

Erratum: Quantum advantage from negativity of work quasiprobability distributions [APS Open Sci. 1, 000093 (2026)]

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There is an omission in Sec. V. Here, the asymptotic formulas given in Eqs. (43), (44), and (46) hold for $k \rightarrow \infty$ as $N \rightarrow \infty$, so that $s/N \rightarrow 0$ as $N \rightarrow \infty$. Actually, to be precise, the example given in Sec. V does not belong to the class $\mathcal{C}$ when $s \neq 0$. Hence, we redefine the class $\mathcal{C}$ so that there can be an eventual finite number (smaller than k) of operators v_X in Eq. (17) having $m \lesssim r^\nu$, while the other operators v_X have $m \sim r^\nu$. Then, of course, all the results, and thus theorem 1, remain unchanged, and the example given in Sec. V belongs to the class $\mathcal{C}$ even if $s \neq 0$.

Furthermore, in Appendix A, we discuss the implications of an eventual violation of Eqs. (A1) and (A2), concerning the works w , w_{t_1} , and w_τ performed in the time intervals $[t_1, \tau]$, $[0, t_1]$, and $[0, \tau]$, respectively. Actually, further investigations show that Eqs. (A1) and (A2), in general, are always satisfied. This trivially follows from the Robertson-Schrödinger uncertainty relation

$$\Delta_\rho^2(A)\Delta_\rho^2(B) \geq \frac{1}{4}|\langle[A, B]\rangle_\rho|^2 + |\text{Cov}_\rho(A, B)|^2, \quad (1)$$

where $\langle A \rangle_\rho = \text{Tr}\{\rho A\}$, $\Delta_\rho^2(A) = \langle A^2 \rangle_\rho - \langle A \rangle_\rho^2$, and $\text{Cov}_\rho(A, B) = \frac{1}{2}\langle\{A, B\}\rangle_\rho - \langle A \rangle_\rho \langle B \rangle_\rho$. In detail, $\sigma_{t_1}^2 = \Delta_\rho^2(A)$, $\sigma_\tau^2 = \Delta_\rho^2(B)$, and $\langle w_\tau w_{t_1} \rangle - \langle w_\tau \rangle \langle w_{t_1} \rangle = \text{Cov}_\rho(A, B)$, with $\rho = \rho(0)$, $A = U_{t_1,0}^\dagger H(t_1) U_{t_1,0} - H(0)$, and $B = U_{\tau,0}^\dagger H(\tau) U_{\tau,0} - H(0)$, so that $|\langle w_\tau w_{t_1} \rangle - \langle w_\tau \rangle \langle w_{t_1} \rangle| \leq \sigma_\tau \sigma_{t_1}$; i.e., Eq. (A1), and thus Eq. (A2), is always satisfied. Then, instead of Eqs. (A1) and (A2), which cannot be considered as Leggett-Garg inequalities, we must consider different inequalities. These inequalities, which we will introduce in the following, are still satisfied for the charging process under consideration, so that the conclusions remain unchanged. To do this, following Ref. [1], we note that the assumption that the work is a macrorealistic variable implies the existence of a probability distribution $p(w_\tau, w_{t_1})$ of the works w_τ and w_{t_1} , which reads

$$p(w_\tau, w_{t_1}) = \sum_{k,n,i} p_{kni} \delta(w_\tau - E_k^\tau + E_i^0) \delta(w_{t_1} - E_n^{t_1} + E_i^0), \quad (2)$$

with $p_{kni} \geq 0$ and $\sum_{k,n,i} p_{kni} = 1$, since macrorealism means that any physical observable (in our case the energy) takes on well-defined values at all points in time throughout the process and it is possible to measure its value without changing the subsequent evolution of the system. In this case, we obtain the probability distribution $p(w)$ of the work w , $p(w) = \int \delta(w - w_\tau + w_{t_1}) p(w_\tau, w_{t_1}) dw_\tau dw_{t_1}$. Then, if there exists a probability distribution $p(w_\tau, w_{t_1})$ as in Eq. (2), the inequalities

$$W_-(a, b) \leq \langle (w_\tau - a)(w_{t_1} - b) \rangle \leq W_+(a, b), \quad (3)$$

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where the average is calculated with respect to $p(w_\tau, w_{t_1})$, or equivalently,

$$2W_-(a, b) \leq \sigma_\tau^2 + \sigma_{t_1}^2 - \sigma^2 + 2(\langle w_\tau \rangle - a)(\langle w_{t_1} \rangle - b) \leq 2W_+(a, b), \quad (4)$$

are satisfied for any values of a and b , where a and b are arbitrary real quantities (having dimensions of energy), and we define

$$W_-(a, b) = \min_{p(w_\tau, w_{t_1}) \geq 0} \langle (w_\tau - a)(w_{t_1} - b) \rangle, \quad W_+(a, b) = \max_{p(w_\tau, w_{t_1}) \geq 0} \langle (w_\tau - a)(w_{t_1} - b) \rangle \quad (5)$$

and the variances $\sigma_t^2 = \langle w_t^2 \rangle - \langle w_t \rangle^2$ for $t = t_1, \tau$ and $\sigma^2 = \langle w^2 \rangle - \langle w \rangle^2$. The maximum and minimum in Eq. (5) are calculated over the probability distributions of the form of Eq. (2). In particular, when the averages of the work are calculated with respect to the quasiprobability distribution $p_q(w)$, the inequalities in Eqs. (3) and (4) do not depend on q . In general, if these inequalities are violated for certain values of the variances σ^2 , $\sigma_{t_1}^2$, and σ_τ^2 , and the mean values $\langle w_{t_1} \rangle$ and $\langle w_\tau \rangle$, then there is no probability distribution $p(w_\tau, w_{t_1})$ as in Eq. (2) capable of reproducing the same values for the variances σ^2 , $\sigma_{t_1}^2$, and σ_τ^2 , and the mean values $\langle w_{t_1} \rangle$ and $\langle w_\tau \rangle$, and thus the macrorealism assumption is broken. For the charging process under consideration in Sec. III A, for $q = 0$ we get

$$p(w_\tau, w_{t_1}) = \frac{1}{2} \delta(w_\tau - E_1 + E_0) (\delta(w_{t_1} - E_+ + E_0) + \delta(w_{t_1} - E_- + E_0)), \quad (6)$$

then the inequalities in Eqs. (3) and (4) are trivially satisfied, since $p(w_\tau, w_{t_1})$ is of the form of Eq. (2). Finally, we note that, in general, these inequalities can be violated for some values of a and b if there is negativity, differently from Eqs. (A1) and (A2), which are always satisfied. To explicitly show this, we focus on a two-level system with energies $E_2^t = \epsilon/2$ and $E_1^t = -\epsilon/2$. For $a = b = 0$, for a probability distribution $p(w_\tau, w_{t_1})$ of the form of Eq. (2), we get $\langle w_\tau w_{t_1} \rangle = (p_{221} + p_{112})\epsilon^2$, then $W_-(0, 0) = 0$ and $W_+(0, 0) = \epsilon^2$. However, if there is negativity, we can achieve a process with a quasiprobability distribution for $q = 0$ of the form of Eq. (2) with $p_{221} + p_{112} < 0$ and thus $\langle w_\tau w_{t_1} \rangle < 0$, which violates the lower bound in Eqs. (3) and (4) for $a = b = 0$.

[1] H. J. D. Miller and J. Anders, Leggett-Garg inequalities for quantum fluctuating work, [Entropy](#) **20**, 200 (2018).