

Analytical singular-value structure of analytic-continuation kernels from Slepian information theory

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Analytic continuation from imaginary-time Green's functions to real-frequency spectra is a central ill-posed inverse problem in quantum many-body physics. We show that the thermal kernel admits an analytical generalized singular-value structure once its purely dynamical part is separated from the statistical weight imposed by the heat bath. The dynamical kernel is the imaginary-bandwidth continuation of Slepian's finite Fourier transform and is governed by the same Sturm-Liouville algebra that yields prolate spheroidal wave functions. Fermionic and bosonic statistics then enter as gauge transformations of the frequency-space inner product, giving a generalized singular system and self-adjoint effective potentials in the corresponding weighted space. The Shannon number, $N_c = \beta\omega_{\max}/\pi$, fixes the upper information capacity of this pure Laplace channel. Finally, the optimal sampling points are obtained as eigenvalues of a Legendre colleague matrix, giving a deterministic compressed-sensing grid without iterative root searches.

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I. INTRODUCTION

Imaginary-time quantum Monte Carlo (QMC) methods provide statistically controlled access to thermal correlation functions, but experimentally relevant spectra live on the real-frequency axis. Their relation is a Fredholm equation of the first kind,

$$G^{\nu}(\tau) = \int_{-\omega_{\max}}^{\omega_{\max}} K^{\nu}(\tau, \omega) A(\omega) d\omega, \quad (1)$$

where $\nu = F, B$ labels fermionic or bosonic statistics. Inverting Eq. (1) is the analytic-continuation problem. It is notoriously unstable: Singular values of the kernel collapse exponentially, so statistical noise in G^{ν} is amplified into uncontrolled spectral structure [1]. The standard responses to this instability have been to impose prior structure: rational Padé continuation [2], maximum entropy and Bayesian inference [1,3], and stochastic analytic continuation or optimization [4,5]. These methods remain indispensable, but their output depends on model selection, entropy or temperature

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schedules, positivity and sum-rule constraints, and diagnostics for distinguishing physical sharp features from amplified noise.

The introduction of sparse modeling changed the numerical viewpoint: Instead of reconstructing all singular directions, one identifies the components that survive the noise level by an L_1 -regularized inverse problem [6,7]. In parallel, the intermediate representation (IR) exposed the low-rank structure of finite-temperature Green's functions and enabled sparse sampling in imaginary time and Matsubara frequency [8–10]. This program has since expanded into discrete Lehmann representations [11,12], hybrid sparse-modeling-Padé strategies [13], causality-preserving Nevanlinna continuations [14], machine-learned optimizers [15], and integrated software ecosystems such as ACFlow [16]. A recent independent work has further identified the finite-Laplace/oblate-spheroidal origin of compression in the standard IR kernel [17]. These developments identify the same finite-dimensional range of the analytic-continuation kernel from different numerical viewpoints. Shinaoka *et al.* obtained the IR basis by numerical singular value decomposition (SVD) of the discretized thermal kernel; Li *et al.* constructed sparse sampling points from zeros of the highest IR basis function; and Kaye, Chen, and Parcollet selected dirichlet lehman representation (DLR) frequencies and imaginary-time nodes by rank-revealing linear algebra. The Slepian formulation below adds a complementary route in which an IR-like compressed space, its singular values, and a compatible node construction are generated from a single Sturm-Liouville algebra.

The present paper gives a closed analytical route to that algorithmic step. We factor the thermal kernel into a universal dynamical transmission factor and a statistics-dependent metric. The former is mapped to Slepian's information theory of time- and band-limited signals; the latter becomes a gauge transformation of the Hilbert-space measure. The result is an exact generalized SVD of the fermionic and bosonic kernels in the weighted metric, together with an algebraic construction of the corresponding optimal sampling grid. The colleague-matrix grid is one deterministic realization of the Slepian basis, while the central object is the semianalytic construction and interpretation of the compressed representation itself. Efficient IR and DLR implementations already generate their grids very rapidly. The purpose of the present construction is instead to show explicitly how the compressed subspace, singular values, and sampling nodes follow from the same differential-equation coefficients.

