

Entanglement in the symmetric subspace: Mapping multipartite to bipartite states

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We propose a technique to investigate multipartite entanglement in the symmetric subspace. Our approach is to map an N -qubit symmetric state onto a bipartite symmetric state of higher local dimension. We show that this mapping preserves separability and allows to characterize the entanglement of the original multipartite state. In particular, we establish a connection between the border rank and the Schmidt rank, and derive lower bounds on entanglement measures. Finally, we reveal the existence of entangled symmetric subspaces, where all bipartite states are entangled.

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I. INTRODUCTION

Entanglement is a key resource in several quantum information processing tasks, ranging from cryptography [1,2] to metrology [3] and measurement-based computation [4], among others. The rapid progress of experimental technology has enabled the generation of entangled states in various physical platforms (see reviews in Refs. [5–8]). However, even for pure states, a complete characterization of multipartite entanglement remains challenging, for at least two reasons: (1) A direct extension of the Schmidt decomposition in terms of multipartite orthogonal product states does not always exist [9] and (2) multipartite maximally entangled states cannot be uniquely defined (see review in Ref. [10]).

A strategy to address these issues is to focus on quantum systems whose state remains unaltered when any two particles are exchanged. Such states live in the *symmetric subspace*, whose dimension is $N + 1$ for N qubits [11,12]. Considering the symmetric subspace can greatly simplify the computational complexity needed to characterize and quantify multipartite entanglement.

One example in this direction is given by the *tensor rank* [13,14], a generalization of the Schmidt rank for multipartite systems. This is related to transformations between equivalence classes under stochastic local operations and classical communication [15,16]. In fact, although determining the tensor rank is known to be NP-hard [17], Ref. [18] showed that, for symmetric states, it can be efficiently estimated by the symmetric tensor rank in the theory of homogeneous polynomials [19].

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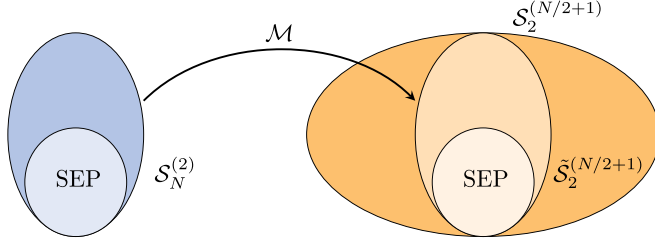


FIG. 1. Pictorial representation of the mapping $\mathcal{M}$ presented in Result 1. This article investigates the entanglement of multipartite states $|\Psi\rangle \in \mathcal{S}_N^{(2)}$ by focusing on their mapped bipartite states $\mathcal{M}(|\Psi\rangle) \in \mathcal{S}_2^{(N/2+1)}$. Here, SEP denotes the set of separable states. Note that, while all separable states in $\mathcal{S}_N^{(2)}$ can be mapped to separable states in $\mathcal{S}_2^{(N/2+1)}$, as shown in Result 2, not all separable states in $\mathcal{S}_2^{(N/2+1)}$ lie in $\tilde{\mathcal{S}}_2^{(N/2+1)}$.

Another example is provided by the geometric measure of entanglement, which quantifies how close a quantum state is to the set of separable states [20,21] (see review in Ref. [22]). This measure has an operational interpretation, being connected to state discrimination via local operations and classical communication (LOCC) [23]. Even for pure states, calculating the geometric measure is generally a hard task [24], since it requires optimizing over the whole set of product states. In contrast, for symmetric states, Ref. [25] has proven that it can be efficiently computed by restricting to the set of symmetric product states only. Furthermore, several theoretical techniques have been developed for symmetric states [26–28], with a particular focus on the Majorana representation [29–32]. Also, symmetric states have been characterized via their nonlocality [33] and spin non-classicality [34], as well as their single-qubit reductions [35,36]. Yet, a comprehensive framework for symmetric entanglement remains elusive.

