

Uncertainty relations via conditional expectation and variance

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The uncertainty principle constitutes a cornerstone of quantum mechanics. However, its conventional formulations do not adequately describe uncertainty in scenarios involving conditional measurement. In this work, we propose uncertainty relations based on conditional variance and conditional expectation. We define the conditional expectation operator $E(\hat{A}|\hat{B})$, its dual operator $E(\hat{B}|\hat{A})$, and a corresponding conditional covariance term $\mathcal{R}_{\text{corr}}$. We demonstrate that even for noncommuting observables $\hat{A}$ and $\hat{B}$, their conditional expectation operators can possess a nonvanishing commutator. This leads to a strengthened uncertainty relation applicable to both pure and mixed states. In addition to the standard term arising from the noncommutativity of the observables themselves, our relation incorporates three distinct contributions: one from the noncommutativity of the conditional expectations, one from quantum coherence, and one from the conditional covariance. These contributions collectively yield a significantly tighter lower bound. Numerical simulations verify the generality and enhanced tightness of the proposed relation. Our results provide a refined theoretical tool for the joint estimation of multiple observables in composite quantum systems, with potential applications in quantum precision metrology and secure quantum key distribution.

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I. INTRODUCTION

The quantum uncertainty principle constitutes a cornerstone of quantum theory. Over the past century, its mathematical formulation has evolved from Heisenberg's original ideas into a rigorous framework founded upon statistical variance [1]. The pursuit of a precise mathematical statement began with Kennard in 1927 [2], who furnished the first rigorous proof for the archetypal conjugate pair, position $\hat{x}$ and momentum $\hat{p}$, establishing the inequality

$$\Delta \hat{x}^2 \Delta \hat{p}^2 \geq \frac{\hbar^2}{4}, \quad (1)$$

for any quantum state. This was decisively generalized by Robertson in 1929 [3], who derived the famous relation for any two Hermitian operators $\hat{A}$ and $\hat{B}$:

$$\Delta \hat{A}^2 \Delta \hat{B}^2 \geq \frac{|\langle [\hat{A}, \hat{B}] \rangle|^2}{4}. \quad (2)$$

This inequality rooted the ultimate limit of measurement precision in the very algebraic structure of the theory—the noncommutativity of observables as captured by the commutator. Shortly

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after, Schrödinger supplemented the expression with an anticommutator term, yielding the tighter Robertson-Schrödinger inequality [4]:

$$\Delta\hat{A}^2\Delta\hat{B}^2 \geq \frac{|[\hat{A}, \hat{B}]|^2}{4} + \frac{|\langle\{\hat{A}, \hat{B}\}\rangle - 2\langle\hat{A}\rangle\langle\hat{B}\rangle|^2}{4}, \quad (3)$$

which also includes how $\hat{A}$ and $\hat{B}$ relate in a given state.

A well-known limitation of this standard product form is the trivial-bound problem: If the state approaches an eigenstate of either $\hat{A}$ or $\hat{B}$, the right-hand side can vanish, leading to a trivial result. Consequently, even for noncommuting observables, the product-form relation may fail to reveal their intrinsic incompatibility—a significant drawback for quantum-information applications that require reliable witnesses of quantumness.

To overcome this issue, attention shifted to the sum of variances, $\Delta\hat{A}^2 + \Delta\hat{B}^2$. A critical breakthrough in this sum-form paradigm was achieved by Maccone and Pati in 2014 [5]. They proved that for any two incompatible observables, a state-dependent uncertainty relation

$$\Delta\hat{A}^2 + \Delta\hat{B}^2 \geq \pm i\langle[\hat{A}, \hat{B}]\rangle + |\langle\psi|\hat{A} \pm i\hat{B}|\psi^\perp\rangle|^2 \quad (4)$$

holds for pure states, where $|\psi^\perp\rangle$ is any state orthogonal to the system state $|\psi\rangle$. The lower bound in this inequality is guaranteed non-negative. Moreover, for noncommuting observables, one can always choose $|\psi^\perp\rangle$ such that the bound is strictly positive, thereby conclusively resolving the trivial-bound problem. This result was later extended to mixed states. For a mixed state $\hat{\rho} = \sum_j p_j |\psi_j\rangle\langle\psi_j|$, if there exists a state $|\psi^\perp\rangle$ orthogonal to all constituent pure states $|\psi_j\rangle$, the following relation holds:

$$\Delta\hat{A}^2 + \Delta\hat{B}^2 \geq \mp i\langle[\hat{A}, \hat{B}]\rangle + \langle(-\hat{A} \pm i\hat{B})|\psi^\perp\rangle\langle\psi^\perp|(-\hat{A} \mp i\hat{B})\rangle. \quad (5)$$

These sum-form relations capture more fundamentally the irreducible “total uncertainty” imposed by noncommutativity. They have exhibited superior performance over the product form in diverse applications, including entanglement detection and the establishment of quantum precision limits in metrology.

Up to now, the variance-based uncertainty theories have formed a comprehensive multilayered research system, and numerous scholars have carried out diversified extensions under the premise of global independent measurements. Luo decomposed variance into classical and quantum components via skew information and constructed uncertainty criteria applicable to mixed states [6]. Li and Qiao further established a monotonic equivalence transformation between variance and Rényi entropy, revealing their intrinsic information-theoretic consistency [7]. Using the Bloch-vector representation, they also derived state-independent uncertainty relations for multiple observables [8], while Shirokov derives generalized uncertainty relations for three or more observables from generalized Cauchy inequalities, which supply extra constraints on pairwise correlation coefficients absent in pairwise Schrödinger uncertainty relations [9]. Generally, such three-observable uncertainty relations can be derived and generalized from the Cauchy-Schwarz inequality [10,11]. Xiao *et al.* put forward a quantitative description of uncertainty intervals using partial Cauchy-Schwarz inequalities [12]. Song *et al.* introduced the G-metric formalism to extend uncertainty relations to $\mathcal{PT}$ -symmetric and $\mathcal{PT}$ -broken non-Hermitian quantum systems [13]. Zheng *et al.* derived tighter lower bounds and second-moment uncertainty constraints for non-Hermitian operators by introducing generalized incompatible auxiliary operators [14]. For continuous-variable and many-body systems, Marian and Marian extended the Einstein-Podolsky-Rosen (EPR) gedanken experiment to multimode continuous-variable setups to quantify long-range quantum correlations [15], while Xu *et al.* investigated uncertainty dynamics in many-body quantum control systems and clarified how entanglement and mixing degree shape variance bounds [16]. Abd-Rabbou and Qiao obtained bidirectional tight uncertainty bounds using the Maligranda inequality [17]. Kimura’s group successively proposed new constraints beyond the Robertson-Schrödinger formalism as well as optimal Robertson-type relations whose coefficients are solely determined by state eigenvalues,

greatly strengthening bounds for mixed states [18,19]. Zhao *et al.* defined symmetrized variance for diagonal observables in pure states, decomposed total variance into observable- and state-dependent components, and derived basis-independent intrinsic quantum quantities [20].