II. SLEPIAN'S FINITE FOURIER TRANSFORM

Before turning to Green's functions, it is useful to recall the classical problem solved by Slepian, Landau, and Pollak [18–21]. Let a signal be observed only in a finite time interval $x \in [-1, 1]$ and be band limited to $y \in [-1, 1]$. The finite Fourier transform

$$(\mathcal{F}_c f)(x) = \int_{-1}^1 e^{icxy} f(y) dy \quad (2)$$

is not diagonal in plane waves because both time and frequency have been truncated. Slepian's key observation was that $\mathcal{F}_c$ commutes with a local Sturm-Liouville operator,

$$\mathcal{D}_z^{(F)} = \frac{\partial}{\partial z} \left[(1 - z^2) \frac{\partial}{\partial z} \right] - c^2 z^2, \quad [\mathcal{F}_c, \mathcal{D}^{(F)}] = 0. \quad (3)$$

Thus, the singular functions of the nonlocal integral transform are the prolate spheroidal wave functions (PSWFs), which can be obtained from the commuting differential equation. The same theory gives the Shannon number $N_c = 2c/\pi$, the number of independent degrees of freedom that can pass through a finite time-bandwidth window.

This classical construction is the reference frame for the present work. The Green-function kernel differs from Eq. (2) in two controlled ways: The unitary Fourier phase e^{icxy} is replaced by the imaginary-time Laplace factor e^{-cxy} , and fermionic or bosonic statistics multiply this universal factor by thermal weights. The first change is an analytic continuation of the Slepian algebra, while

the second is a gauge transformation of the frequency-space metric. The kernel symmetrization and the Slepian mapping are written explicitly in the following section.

III. SLEPIAN MAPPING AND SHANNON LIMIT

Introduce

$$x = \frac{2\tau}{\beta} - 1, \quad y = \frac{\omega}{\omega_{\max}}, \quad c = \frac{\beta\omega_{\max}}{2}. \quad (4)$$

The physical kernels on $[-1, 1]^2$ separate as

$$K^F(x, y) = \frac{1}{2} \frac{e^{-cxy}}{\cosh(cy)}, \quad (5)$$

$$K^B(x, y) = \frac{\omega_{\max}}{2} \frac{ye^{-cxy}}{\sinh(cy)}. \quad (6)$$

All dependence on quantum statistics is confined to the y -space weights. The common factor

$$\mathcal{K}_c(x, y) = e^{-cxy} \quad (7)$$

is the pure imaginary-time phase-transfer channel. It satisfies

$$\mathcal{D}_x^{(L)} \mathcal{K}_c(x, y) = \mathcal{D}_y^{(L)} \mathcal{K}_c(x, y), \quad (8)$$

where

$$\mathcal{D}_z^{(L)} = \frac{\partial}{\partial z} \left[(1 - z^2) \frac{\partial}{\partial z} \right] + c^2 z^2. \quad (9)$$

This is the imaginary-bandwidth continuation of the Slepian Sturm-Liouville operator. Hence, $\mathcal{K}_c$ has the bilinear expansion

$$e^{-cxy} = \sum_{n=0}^{\infty} \mu_n \psi_n(x; c) \psi_n(y; c), \quad (10)$$

where ψ_n are PSWFs analytically continued in the bandwidth parameter, equivalently oblate-spheroidal functions. The information capacity of this pure channel is therefore set by the Slepian-Landau-Pollak counting law,

$$N_c = \frac{2c}{\pi} = \frac{\beta\omega_{\max}}{\pi}. \quad (11)$$

Components with $n \gg N_c$ encode the exponentially small tail of the channel and are therefore the modes most vulnerable to numerical and QMC noise.

Here, μ_n denotes the eigenvalue of the compact operator

$$(\mathcal{K}_c f)(x) = \int_{-1}^1 e^{-cxy} f(y) dy \quad (12)$$

associated with the eigenfunction ψ_n . Its sign depends on the parity convention, while the generalized singular values of the physical kernels are proportional to $|\mu_n|$.