In this article, we present a systematic approach to address N -qubit symmetric entanglement. Our idea is to map a multipartite state into a bipartite state of higher local dimensions and analyze the properties of the latter, as illustrated in Fig. 1. This mapping is based on an embedding between symmetric subspaces, which was first pointed out in Ref. [12], and more recently considered in Refs. [37,38], in the context of the separability problem for multipartite symmetric states. Remarkably, our technique can be extended to high-dimensional mixed states (more details will appear elsewhere [39]).

We first recall the structure of the symmetric subspace, both in the multipartite and in the bipartite cases. Then, we present our mapping (Result 1) and prove that it preserves the separability of quantum states (Result 2). Also, we investigate how this mapping transforms the entanglement of a multipartite symmetric state in terms of the symmetric border rank (Result 3), providing analytically computable lower bounds to the geometric measure (Result 4) and another entanglement measure (Results 5) based on the nuclear norm. Finally, we show that the complementary symmetric subspace to the mapped subspace is entangled (Result 6).

II. SYMMETRIC SUBSPACE

Let $\mathcal{H}_N^{(d)} = (\mathbb{C}^d)^{\otimes N}$ be the N -qudit Hilbert space with dimension d^N , where each single-qudit Hilbert space $\mathbb{C}^d$ is spanned by the orthonormal bases $\{|i\rangle_d\}_{i=0}^{d-1}$. The symmetric subspace, denoted by $\mathcal{S}_N^{(d)}$, with dimension $\binom{N+d-1}{d-1}$, corresponds to the convex set spanned by the pure states that are invariant under any permutation of the parties.

For any $N \geq 2$ and $d = 2$ (i.e., N qubits), the subspace $\mathcal{S}_N^{(2)}$ has dimension $N + 1$ and is spanned by the orthonormal Dicke states, i.e., $\{|D_N^k\rangle\}_{k=0}^N$, defined as

$$|D_N^k\rangle = \binom{N}{k}^{-1/2} \sum_{\pi_N} \pi_N(|0\rangle^{\otimes(N-k)} |1\rangle^{\otimes k}), \quad (1)$$

where k denotes the number of excitations, and the sum runs over all possible permutations π_N acting on N qubits. Here and in the following, we omit the subscript for the qubit basis and set $|0\rangle \equiv |0\rangle_2$ and $|1\rangle \equiv |1\rangle_2$. For any $d \geq 2$ and $N = 2$ (i.e., two qudits), the subspace $\mathcal{S}_2^{(d)}$ has dimension $d(d+1)/2$ and is spanned by the orthonormal states $\{|\psi_{ij}^{(d)}\rangle\}_{i,j=0}^{d-1}$, given by

$$|\psi_{ii}^{(d)}\rangle = |ii\rangle_d, \quad |\psi_{ij}^{(d)}\rangle = \frac{|ij\rangle_d + |ji\rangle_d}{\sqrt{2}}, \quad i \neq j, \quad (2)$$

where $|ij\rangle_d = |i\rangle_d \otimes |j\rangle_d$. For example, the Greenberger-Horne-Zeilinger (GHZ) state in N qubits, $|\text{GHZ}_N\rangle \equiv (1/\sqrt{2})(|D_N^0\rangle + |D_N^N\rangle)$, lives in the subspace $\mathcal{S}_N^{(2)}$, while the two-qudit maximally entangled state, $|\psi_d^+\rangle = (1/\sqrt{d}) \sum_{i=0}^{d-1} |\psi_{ii}^{(d)}\rangle$, lives in the subspace $\mathcal{S}_2^{(d)}$.

III. MAPPING

Let us present a mapping that allows to cast an N -qubit symmetric state for any even N as a high-dimensional bipartite symmetric state.