Notably, the previous works ignore the conditional correlations of measurement outcomes and fail to describe sequential conditional measurement protocols. In 2018, Sazim *et al.* developed a variance-algebra-based conditional uncertainty framework to derive inequalities and construct entanglement and quantum-steering criteria. However, its conditional variance originates from fluctuations of linearly superposed observables and contradicts practical sequential-measurement physics [21].

Furthermore, Gour *et al.* established measure-independent conditional majorization theory to obtain universal conditional uncertainty constraints for bipartite systems. Yet it has no operator-level optimal-estimation models and cannot reflect uncertainty limits originating from noncommuting operators [22].

Despite these advances, a significant gap remains: Existing forms describe limits for separate, total measurements. Real-world quantum tasks—such as weak measurement or the control of coupled systems—often rely on conditional measurements, where the outcome of one measurement ($\hat{B}$) informs the estimation or control of another ($\hat{A}$). Conventional variance $\Delta\hat{A}^2$ quantifies the total spread, but does not reflect the uncertainty remaining in $\hat{A}$ once $\hat{B}$ is known. Current theory lacks a satisfactory description of this conditional uncertainty, rendering traditional relations inadequate for analyzing complex modern protocols.

To bridge this gap, we introduce a set of methods that utilize conditional variance and conditional expectation to depict the uncertain relations. Its central idea is to decompose the traditional global variance into the sum of “conditional variance” and “conditional-expectation fluctuation,” thereby disentangling the mutual information and correlation structure between measurements. Following the formulation pioneered by Maccone and Pati, we derive a family of uncertainty relations for the sum of conditional variances, applicable to both pure and mixed states, and demonstrate through numerical experiments that they provide tighter, more informative bounds than conventional alternatives.

II. UNCERTAINTY RELATIONS BASED ON THE SUM OF CONDITIONAL VARIANCES

To overcome the limitations inherent in global uncertainty formulations, we develop a theoretical framework grounded in the concept of conditional expectation. This approach provides a natural tool for characterizing uncertainty in scenarios involving sequential or conditional measurements. The notion of quantum conditional expectation, with roots in the work of Holevo on quantum measurement and estimation theory [23], serves as our starting point. Here, we formulate a well-defined conditional expectation operator for a pair of noncommuting observables $\hat{A}$ and $\hat{B}$ using their spectral decompositions.

For an observable $\hat{B}$ with discrete spectral decomposition $\hat{B} = \sum_j b_j |b_j\rangle\langle b_j|$, where $\{|b_j\rangle\}$ forms a complete orthonormal basis, the conditional expectation operator of observable $\hat{A}$ given $\hat{B}$ is defined as

$$E(\hat{A}|\hat{B}) := \sum_j \langle b_j|\hat{A}|b_j\rangle |b_j\rangle\langle b_j|. \quad (6)$$

Physically, if an ideal measurement of $\hat{B}$ yields outcome b_j , the postmeasurement state is $|b_j\rangle$. The value $\langle b_j|\hat{A}|b_j\rangle$ is then the optimal (minimum mean-square-error) estimate of $\hat{A}$ based solely on that outcome. Thus, $E(\hat{A}|\hat{B})$ encodes, diagonally in the $\hat{B}$ basis, the best estimate of $\hat{A}$ after a perfect $\hat{B}$ measurement. In a symmetric manner, we define the dual operator $E(\hat{A}|\hat{B})$ using the eigenbasis of $\hat{A}$.

A central question is whether these two estimation operators commute. Their commutator

$$[E(\hat{A}|\hat{B}), E(\hat{B}|\hat{A})] = E(\hat{A}|\hat{B})E(\hat{B}|\hat{A}) - E(\hat{B}|\hat{A})E(\hat{A}|\hat{B}) \quad (7)$$

need not vanish even when the original observables are noncommuting. This reflects a deeper incompatibility: The optimal estimation of $\hat{A}$ given full information about $\hat{B}$ and the optimal estimation of $\hat{B}$ given full information about $\hat{A}$ are themselves mutually exclusive procedures, because each relies on a different measurement basis. The nonzero commutator in Eq. (7) quantifies this informational exclusivity, revealing a hierarchical layer of quantum limitations beyond the usual Heisenberg principle [24].

From the conditional expectation, we naturally define the conditional variance:

$$(\Delta\hat{A}|\hat{B})^2 = \langle(\hat{A} - E(\hat{A}|\hat{B}))^2\rangle, \quad (8)$$

which measures the residual uncertainty of $\hat{A}$ after the outcome of a $\hat{B}$ measurement is taken into account. Unlike the ordinary variance $\langle(\hat{A} - \langle\hat{A}\rangle)^2\rangle$, the conditional variance isolates the uncertainty that cannot be reduced by knowledge of $\hat{B}$ within the chosen measurement context [25].

To capture correlations between the estimation errors of $\hat{A}$ and $\hat{B}$ in this conditional setting, we introduce a conditional covariance quantity

$$\text{cov}(\hat{A}|\hat{B}, \hat{B}|\hat{A}) := \frac{1}{2}\langle(\hat{A} - E(\hat{A}|\hat{B}))(\hat{B} - E(\hat{B}|\hat{A}))\rangle. \quad (9)$$

This definition employs the square of the symmetrized product of the two estimation errors, providing a second-moment measure of how the errors covary when each observable is optimally estimated conditioned on the other. In quantum mechanics, measurement backaction and noncommutativity often induce such correlated fluctuations; the quantity in Eq. (9) systematically accounts for them.

Equipped with these constructs, we can now state our main uncertainty relations for the sum of conditional variances.