The Legendre basis is used for a standard reason in the computation of prolate and oblate-spheroidal wave functions. When $c = 0$, Eq. (9) reduces to the Legendre Sturm-Liouville problem, so the eigenfunctions are exactly Legendre polynomials. For $c \neq 0$, the multiplication operator x^2 has a sparse representation because of the three-term recurrence relation of $P_m(x)$; it couples only neighboring parities, $m \rightarrow m, m \pm 2$. Thus, the differential eigenproblem becomes a parity-separated tridiagonal matrix problem after the expansion.

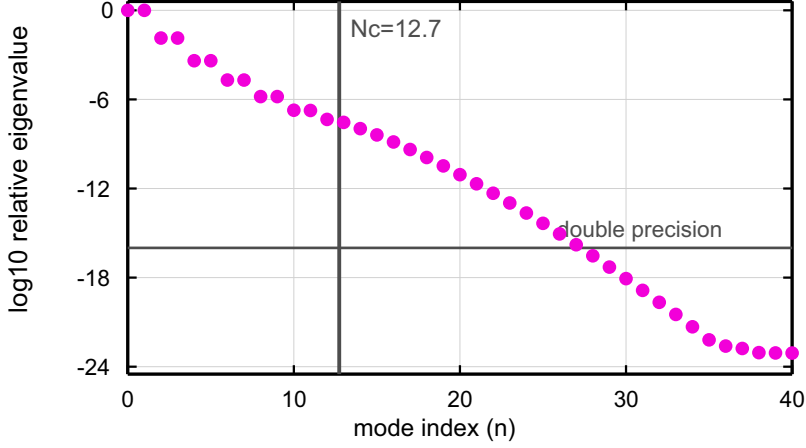


FIG. 1. Eigenvalue cliff of the pure Laplace-Slepian kernel $\mathcal{K}_c(x, y) = e^{-c|xy|}$ for $c = 20$. The plotted quantity is the ordered magnitude $|\mu_n|/|\mu_0|$ obtained from the Legendre coefficients of the analytically continued PSWFs. The gray vertical line marks the Shannon number $N_c = 2c/\pi = 12.73$, and the gray horizontal line marks 10^{-16} .

To obtain the functions ψ_n , we solve the eigenproblem of Eq. (9) in this Legendre polynomial basis,

$$\psi_n(x; c) = \sum_{m=0}^M d_m^{(n)} P_m(x), \quad (13)$$

where M is the Legendre cutoff used to represent the functions. Substitution into Eq. (9) gives a parity-separated tridiagonal eigenproblem,

$$A_m d_{m-2}^{(n)} + B_m d_m^{(n)} + C_m d_{m+2}^{(n)} = \chi_n d_m^{(n)}. \quad (14)$$

The cutoff M is only a numerical representation parameter and should not be confused with the Shannon number N_c in Eq. (11). The coefficients $d_m^{(n)}$ determine both the basis functions and the pure-kernel eigenvalues. Evaluating the integral equation at $x = 0$ for even parity and differentiating it at $x = 0$ for odd parity gives

$$\mu_n = \frac{2d_0^{(n)}}{\sum_{m=0}^M d_m^{(n)} P_m(0)} \quad (n \text{ even}), \quad (15)$$

$$\mu_n = -\frac{2c d_1^{(n)}}{3 \sum_{m=0}^M d_m^{(n)} P'_m(0)} \quad (n \text{ odd}). \quad (16)$$

Equations (14)–(16) summarize the semianalytic construction used here: A local tridiagonal eigenproblem gives the Legendre coefficients, and the same coefficients give the eigenvalues μ_n by algebraic formulas.

The advantage of Eqs. (14)–(16) is that the basis functions, the singular spectrum of the pure Laplace channel, and the colleague-matrix nodes below are all obtained from one set of Legendre coefficients.

