified with the constant GKLS rate γ .

On the Gibbs family, $\omega_{\text{eff}} = \omega$, and Eq. (D18) reduces to

$$\alpha = \frac{T_0\gamma(v - v_{\text{th}})}{(v^2 - \frac{1}{4})(g\omega - \omega_{\mathcal{I}})}. \quad (\text{D19})$$

Using

$$\frac{\partial\omega_{\mathcal{I}}}{\partial v} = -\frac{T_0}{v^2 - \frac{1}{4}}, \quad (\text{D20})$$

one finds $\alpha \rightarrow \gamma$ near the bath target. Matching the full covariance trajectory away from that regime requires more than matching the entropy equation alone. This is why the driven law is described as a local continuation rather than an exact change of variables.

4. Moving Gibbs target and source of transverse excitation

For fixed ω , the kernel representation of the Gibbs family is

$$\mathcal{M}_{\text{G}} = \{\mathcal{I} = \lambda H\}, \quad g_0 = 0, \quad g_+ = \omega^2 g_-. \quad (\text{D21})$$

When $\omega(t)$ varies, the corresponding family moves with the Hamiltonian. Defining

$$\delta\mathcal{I} = \mathcal{I} - gH, \quad (\text{D22})$$

Equation (72) gives

$$D_t\delta\mathcal{I} + \alpha\delta\mathcal{I} = -D_t(gH) = -\dot{g}H - g\dot{H}. \quad (\text{D23})$$

The first source moves the target along the Gibbs direction. The second contains a transverse pair component. With the instantaneous oscillator operator a_t ,

$$\dot{H} = \frac{\dot{\omega}}{2} (a_t^2 + a_t^{\dagger 2} + 2a_t^{\dagger}a_t + 1). \quad (\text{D24})$$

Thus varying T_{bath} at fixed ω does not by itself generate squeezing, whereas varying ω generically does. In this local description, the Hamiltonian drive generates the transverse deformation and the bath relaxes it.

5. Bath-correlation and secular conditions

For a linear interaction proportional to x , the system operator in the invariant basis has a matrix-element prefactor

$$A(t) = \sqrt{\frac{g_-(t)}{2\omega_{\mathcal{I}}(t)}}. \quad (\text{D25})$$

Define

$$\Delta_{\mathcal{I}} = \frac{g\omega_{\text{eff}}}{\omega_{\mathcal{I}}} - 1. \quad (\text{D26})$$

Then,

$$\frac{\dot{\omega}_{\mathcal{I}}}{\omega_{\mathcal{I}}} = \alpha \Delta_{\mathcal{I}}, \quad \tau_{\mathcal{I}} = \frac{1}{\alpha |\Delta_{\mathcal{I}}|}. \quad (\text{D27})$$

A local Born-Markov construction requires the resonant data to vary negligibly during the bath correlation time:

$$\tau_B \ll \min(\alpha^{-1}, \tau_{\mathcal{I}}, \tau_A), \quad \tau_A = \left| \frac{\dot{A}}{\ddot{A}} \right|. \quad (\text{D28})$$

The amplitude variation is

$$\frac{\dot{A}}{A} = -\frac{g_0}{g_-} - \frac{\alpha}{2} \left[\frac{g_- - g}{g_-} + \Delta_{\mathcal{I}} \right]. \quad (\text{D29})$$

Near the bath target in a weak-squeezing regime, $|g_0/g_-| = O(\alpha)$, so $\tau_A = O(\alpha^{-1})$ and the condition reduces parametrically to the ordinary weak-coupling Markov hierarchy. Under strong parametric driving, however, the g_0/g_- term must be checked separately.

If a secular approximation is also imposed, the coarse-graining time must obey

$$\max(\tau_B, \omega_{\mathcal{I}}^{-1}) \ll \Delta t \ll \min(\alpha^{-1}, \tau_{\mathcal{I}}, \tau_A). \quad (\text{D30})$$

Outside this window, a microscopic driven-bath treatment may require different jump operators or a nonlocal memory kernel.

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