Theorem 1. For pure states, we have derived the uncertainty relation in the variance-sum form

$$\begin{aligned} (\Delta\hat{A}|\hat{B})^2 + (\Delta\hat{B}|\hat{A})^2 &\geq \pm i\langle[\hat{A}, \hat{B}]\rangle \pm i\langle[E(\hat{A}|\hat{B}), E(\hat{B}|\hat{A})]\rangle + |\langle\psi|(\hat{A} \pm i\hat{B})|\psi^\perp\rangle \\ &\quad - \langle\psi|(E(\hat{A}|\hat{B}) \pm E(\hat{B}|\hat{A}))|\psi^\perp\rangle|^2 + \frac{|\text{cov}(\hat{A}|\hat{B}, \hat{B}|\hat{A})|^2}{(\Delta\hat{A}|\hat{B})^2 + (\Delta\hat{B}|\hat{A})^2}. \end{aligned} \quad (10)$$

Proof. Now, within the context of two-dimensional pure states, we substitute Eqs. (6) and (8) into Eq. (4) from Maccone and Pati to conduct the theoretical derivation.

Define the residual operator:

$$\hat{\mathcal{O}}_1 := \hat{A} - E(\hat{A}|\hat{B}), \quad \hat{\mathcal{O}}_2 := \hat{B} - E(\hat{B}|\hat{A}). \quad (11)$$

Substituting into Eq. (4) yields the following expression:

$$\Delta\hat{\mathcal{O}}_1^2 + \Delta\hat{\mathcal{O}}_2^2 \geq \pm i\langle[\hat{\mathcal{O}}_1, \hat{\mathcal{O}}_2]\rangle + |\langle\psi|\hat{\mathcal{O}}_1 \pm i\hat{\mathcal{O}}_2|\psi^\perp\rangle|^2. \quad (12)$$

Expanding the commutator,

$$\Delta\hat{\mathcal{O}}_1^2 + \Delta\hat{\mathcal{O}}_2^2 = \langle\hat{\mathcal{O}}_1^2\rangle + \langle\hat{\mathcal{O}}_2^2\rangle = \langle(\hat{A} - E(\hat{A}|\hat{B}))^2\rangle + \langle(\hat{B} - E(\hat{B}|\hat{A}))^2\rangle = (\Delta\hat{A}|\hat{B})^2 + (\Delta\hat{B}|\hat{A})^2, \quad (13)$$

$$\langle[\hat{\mathcal{O}}_1, \hat{\mathcal{O}}_2]\rangle = \langle[\hat{A}, \hat{B}]\rangle + \langle[E(\hat{A}|\hat{B}), E(\hat{B}|\hat{A})]\rangle - \langle[\hat{A}, E(\hat{B}|\hat{A})]\rangle - \langle[E(\hat{A}|\hat{B}), \hat{B}]\rangle. \quad (14)$$

Because $E(\hat{B}|\hat{A})$ is a function of the projectors $|a_k\rangle\langle a_k|$ of $\hat{A}$, it commutes with $\hat{A}$; similarly, $E(\hat{A}|\hat{B})$ commutes with $\hat{B}$. Hence, the last two terms vanish, leaving

$$\langle[\hat{\mathcal{O}}_1, \hat{\mathcal{O}}_2]\rangle = \langle[\hat{A}, \hat{B}]\rangle + \langle[E(\hat{A}|\hat{B}), E(\hat{B}|\hat{A})]\rangle. \quad (15)$$

The interference term becomes

$$\begin{aligned}\langle \psi | \hat{\mathcal{O}}_1 \pm i \hat{\mathcal{O}}_2 | \psi^\perp \rangle &= \langle \psi | (\hat{A} \pm i \hat{B} - E(\hat{A}|\hat{B}) \mp E(\hat{B}|\hat{A})) | \psi^\perp \rangle \\ &= \langle \psi | (\hat{A} \pm i \hat{B}) | \psi^\perp \rangle - \langle \psi | (E(\hat{A}|\hat{B}) \pm E(\hat{B}|\hat{A})) | \psi^\perp \rangle.\end{aligned}\quad (16)$$

Finally, the first uncertainty relation based on conditional variance and in pure-state condition is obtained:

$$\begin{aligned}(\Delta \hat{A}|\hat{B})^2 + (\Delta \hat{B}|\hat{A})^2 &\geq \pm i \langle [\hat{A}, \hat{B}] \rangle \pm i \langle [E(\hat{A}|\hat{B}), E(\hat{B}|\hat{A})] \rangle + |\langle \psi | (\hat{A} \pm i \hat{B}) | \psi^\perp \rangle \\ &\quad - \langle \psi | (E(\hat{A}|\hat{B}) \pm E(\hat{B}|\hat{A})) | \psi^\perp \rangle|^2.\end{aligned}\quad (17)$$

The lower bound presented in Eq. (17) consists of three items related to the noncommutativity of the observables themselves, the noncommutativity of the conditional expectation operators, and quantum coherence, respectively. However, this bound only characterizes the independent fluctuation constraints of the two conditional estimation errors separately, and does not account for the contribution of correlated fluctuation effects between the two sets of errors to the total uncertainty. In practical conditional measurement scenarios, the noncommutative nature of observables and measurement backaction render the errors of the two conditional estimates nonindependent: When the conditional estimation of $\hat{A}$ based on the measurement outcomes of $\hat{B}$ exhibits a large deviation, the deviation of the conditional estimation of $\hat{B}$ based on the measurement outcomes of $\hat{A}$ also exhibits systematic correlated variations. This second-order correlation between errors is an inherent component of quantum uncertainty and imposes an additional constraint on the total conditional uncertainty. Since Eq. (17) fails to capture this effect, its lower bound might be further tightened.

To quantify the correlation strength of the aforementioned conditional estimation errors, we define the conditional covariance as

$$\text{cov}(\hat{A}|\hat{B}, \hat{B}|\hat{A}) = \frac{1}{2} \langle (\hat{\mathcal{O}}_1 \hat{\mathcal{O}}_2)^2 \rangle. \quad (18)$$

It characterizes the second-order amplitude of joint fluctuations of the two residual operators and physically corresponds to the probability weight of simultaneous large deviations in the two conditional estimates.

On this basis, we construct the covariance correction term

$$\mathcal{R}_{\text{corr}} = \frac{|\text{cov}(\hat{A}|\hat{B}, \hat{B}|\hat{A})|^2}{(\Delta \hat{A}|\hat{B})^2 + (\Delta \hat{B}|\hat{A})^2} = \frac{|\text{cov}|^2}{S}, \quad (19)$$

where $S = (\Delta \hat{A}|\hat{B})^2 + (\Delta \hat{B}|\hat{A})^2$ denotes the sum of conditional variances and $\text{cov} = \text{cov}(\hat{A}|\hat{B}, \hat{B}|\hat{A})$. Physically, this correction term represents the ratio of the conditional error correlation strength to the total conditional uncertainty, corresponding to the additional uncertainty constraint contributed by error correlation.