The Laplace version of this statement is less familiar than Slepian's original Fourier problem because the kernel is no longer unitary. The present formulation makes the point directly visible: The eigenvalues μ_n of Eq. (10) can be evaluated from the PSWF coefficients through Eqs. (15) and (16). Figure 1 shows the ordered magnitudes for the same parameter used below, $c = 20$ ($\beta\omega_{\max} = 40$). The sign alternation of μ_n is absorbed into the right singular-vector convention;

the generalized singular values of the fermionic and bosonic kernels are obtained from $|\mu_n|$ by the statistical prefactors in Eq. (22). Around the Shannon index $N_c = 12.73$, the spectrum has already fallen by more than seven orders of magnitude and then rapidly enters the exponentially small tail.

IV. GAUGE-TRANSFORMED GENERALIZED SVD

Equations (5) and (6) show that particle statistics act as a change of metric in frequency space. Define weighted inner products

$$\langle f, g \rangle_v = \int_{-1}^1 f(y)g(y)\rho^v(y) dy, \quad (17)$$

where

$$\rho^F = \cosh^2(cy), \quad \rho^B = \frac{\sinh^2(cy)}{y^2}, \quad (18)$$

with the regular limit understood at $y = 0$ for bosons. Combining Eq. (10) with Eq. (18) gives the generalized SVD

$$K^v(x, y) = \sum_{n=0}^{\infty} S_n^v U_n(x) V_n^v(y), \quad (19)$$

with

$$U_n(x) = \psi_n(x; c), \quad (20)$$

$$V_n^F(y) = \frac{\text{sgn}(\mu_n)\psi_n(y; c)}{\cosh(cy)}, \quad S_n^F = \frac{|\mu_n|}{2}, \quad (21)$$

$$V_n^B(y) = \frac{\text{sgn}(\mu_n)y\psi_n(y; c)}{\sinh(cy)}, \quad S_n^B = \frac{\omega_{\max}|\mu_n|}{2}. \quad (22)$$

Thus, numerical kernel diagonalization in this weighted metric is replaced by the spectral algebra of a second-order differential operator; it is complementary to the standard SVD, whose effective rank at a fixed tolerance can be smaller after thermal weighting [17].

For example, in the fermionic case, the right functions are orthonormal in the weighted metric because

$$\int_{-1}^1 V_n^F(y) V_m^F(y) \rho^F(y) dy = \int_{-1}^1 \psi_n(y; c) \psi_m(y; c) dy = \delta_{nm}. \quad (23)$$

Moreover,

$$\int_{-1}^1 K^F(x, y) V_n^F(y) \rho^F(y) dy = S_n^F U_n(x), \quad (24)$$

and the adjoint relation follows in the same metric. The bosonic case is identical after replacing the metric by ρ^B . This is why we call Eq. (19) a generalized SVD: It is exact in the natural Hilbert space generated by the thermal statistical weight, while the conventional IR is the standard SVD of the already weighted thermal kernel.

The same gauge transformation gives the physical interpretation. Writing $\psi_n(y; c) = g^v(y) V_n^v(y)$ with $g^F = \cosh(cy)$ and $g^B = \sinh(cy)/y$, the right singular functions obey a self-adjoint Sturm-Liouville equation,

$$\frac{1}{\rho^v} \frac{d}{dy} \left[(1 - y^2) \rho^v \frac{dV_n^v}{dy} \right] + \mathcal{V}^v(y) V_n^v = \chi_n V_n^v. \quad (25)$$

For fermions,

$$\mathcal{V}^F(y) = c^2 - 2cy \tanh(cy), \quad (26)$$

up to the irrelevant constant shift fixed by the convention for χ_n . The thermal bath therefore appears as a gauge field that reshapes the frequency-space measure and suppresses statistically inaccessible high-energy information. Fermions and bosons differ only through this metric and its associated effective potential; the singular-value cliff itself is governed by the universal Slepian channel. The weighted orthogonality in Eq. (23), the operator action in Eq. (24), and the local equation in Eq. (25) give the integral and differential forms of the same generalized singular system.

V. SAMPLING FROM A COLLEAGUE MATRIX

The analytical basis also fixes where imaginary-time data should be measured. Truncating at $N = \lceil N_c \rceil$, the extended Gaussian quadrature nodes are the zeros of the boundary basis function,

$$\psi_N(x_i; c) = 0, \quad i = 1, \dots, N, \quad (27)$$

so uniform sampling is not information-theoretically optimal. These nodes can be obtained without Newton iterations or prior function evaluations. Expand

$$\psi_N(x; c) \simeq \sum_{m=0}^M d_m P_m(x), \quad (28)$$

where P_m are Legendre polynomials and the coefficients d_m are already produced by the parity-separated tridiagonal eigenproblem associated with Eq. (9). Using

$$xP_m(x) = \alpha_m P_{m+1}(x) + \gamma_m P_{m-1}(x), \quad (29)$$

where

$$\alpha_m = \frac{m+1}{2m+1}, \quad \gamma_m = \frac{m}{2m+1}, \quad (30)$$

these coefficients form the Jacobi matrix J for multiplication by x on $\mathbf{P} = (P_0, \dots, P_{M-1})^T$. At a zero of Eq. (28), $P_M(x_i) = -d_M^{-1} \sum_{m=0}^{M-1} d_m P_m(x_i)$. Eliminating P_M closes the recurrence into the colleague eigenproblem

$$C\mathbf{P}(x_i) = x_i \mathbf{P}(x_i), \quad (31)$$

where

$$C = J - \frac{\alpha_{M-1}}{d_M} \mathbf{e}_{M-1} (d_0, \dots, d_{M-1}). \quad (32)$$

Here, $\mathbf{e}_{M-1} = (0, \dots, 0, 1)^T$ is the last unit vector in the M -dimensional Legendre coefficient space. The physical nodes are the N eigenvalues of C in $[-1, 1]$, and their quadrature weights follow from the corresponding eigenvectors by the usual Golub-Welsch moment construction [22,23]. The colleague-matrix formulation [24,25] replaces heuristic root searches by a deterministic finite-dimensional eigenproblem. In practice, one first solves the tridiagonal Legendre problem for ψ_N with a cutoff M , forms C from the same coefficient vector $\mathbf{d}$, diagonalizes C , and keeps the real eigenvalues in the physical interval as the imaginary-time nodes. The dimension of C is controlled by the Legendre representation cutoff M , while the number of retained nodes is chosen from the desired rank N , typically near the Shannon scale.

This construction should be compared directly with the existing sparse-grid strategies. Li *et al.* also use zeros of a highest-order basis function, but the basis function is the numerically computed IR singular function. Kaye, Chen, and Parcollet use rank-revealing linear algebra to choose DLR frequencies and interpolation nodes. The colleague matrix is best viewed as an algebraic realization tied to the Slepian basis rather than as a generally cheaper root finder; all these grids are inexpensive

at the ranks of interest. Its significance is that the same Legendre coefficients that generate the Slepian basis and the algebraic eigenvalues μ_n also generate the sampling nodes. In this sense, the basis, singular spectrum, and grid are linked by the same finite-dimensional calculation.

VI. RELATION TO SPARSE-MODELING RECONSTRUCTION

The grid samples are used in two distinct ways. First, for interpolation or compression of the imaginary-time Green's function, one solves a small linear system

$$G(x_i) \simeq \sum_{n=0}^{M-1} a_n U_n(x_i) \quad (33)$$

and evaluates the resulting expansion on a dense reference grid. Second, for spectral reconstruction, the sampled Green's function is passed to a positivity- and sum-rule-constrained inverse solver. This distinction is important because the sampling grid alone cannot remove the ill-posedness of analytic continuation.