Due to $|\text{cov}|^2 \geq 0$ and $S \geq 0$, one can obtain the non-negativity of $\mathcal{R}_{\text{corr}}$ from Eq. (19). For the special case with $S = 0$, it implies error-free conditional estimation for both observables. In this case, the error correlation vanishes simultaneously and the covariance equals zero, so $\mathcal{R}_{\text{corr}}$ can be consistently defined as 0 via a limiting process. Therefore, for any physical quantum state, one can obtain

$$\mathcal{R}_{\text{corr}} \geq 0. \quad (20)$$

This non-negativity directly guarantees that adding $\mathcal{R}_{\text{corr}}$ to the right-hand side of Eq. (17) necessarily yields a tighter lower bound.

From physical intuition, the additional constraint introduced by error correlation cannot exceed the total conditional uncertainty itself; otherwise, it would contradict basic physical reasoning. We analyze its boundedness via standard operator moment inequalities. First, by the quantum Cauchy-Schwarz inequality, we have

$$|\langle \hat{\mathcal{O}}_1^2 \hat{\mathcal{O}}_2^2 \rangle|^2 \leq \langle \hat{\mathcal{O}}_1^4 \rangle \langle \hat{\mathcal{O}}_2^4 \rangle, \quad (21)$$

for arbitrary Hermitian operators. Combined with the off-diagonal structure of the residual operators—each contains only off-diagonal elements in the eigenbasis of the other observable—it can be further shown that

$$|(\langle \hat{O}_1 \hat{O}_2 \rangle)| \leq |\langle \hat{O}_1^2 \hat{O}_2^2 \rangle|. \quad (22)$$

Thus, the constraint in Eq. (21) can be transferred to the product term corresponding to the conditional covariance.

Second, by the arithmetic-geometric mean inequality, one attains

$$\langle \hat{O}_1^2 \rangle \langle \hat{O}_2^2 \rangle \leq \left(\frac{\langle \hat{O}_1^2 \rangle + \langle \hat{O}_2^2 \rangle}{2} \right)^2 = \frac{S^2}{4}, \quad (23)$$

for non-negative conditional variances.

Specifically, in two-dimensional pure-state systems, traceless Hermitian operators satisfy $\langle \hat{O}_i^4 \rangle = \langle \hat{O}_i^2 \rangle^2$ for $i = 1, 2$. Thereby, one can analytically derive that

$$S - \frac{|\text{cov}|^2}{S} = \frac{S^2 - |\text{cov}|^2}{S} \geq 0, \quad (24)$$

which is equivalent to $\mathcal{R}_{\text{corr}} \leq S$. This result indicates that the correction term from error correlation does not exceed the total conditional uncertainty in two-dimensional systems, verifying that the correction term has a reasonable magnitude and is physically self-consistent.

For higher-dimensional pure states and mixed-state systems, since the fourth-order moments of Hermitian operators satisfy $\langle \hat{O}_i^4 \rangle \geq \langle \hat{O}_i^2 \rangle^2$, the aforementioned concise scaling approach is no longer applicable, and a universal analytical upper bound cannot be obtained via simple moment inequalities. Nevertheless, all orders of moments of operators in finite-dimensional Hilbert spaces are finite, ensuring that $\mathcal{R}_{\text{corr}}$ always remains a well-defined finite quantity with clear physical meaning. As verified by subsequent large-scale numerical simulations covering the full parameter space, the inequality incorporating the $\mathcal{R}_{\text{corr}}$ correction term holds strictly for any physical quantum state of arbitrary dimension, and the tightness of the lower bound is significantly improved compared with the fundamental inequality [Eq. (17)].

It quantifies the strength of amplitude correlations between $\hat{O}_1$ and $\hat{O}_2$, the probability of large errors occurring simultaneously. This is crucial for capturing quantum non-Gaussianity and the correlated fluctuations induced by entanglement. Also, it significantly narrows the gap with the actual uncertainty and yields a more accurate quantum limit. By adding a $\mathcal{R}_{\text{corr}}$ term to Eq. (17), we obtain the second uncertainty relation [Eq. (10)] based on the sum of conditional variances under the condition of pure states.

Now, let us analyze Eq. (10); the term $i[\hat{A}, \hat{B}]$ represents the contribution from the standard quantum noncommutativity of the observables. This fundamental limit, inherent in any pair of noncommuting operators, persists as a core source of uncertainty even within the conditional estimation framework. The second term, $i[E(\hat{A}|\hat{B}), E(\hat{B}|\hat{A})]$ arises from the noncommutativity of the conditional-expectation operators. It reflects a deeper, information-theoretic incompatibility: The optimal estimation of $\hat{A}$ conditioned on a perfect measurement of $\hat{B}$ and the optimal estimation of $\hat{B}$ conditioned on a perfect measurement of $\hat{A}$ constitute two mutually exclusive perspectives. This context-dependent noncommutativity introduces an additional layer of quantum uncertainty not captured by conventional relations. Next, the third contribution is the interference term, $|\langle \psi | (\hat{A} \pm i\hat{B}) | \psi^\perp \rangle - \langle \psi | (E(\hat{A}|\hat{B}) \pm E(\hat{B}|\hat{A})) | \psi^\perp \rangle|^2$, which quantifies the role of quantum coherence. Here, $\langle \psi | (\hat{A} \pm i\hat{B}) | \psi^\perp \rangle$ transition amplitude for the operator $\hat{A} \pm i\hat{B}$, encoding the system's intrinsic coherence. Its counterpart, $\langle \psi | (E(\hat{A}|\hat{B}) \pm E(\hat{B}|\hat{A})) | \psi^\perp \rangle$, represents the corresponding amplitude predicted solely by the conditional (i.e., effectively classical) expectations. Their difference thus measures the excess uncertainty stemming from genuine quantum coherence that cannot be inferred from the conditional data alone. This term vanishes only when the system's behavior is fully consistent with the conditional expectations, highlighting its purely quantum origin. Finally, the

conditional covariance term $\mathcal{R}_{\text{corr}}$ captures the contribution from correlations between the conditional fluctuations of $\hat{A}$ and $\hat{B}$. It quantifies how the errors in the two conditional estimates covary, adding a positive, state-dependent tightening to the bound. Alongside noncommutativity and coherence, these correlated conditional fluctuations form a distinct and significant source of uncertainty, particularly relevant in entangled or non-Gaussian states.