The analytical singular system also clarifies the noise filtering mechanism of singular-value-embedded sparse modeling [6,7]. A naive projection-divide procedure estimates $\rho_n = g_n/S_n$ and therefore amplifies noise by $1/S_n$ near the singular-value cliff. The sparse-modeling formulation keeps the singular values inside the forward matrix,

$$\min_{\rho} \left[\frac{1}{2} \|G - \Phi S \rho\|_2^2 + \lambda \|\rho\|_1 \right], \quad (34)$$

with positivity and sum-rule constraints when needed. If the sampling grid and quadrature weights give the orthogonality $\Phi^T W \Phi = I$, the unconstrained weighted problem decouples into

$$\min_{\rho_n} \left[\frac{1}{2} (g_n - s_n \rho_n)^2 + \lambda |\rho_n| \right], \quad (35)$$

where $g_n = \sum_i w_i G(x_i) U_n(x_i)$, and its closed solution is

$$\rho_n = \frac{1}{s_n} \text{ST} \left(g_n, \frac{\lambda}{s_n} \right), \quad (36)$$

where

$$\text{ST}(g, \theta) = \text{sgn}(g) \max(|g| - \theta, 0). \quad (37)$$

The effective threshold grows as s_n decreases, so modes beyond the cliff are suppressed continuously rather than divided out. This closed-form expression is included here only to explain the filtering mechanism of the sparse-modeling formulation; the constrained reconstructions used below still solve the positivity- and sum-rule-constrained problem numerically. Positivity and sum rules recouple the variables, in which case alternating direction of multiplier method (ADMM) remains the appropriate constrained solver, but Eq. (36) gives the algebraic reason why singular-value-embedded sparse modeling is robust.

VII. NUMERICAL SAMPLING TESTS

We finally illustrate the sampling step in a controlled forward problem. A model spectrum $A(\omega)$ consisting of four Gaussian peaks, plotted against $y = \omega/\omega_{\max}$, is transformed by the fermionic kernel with $c = 20$, or $\beta\omega_{\max} = 40$, for which $N_c = \beta\omega_{\max}/\pi = 12.73$. The dense reference uses $N_{\text{dense}} = 201$ uniformly spaced imaginary-time points. Keeping only M colleague-matrix observation nodes, Fig. 3 shows that the error is large when $M < N_c$, but drops to 3.4×10^{-3} at $M = 13$ and to 6.4×10^{-4} at $M = 16$, corresponding to a $201/13 = 15.5$ compression factor at the Shannon scale. This should be distinguished from the final reconstruction accuracy of $A(\omega)$: Singular-value-embedded sparse modeling treats the ill-conditioned directions in an information-theoretically sound way by keeping S_n in the forward matrix instead of dividing by it [6,7]. The

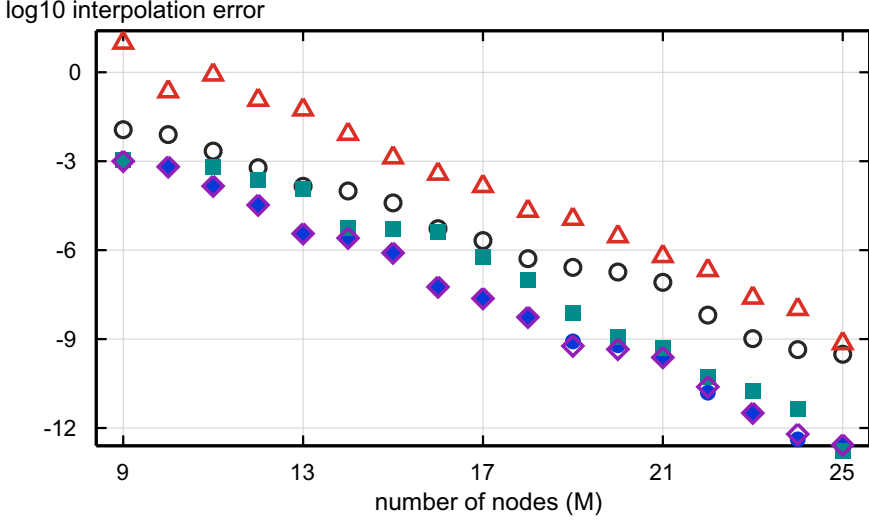


FIG. 2. Interpolation error of the imaginary-time Green’s function for several node choices at $\beta\omega_{\max} = 40$. Black open circles denote the uniform grid, blue filled circles the Li-style IR grid, cyan squares the DLR/pyd1r grid, red triangles the pure Slepian colleague grid, and purple diamonds the standard-metric Slepian grid. The plotted error is $\|G_{\text{interp}} - G_{\text{dense}}\|_2 / \|G_{\text{dense}}\|_2$ in each method’s own interpolation basis. The standard-metric Slepian grid is obtained by diagonalizing the small matrix $B_{mn} = s_m s_n \int V_m(y) V_n(y) dy$ inside the truncated Slepian subspace.

Slepian construction therefore does not evade the Shannon limit; it identifies the pure-channel limit analytically and converts it into a deterministic sampling prescription. Existing uniform data can still be projected onto the analytical U_n, S_n and processed by the same ADMM scheme, so the basis is useful both for planned compressed measurements and for postprocessing data that have already been sampled.

We compare five choices under the same forward problem: a uniform grid, the default sparse-ir imaginary-time grid of Li *et al.*, the official pyd1r DLR interpolation grid, the pure Slepian colleague grid, and a standard-metric Slepian grid. The distinction between the last two is the inner product used in frequency space. The generalized SVD uses the weighted inner product. The conventional fermionic IR SVD instead treats the physical kernel $K^F(x, y)$ as an operator from flat $L^2([-1, 1], dy)$ to flat $L^2([-1, 1], dx)$. To compare with that convention, we restrict the flat-metric normal operator to the truncated Slepian subspace. In the Slepian basis, this is the small matrix

$$B_{mn} = s_m s_n \int_{-1}^1 V_m(y) V_n(y) dy, \quad (38)$$

where s_m are the Slepian generalized singular values and V_m are the corresponding right functions. This matrix is introduced here because it is the explicit finite-dimensional operation used in the numerical comparison: It converts the weighted Slepian singular system into the flat frequency metric used by the standard IR SVD. Diagonalizing B gives an orthogonal rotation of the Slepian functions. The rotated functions remain Legendre series, so their nodes are again obtained by a colleague matrix. This is the precise meaning of the standard-metric Slepian grid: It is the Slepian subspace expressed in the same flat frequency inner product used by the usual IR SVD.

Figure 2 shows the interpolation error as a function of the number of nodes. The DLR/ pyd1r symbols are plotted at the ranks generated by scanning the DLR tolerance. The pure Slepian colleague grid is less accurate at $M = 13$ because it is tied to the weighted Slepian singular system of Sec. IV, whereas the Li-style IR grid is optimized for the flat-metric physical kernel. After the