Theorem 2. Let $\hat{\rho} = \sum_j p_j |\psi_j\rangle\langle\psi_j|$ be a mixed state and choose a state $|\psi^\perp\rangle$ orthogonal to every constituent $|\psi_j\rangle$ (this requires the Hilbert-space dimension to be at least three). Then,

$$\begin{aligned} (\Delta\hat{A}|\hat{B})^2 + (\Delta\hat{B}|\hat{A})^2 \geq & \mp i\langle[\hat{A}, \hat{B}]\rangle \mp i\langle[E(\hat{A}|\hat{B}), E(\hat{B}|\hat{A})]\rangle + \langle((- \hat{A} \pm i\hat{B}) + (E(\hat{A}|\hat{B}) \mp iE(\hat{B}|\hat{A})))|\psi^\perp\rangle \\ & \times \langle\psi^\perp|((- \hat{A} \mp i\hat{B}) + (E(\hat{A}|\hat{B}) \pm iE(\hat{B}|\hat{A})))\rangle + \frac{|\text{cov}(\hat{A}|\hat{B}, \hat{B}|\hat{A})|^2}{(\Delta\hat{A}|\hat{B})^2 + (\Delta\hat{B}|\hat{A})^2}. \end{aligned} \quad (25)$$

Proof. The argument follows the same pattern as for Theorem 1, but starts from the mixed-state generalization of the Maccone-Pati sum relation in Eq. (5):

$$\hat{O}_1^2 + \hat{O}_2^2 \geq \mp i\langle[\hat{O}_1, \hat{O}_2]\rangle + \langle(-\hat{O}_1 \pm i\hat{O}_2)|\psi^\perp\rangle\langle\psi^\perp|(-\hat{O}_1 \mp i\hat{O}_2)\rangle + \mathcal{R}_{\text{corr}}. \quad (26)$$

Substituting the residual operators in Eq. (11) and adding the non-negative term $\mathcal{R}_{\text{corr}}$ defined in Eq. (19) yields, after expansion, we can obtain the inequality [Eq. (25)]. The necessity of an orthogonal state $|\psi^\perp\rangle$ that is orthogonal to all pure components of $\hat{\rho}$ restricts the validity to dimensions three or higher.

Now, we established Theorem 2 for mixed states with a nontrivial null space, based on the assumption of a common orthogonal state. While this theorem features a compact form and clear physical interpretation, its validity is constrained by the requirement that the density matrix has a nonempty null space. Specifically, for mixed states with $\text{rank}(\hat{\rho}) < d$ in a Hilbert space of dimension $d \geq 3$, a common orthogonal state $|\psi^\perp\rangle$ exists and Eq. (25) holds. However, for full-rank mixed states in two-dimensional (single-qubit) systems, the null space of the density matrix contains only the zero vector, so no nontrivial common orthogonal state can be found, and Theorem 2 does not apply directly.

To fill this gap in applicability, we construct a universal uncertainty relation valid for mixed states of arbitrary dimension and arbitrary rank, based on the pure-state result of Theorem 1 and the convex decomposition of quantum states. This result complements Theorem 2 in applicable scenarios, and together they provide full coverage from pure states to full-rank mixed states.

III. EXAMPLES

A. Two-dimensional arbitrary pure-state verification

We have established an uncertainty relation applicable to pure states, along with its extension to mixed states. For clarity, we first consider a pure state of a single-qubit system as an example. This demonstrates that Eq. (10) holds for arbitrary two-dimensional states and provides a tighter inequality than Eq. (17). Consider the Hermitian operators

$$\hat{A} = \hat{\sigma}_z, \quad \hat{B} = \cos\theta \hat{\sigma}_z + \sin\theta \hat{\sigma}_x. \quad (27)$$

We select an arbitrary quantum state and its orthogonal state:

$$|\psi\rangle = \cos\alpha |0\rangle + \sin\alpha |1\rangle, \quad |\psi^\perp\rangle = -\sin\alpha |0\rangle + \cos\alpha |1\rangle. \quad (28)$$

Within this setup, the proposed inequality can be verified for any state represented on the surface of the Bloch sphere [26]. From Figs. 1 and 2, it can be concluded that the left-hand side of Eq. (10) is always greater than the right-hand side on the Bloch sphere (for any pure state), and because the additional term $\mathcal{R}_{\text{corr}}$ was added in Eq. (10), the bound is tighter than the bound in Eq. (17).

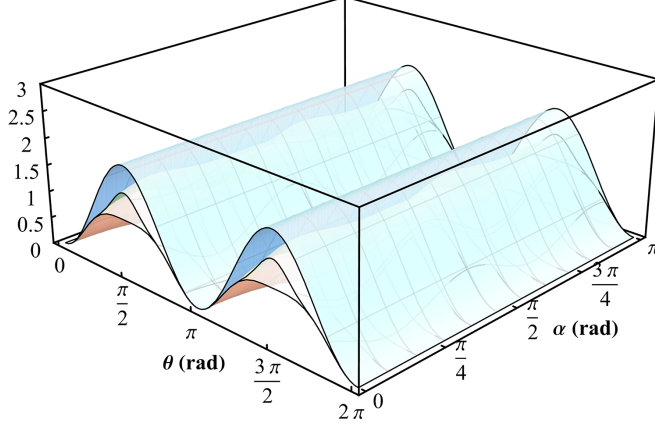


FIG. 1. Quantum expressions vs θ and α under the positive sign parameter configuration, the exact sum of conditional variances (blue surface) is compared with two theoretical lower bounds: one from Eq. (17) (orange surface) and the other from Eq. (10) (green surface). The bound including the $\mathcal{R}_{\text{corr}}$ term (green) lies closer to the exact value, demonstrating its role in tightening the uncertainty relation.