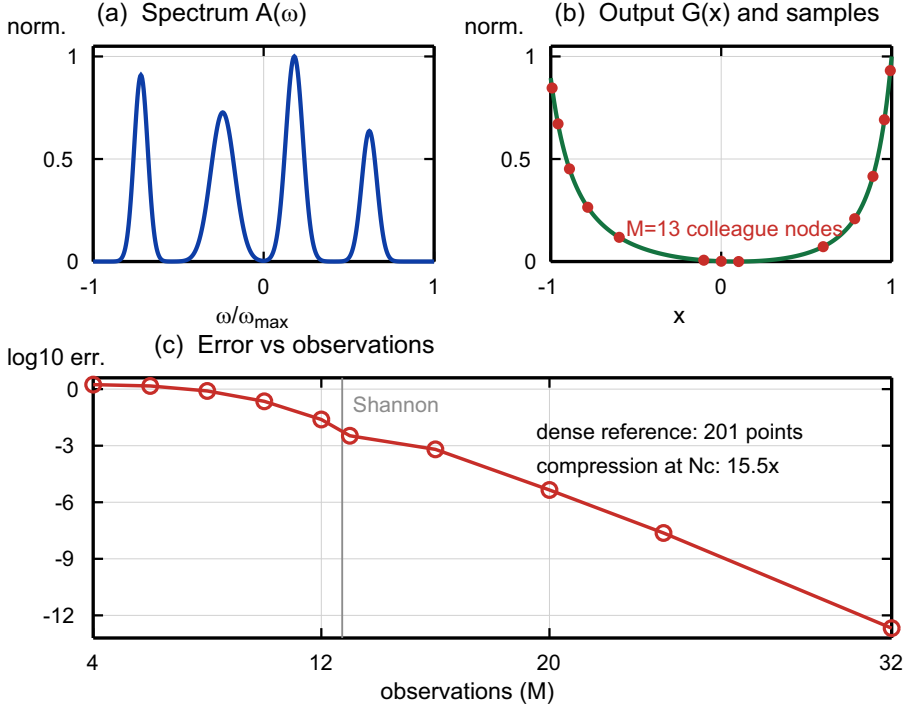


FIG. 3. Numerical sampling experiment. (a) Original spectrum $A(\omega)$ made of four Gaussian peaks, shown as a function of $y = \omega/\omega_{\max}$. (b) Imaginary-time output obtained from the fermionic kernel; red points are the $M = \lceil N_c \rceil = 13$ colleague-matrix observation nodes. (c) Relative reconstruction error of $G(x)$ as the number of observations M is varied. The sharp improvement at the Shannon scale shows that the compressed grid preserves the information transmitted by the kernel.

flat-metric matrix B is diagonalized, the standard-metric Slepian grid gives the same interpolation accuracy as the Li-style IR grid to the displayed precision: $\delta G = 3.61 \times 10^{-6}$ at $M = 13$ and $\delta G = 3.30 \times 10^{-12}$ at $M = 23$. Thus, the numerical comparison supports a specific statement: The usual IR subspace can be reconstructed, for this benchmark, by rotating the analytical Slepian subspace with the flat-metric Gram matrix.

The same sampled data were also passed to the same positivity- and sum-rule-constrained ADMM solver on a fixed frequency grid. The relative spectral error stayed in the narrow range 0.70–0.73 for all sparse grids shown here, and a dense uniform baseline with $M = 201$ gave 0.672 with the same solver. These numbers are not shown in Fig. 2 because they diagnose the subsequent analytic-continuation solver rather than the interpolation grid. They confirm that the grid comparison does not remove the intrinsic information loss of analytic continuation.

For the pure-colleague grid, moment-matched quadrature weights become positive once a modest safety margin above the Shannon index is used. Then, the weighted matrix $W^{1/2}\Phi$ is nearly orthogonal, with condition number 1.01 at $M = 20$ and 1.00 at $M = 23$. At $M = 13$, the pure-colleague weights are not all positive, so we use unweighted least squares for the interpolation error in Fig. 3. This is a finite-rank numerical detail rather than a separate reconstruction method.

VIII. CONCLUSION AND OUTLOOK

Analytic continuation is difficult because thermal imaginary-time evolution is an information channel with exponentially small singular directions. By separating the universal Laplace

transmission factor from the statistical metric, we have shown that this channel is analytically controlled by Slepian's time-bandwidth theory. Fermionic and bosonic kernels acquire their physical distinctions through gauge-transformed measures and effective potentials, while the Shannon number $N_c = \beta\omega_{\max}/\pi$ bounds the capacity of the pure Laplace channel. This perspective also clarifies why singular-value-embedded ADMM is effective: It regularizes the information that survives the kernel rather than trying to recover modes that are not transmitted. The numerical comparison shows that the pure Slepian colleague grid captures the correct information scale, and that the standard-metric Slepian grid reproduces Li-style IR interpolation accuracy while giving comparable final spectral errors to Li-style IR and DLR/pyd1r sampling. The central message is that an IR-like compressed subspace can be reconstructed from an analytical Slepian basis plus the finite flat-metric Gram matrix in Eq. (38). The same Legendre coefficients also provide the pure-channel eigenvalues and the colleague-matrix nodes. The result is therefore a unified semianalytic explanation of three objects that are usually introduced separately: compressed basis functions, singular values, and sparse sampling nodes. It provides a mathematical basis for compressed imaginary-time measurements, in which one chooses coordinates spanning the transmitted information while recognizing that the exponentially unstable complement is not recoverable by any grid.

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

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