Calculations based on this example reveal that

$$[\hat{A}, \hat{B}] = \begin{pmatrix} 0 & 2 \sin \theta \\ -2 \sin \theta & 0 \end{pmatrix}, \quad [E(\hat{A}|\hat{B}), E(\hat{B}|\hat{A})] = \begin{pmatrix} 0 & -\frac{\sin \theta + \sin 3\theta}{2} \\ \frac{\sin \theta + \sin 3\theta}{2} & 0 \end{pmatrix}. \quad (29)$$

This illustrates that quantum incompatibility extends beyond a mere property of measurement outcomes; it is fundamentally embedded within the very processes and strategies of information acquisition. Adopting one probing strategy—for instance, measuring $\hat{B}$ first and then estimating $\hat{A}$ —inherently excludes or disrupts the effectiveness of an alternative strategy, such as measuring $\hat{A}$ first and then estimating $\hat{B}$. This remains true even when both strategies aim only for “optimal estimation” rather than exact values.

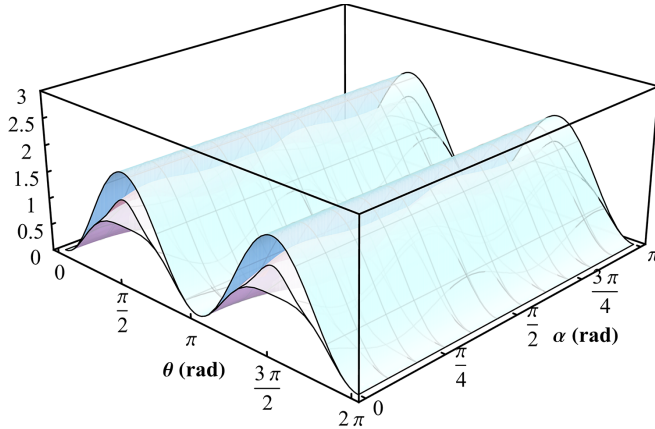


FIG. 2. Quantum expressions vs θ and α under the negative sign parameter configuration, the same comparison under the negative sign configuration, with the bounds from Eq. (17) (purple surface) and Eq. (10) (red surface). In both configurations, the enhanced bound remains valid over the entire Bloch sphere, confirming the universality of the relation for any single-qubit pure state.

This exclusivity constitutes the physical origin of the new term $i\langle[E(\hat{A}|\hat{B}), E(\hat{B}|\hat{A})]\rangle$ in our uncertainty relation. The term quantifies this deeper, information-theoretic form of incompatibility, resulting in a lower bound that is both tighter and more informative than those derived from conventional relations.

Furthermore, we analyze the difference between Eqs. (10) and (17), which is readily $\mathcal{R}_{\text{corr}}$. Substitute the observables $\hat{A} = \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\hat{B} = \cos \theta \hat{\sigma}_z + \sin \theta \hat{\sigma}_x$, and the arbitrary pure state $|\psi\rangle = \cos \alpha |0\rangle + \sin \alpha |1\rangle$ into $\mathcal{R}_{\text{corr}}$.

Substituting the derived analytical results, the numerator reads

$$|\text{cov}(\hat{A}|\hat{B}, \hat{B}|\hat{A})|^2 = \left(\frac{1}{2} \sin^4 \theta \cos 2\theta\right)^2 = \frac{1}{4} \sin^8 \theta \cos^2 2\theta, \quad (30)$$

and the denominator takes the form

$$(\Delta\hat{A}|\hat{B})^2 + (\Delta\hat{B}|\hat{A})^2 = 2 \sin^2 \theta. \quad (31)$$

Combining the numerator and denominator, we obtain the closed-form expression of the correlation tightening term:

$$\mathcal{R}_{\text{corr}} = \frac{\frac{1}{4} \sin^8 \theta \cos^2 2\theta}{2 \sin^2 \theta} = \frac{\sin^6 \theta \cos^2 2\theta}{8}. \quad (32)$$

It reveals that $\mathcal{R}_{\text{corr}}$ equals zero at $\theta = 0, \pi, \pi/4, 3\pi/4$ and reaches its maximum value $1/8$ at $\theta = \pi/2$. At $\theta = 0, \pi$, the observables $\hat{A}$ and $\hat{B}$ commute completely without intrinsic quantum incompatibility, so no correlation exists between conditional estimation errors and the covariance correction term vanishes. At $\theta = \pi/4, 3\pi/4$, the two observables remain noncommutative, yet the second-order cross correlation of two types of conditional estimation errors cancels out exactly, only eliminating the correction term while the fundamental uncertainty bound stays positive. When $\theta = \pi/2$, $\hat{A}$ and $\hat{B}$ are fully complementary conjugate Pauli operators, the conditional error correlation induced by measurement backaction becomes the strongest, and this correction term tightens the lower bound of the uncertainty relation to the greatest extent.

B. Three-dimensional mixed-state verification

For the mixed-state case described by Eq. (25), the two-dimensional Hilbert space is insufficient because an appropriate orthogonal state $|\psi^\perp\rangle$ must be chosen. We therefore consider mixed states within a three-dimensional system framework and systematically verify the validity of the inequality through the following two approaches.

We select the Hermitian operators

$$\hat{A} = \hat{S}_z, \quad \hat{B} = \cos \theta \hat{S}_z + \sin \theta \hat{S}_x, \quad (33)$$

where

$$\hat{S}_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \hat{S}_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad (34)$$

along with the density matrix and the orthogonal state

$$\hat{\rho} = 0.4 \times |0\rangle\langle 0| + 0.6 \times \frac{1}{2}(|0\rangle + |1\rangle)(\langle 0| + \langle 1|), \quad |\psi^\perp\rangle = |\hat{\rho}^\perp\rangle = |2\rangle. \quad (35)$$

Figure 3 illustrates the variation of the uncertainty relation specifically, the left-hand side (the actual sum of conditional variances) and the right-hand side (the theoretical lower bound)—as the parameter θ varies over the interval $[0, 2\pi]$. We found that no matter which symbol (positive sign or negative sign) is chosen, it does not affect the value of bound, and the difference between the left and right sides of the inequality is always greater than zero. The results demonstrate that the inequality holds strictly over the entire parameter range.

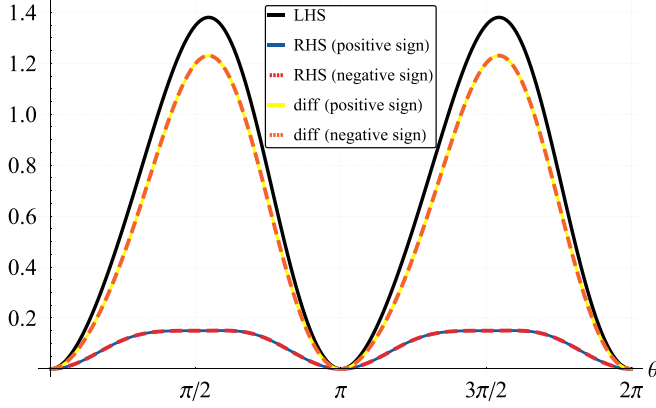


FIG. 3. Verification of the quantum uncertainty relation in special mixed states. A diagram illustrating the relations between the two sides of the inequality as a function of θ for a specific three-dimensional mixed state. The blue line and the red dotted line, respectively, reflect our bounds in the cases of positive and negative signs. The green line and the orange dotted lines indicate that the value of the left side minus the right side of the inequality is always greater than zero; that is, in this mixed state, the inequality always holds.

To examine whether the inequality holds universally, we perform a large-scale numerical verification: We discretize the interval $[0, 2\pi]$ into 10^4 uniform values of θ and generate 10^4 independent random rank-2 three-dimensional mixed states for each θ , leading to a total of 10^8 inequality evaluations. A reference state orthogonal to all pure components of each density matrix is constructed via singular value decomposition. We compute the difference between the left-hand and right-hand sides of the inequality strictly following the definitions of conditional expectation, conditional variance, and the covariance correction term proposed in this work. The statistical results from Fig. 4 show that the difference remains non-negative, supporting the validity of Eq. (25).

In practice, conventional uncertainty relations, such as the Robertson relation [3], often become trivial for mixed states: Their lower bound, $|\langle[\hat{A}, \hat{B}]\rangle|/2$, may approach zero and thus cease to provide physically informative constraints. In contrast, the conditional expectation commutator term $\langle[E(\hat{A}|\hat{B}), E(\hat{B}|\hat{A})]\rangle$ and the orthogonal-projection term introduced in our framework generally

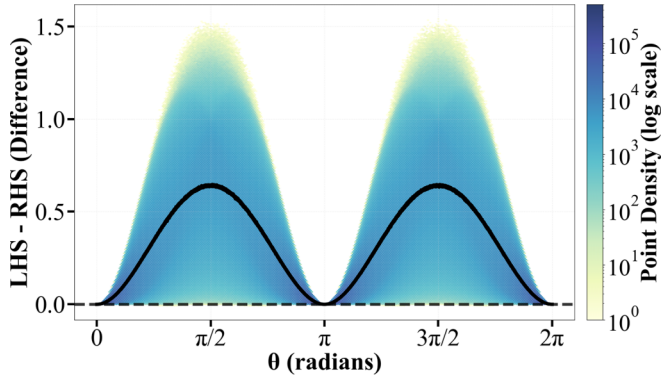


FIG. 4. Quantum inequality validation: Density scatter plot of difference vs θ , we conducted a total of 10^8 tests. The solid line represents the average change of the difference between the left and right sides of the inequality with respect to the θ angle under different random mixed states. The dotted line represents the 0 line. From the figure, it can be seen that all the differences are above the 0 line, serving to indirectly verify whether Eq. (25) is universally and consistently valid.

remain nonzero and contribute positively even for mixed states. Moreover, the newly defined covariance term inherently encodes information about state mixing. Therefore, when describing the intrinsic uncertainty of high-dimensional, open, or entangled quantum systems, the lower bound supplied by Eq. (26) is both tighter and more informative. It is robust [27] against state purity—an essential feature for real-world quantum systems [28], which are invariably subject to decoherence—and consequently offers more descriptive power for practical quantum scenarios.

IV. CONCLUSIONS

To summarize, this study investigates the uncertainty relations among multiple observables in multipartite systems and, for the first time, proposes a novel uncertainty framework based on conditional expectation and conditional variance, overcoming the limitations of traditional global variance analysis in conditional measurements and multipartite correlated systems. We systematically construct the conditional expectation operators $E(\hat{A}|\hat{B})$ and $E(\hat{B}|\hat{A})$, thoroughly explore their commutation structure and physical significance, and subsequently derive generalized uncertainty relations applicable to both pure and mixed states. The research reveals that even when $\hat{A}$ and $\hat{B}$ do not commute, their conditional expectation operators can still exhibit nontrivial commutation relations ($[E(\hat{A}|\hat{B}), E(\hat{B}|\hat{A})] \neq 0$). This reflects that information extraction processes based on different measurement contexts are themselves mutually exclusive, thereby deepening the understanding of the hierarchical structure of “measurement-information uncertainty”.

This work advances the understanding of quantum uncertainty relations. By decomposing the conventional global variance into the sum of conditional variance and the fluctuations of conditional expectations, the framework disentangles the mutual information and correlation structure inherent in quantum measurements. This provides a more nuanced characterization of uncertainty in the context of conditional measurements. Building upon the sum-form uncertainty relation introduced by Maccone and Pati, we strengthen the bound by incorporating a conditional covariance term $\mathcal{R}_{\text{corr}}$, which significantly enhances tightness and holds for arbitrary-dimensional pure and mixed states. To demonstrate broad applicability, numerical simulations—covering two-dimensional pure states, three-dimensional mixed states, and large-scale random states—confirm the universal validity and superiority of the proposed relation. Notably, it maintains a nontrivial lower bound even for mixed states, thereby overcoming the tendency of traditional Robertson relations to become trivial in such scenarios. The framework of conditional expectation and conditional variance established in this work can be naturally generalized to multiobservable scenarios. A mathematically rigorous extension can be straightforwardly realized by summing all pairwise uncertainty relations, while tighter formulations based on joint conditional expectation constitute a promising direction for future investigation. The research provides a more precise theoretical tool for the simultaneous estimation of multiple observables in multipartite quantum systems, with promising applications in practical quantum information tasks such as quantum precision measurement, quantum key distribution, entanglement detection, and quantum control. Future studies may further explore the potential of this framework in complex systems including non-Markovian processes [29–31], quantum networks [32–34], and distributed sensing [35,36].

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

- [1] W. Heisenberg, Über den anschaulichen Inhalt der quantentheoretischen Kinematik und Mechanik, *Z. Phys.* **43**, 172 (1927).
- [2] E. H. Kennard, Zur Quantenmechanik einfacher Bewegungstypen, *Z. Phys.* **44**, 326 (1927).
- [3] H. P. Robertson, The uncertainty principle, *Phys. Rev.* **34**, 163 (1929).
- [4] L. Skála, Internal structure of the Heisenberg and Robertson-Schrödinger uncertainty relations: Multidimensional generalization, *Phys. Rev. A* **88**, 042118 (2013).
- [5] L. Maccone and A. K. Pati, Stronger uncertainty relations for all incompatible observables, *Phys. Rev. Lett.* **113**, 260401 (2014).
- [6] S. Luo, Heisenberg uncertainty relation for mixed states, *Phys. Rev. A* **72**, 042110 (2005).
- [7] J.-L. Li and C.-F. Qiao, Equivalence theorem of uncertainty relations, *J. Phys. A: Math. Theor.* **50**, 03LT01 (2017).
- [8] J.-L. Li and C.-F. Qiao, Reformulating the quantum uncertainty relation, *Sci. Rep.* **5**, 12708 (2015).
- [9] M. I. Shirokov, Interpretation of uncertainty relations for three or more observables, [arXiv:quant-ph/0404165](https://arxiv.org/abs/quant-ph/0404165).
- [10] M. Arav, F. J. Hall, and Z. Li, A Cauchy–Schwarz inequality for triples of vectors, *Math. Inequal. Appl.* **11**, 629 (2008).
- [11] C. Lupu and D. Schwarz, Another look at some new Cauchy–Schwarz type inner product inequalities, *Appl. Math. Comput.* **231**, 463 (2014).
- [12] Y. Xiao, N. Jing, B. Yu, S.-M. Fei, and X. Li-Jost, Near-optimal variance-based uncertainty relations, *Front. Phys.* **10**, 846330 (2022).
- [13] X.-F. Song, Y.-F. Ren, S. Liu, X.-H. Chen, and Y. Turek, Stronger sum uncertainty relations for non-Hermitian operators, *Phys. Rev. A* **111**, 012221 (2025).
- [14] X. Zheng, S.-Q. Ma, G.-F. Zhang, H. Fan, and W.-M. Liu, Unified and exact framework for variance-based uncertainty relations, *Sci. Rep.* **10**, 150 (2020).
- [15] P. Marian and T. A. Marian, Einstein-Podolsky-Rosen uncertainty limits for bipartite multimode states, *Phys. Rev. A* **103**, 062224 (2021).
- [16] J. Xu, X. Zheng, A.-L. Ji, and G.-F. Zhang, Dynamics investigation of the quantum-control-assisted multipartite uncertainty relation in Heisenberg model with Dzyaloshinski-Moriya interaction, *Phys. Scr.* **98**, 065106 (2023).
- [17] M. Y. Abd-Rabbou and C.-F. Qiao, Strengthening the uncertainty and the reverse uncertainty relation limits, *Quantum Inf. Process.* **24**, 387 (2025).
- [18] G. Kimura, A. Mayumi, H. Ohno, J. Lee, and D. Chruściński, Beyond Robertson–Schrödinger: a general uncertainty relation unveiling hidden noncommutative trade-offs, *npj Quantum. Inf.* (2026), doi: [10.1038/s41534-026-01374-0](https://doi.org/10.1038/s41534-026-01374-0).
- [19] G. Kimura, A. Mayumi, and H. Yamashita, Tight generalization of Robertson-type uncertainty relations, [arXiv:2505.19861v2](https://arxiv.org/abs/2505.19861v2).
- [20] M.-J. Zhao, L. Zhang, and S.-M. Fei, Standard symmetrized variance with applications to coherence, uncertainty, and entanglement, *Phys. Rev. A* **106**, 012417 (2022).
- [21] S. Sazim, S. Adhikari, A. K. Pati, and P. Agrawal, Mutual uncertainty, conditional uncertainty, and strong subadditivity, *Phys. Rev. A* **98**, 032123 (2018).
- [22] G. Gour, A. Grudka, M. Horodecki, W. Kłobus, J. Łodyga, and V. Narasimhachar, Conditional uncertainty principle, *Phys. Rev. A* **97**, 042130 (2018).
- [23] A. S. Holevo, The standard statistical model of quantum mechanics, in *Statistical Structure of Quantum Theory*, Lecture Notes in Physics Monographs Vol. 67 (Springer, Berlin, 2001), pp. 1–32.

- [24] J. Oppenheim and S. Wehner, The uncertainty principle determines the nonlocality of quantum mechanics, [Science](#) **330**, 1072 (2010).
- [25] K. Pearson, Mathematical contributions to the theory of evolution—On a form of spurious correlation which may arise when indices are used in the measurement of organs, [Proc. R. Soc. Lond.](#) **60**, 489 (1897).
- [26] F. Bloch, Nuclear induction, [Phys. Rev.](#) **70**, 460 (1946).
- [27] G. Vidal and R. Tarrach, Robustness of entanglement, [Phys. Rev. A](#) **59**, 141 (1999).
- [28] C. Napoli, T. R. Bromley, M. Cianciaruso, M. Piani, N. Johnston, and G. Adesso, Robustness of coherence: An operational and observable measure of quantum coherence, [Phys. Rev. Lett.](#) **116**, 150502 (2016).
- [29] H.-P. Breuer, E.-M. Laine, and J. Piilo, Measure for the degree of non-Markovian behavior of quantum processes in open systems, [Phys. Rev. Lett.](#) **103**, 210401 (2009).
- [30] Á. Rivas, S. F. Huelga, and M. B. Plenio, Entanglement and non-Markovianity of quantum evolutions, [Phys. Rev. Lett.](#) **105**, 050403 (2010).
- [31] H.-P. Breuer, E.-M. Laine, J. Piilo, and B. Vacchini, Non-Markovian dynamics in open quantum systems, [Rev. Mod. Phys.](#) **88**, 021002 (2016).
- [32] J. I. Cirac, P. Zoller, H. J. Kimble, and H. Mabuchi, Quantum state transfer and entanglement distribution among distant nodes in a quantum network, [Phys. Rev. Lett.](#) **78**, 3221 (1997).
- [33] S. Wehner, D. Elkouss, and R. Hanson, Quantum internet: A vision for the road ahead, [Science](#) **362**, eaam9288 (2018).
- [34] H. J. Kimble, The quantum internet, [Nature \(London\)](#) **453**, 1023 (2008).
- [35] V. Giovannetti, S. Lloyd, and L. Maccone, Quantum metrology, [Phys. Rev. Lett.](#) **96**, 010401 (2006).
- [36] E. M. Kessler, P. Kómár, M. Bishof, L. Jiang, A. S. Sørensen, J. Ye, and M. D. Lukin, Heisenberg-limited atom clocks based on entangled qubits, [Phys. Rev. Lett.](#) **112**, 190403 (2014).