Result 1. For any even N , there is a mapping $\mathcal{M} : \mathcal{S}_N^{(2)} \rightarrow \tilde{\mathcal{S}}_2^{(N/2+1)}$ such that

$$\mathcal{M}(|D_N^k\rangle) = \sum_{i \leq j=0}^{N/2} \delta_{k,i+j} \mu_{ij} |\psi_{ij}^{(N/2+1)}\rangle, \quad (3)$$

where $|D_N^k\rangle \in \mathcal{S}_N^{(2)}$ is the Dicke state of Eq. (1), $|\psi_{ij}^{(N/2+1)}\rangle \in \mathcal{S}_2^{(N/2+1)}$ is the bipartite state of Eq. (2), and $\delta_{a,b}$ is the Kronecker delta. Here, μ_{ij} is a factor given by

$$\mu_{ii} = \frac{\binom{N/2}{i}}{\sqrt{\binom{N}{2i}}}, \quad \mu_{ij} = \sqrt{2 \frac{\binom{N/2}{i} \binom{N/2}{j}}{\binom{N}{i+j}}}, \quad i \neq j. \quad (4)$$

The subspace $\tilde{\mathcal{S}}_2^{(N/2+1)}$ is spanned by the states $\{\mathcal{M}(|D_N^k\rangle)\}_{k=0}^N$, with dimension $N+1$.

The proof is given in Appendix A. We have three remarks on Result 1. First, this mapping does not correspond to an isomorphism between symmetric subspaces, but rather to an embedding, i.e.,

$$\tilde{\mathcal{S}}_2^{(N/2+1)} \subset \mathcal{S}_2^{(N/2+1)}. \quad (5)$$

This pictorial representation is illustrated in Fig. 1. Second, since any $|\Psi\rangle \in \mathcal{S}_N^{(2)}$ can be written as $|\Psi\rangle = \sum_k c_k |D_N^k\rangle$ for some coefficients c_k , one can always find the mapped state $\mathcal{M}(|\Psi\rangle) = \sum_k c_k \mathcal{M}(|D_N^k\rangle) \in \tilde{\mathcal{S}}_2^{(N/2+1)}$. Finally we prove in Appendix B that, for any $|\Psi_1\rangle, |\Psi_2\rangle \in \mathcal{S}_N^{(2)}$, the mapping $\mathcal{M}$ preserves the inner product, i.e.,

$$F(|\Psi_1\rangle, |\Psi_2\rangle) = F[\mathcal{M}(|\Psi_1\rangle), \mathcal{M}(|\Psi_2\rangle)], \quad (6)$$

where $F(|\Psi_1\rangle, |\Psi_2\rangle) \equiv |\langle\Psi_1|\Psi_2\rangle|^2$ denotes the fidelity between two pure states. As a consequence, $\{\mathcal{M}(|D_N^k\rangle)\}$ forms an orthonormal basis for the subspace $\tilde{\mathcal{S}}_2^{(N/2+1)}$.

IV. SEPARABILITY

Here, we show that the mapping of Eq. (3) preserves the separability of quantum states; i.e., any separable state in $\mathcal{S}_N^{(2)}$ is mapped to a bipartite separable state in $\tilde{\mathcal{S}}_2^{(N/2+1)}$. Let us recall that a pure state $|\Phi_{\text{sep}}\rangle \in \mathcal{H}_N^{(2)}$ is said (fully) separable if it can be cast as $|\Phi_{\text{sep}}\rangle = \bigotimes_{i=1}^N |\Phi_i\rangle$ with $|\Phi_i\rangle \in \mathbb{C}^2$ for $i = 1, \dots, N$. Otherwise, it is said entangled. In particular, a symmetric state is either genuinely multipartite entangled or fully separable [40–42], i.e., $|\Phi_{\text{sep}}\rangle = |\Phi\rangle^{\otimes N}$ with $|\Phi\rangle \in \mathbb{C}^2$. Hence, we find the following result:

Result 2. Let $|\Phi\rangle \in \mathcal{S}_N^{(2)}$ be a multipartite state and $\mathcal{M}(|\Phi\rangle) \in \tilde{\mathcal{S}}_2^{(N/2+1)}$ be its mapped bipartite state. If $|\Phi\rangle$ is separable, then $\mathcal{M}(|\Phi\rangle)$ is also separable. Conversely, if $\mathcal{M}(|\Phi\rangle)$ is entangled, then $|\Phi\rangle$ is also entangled.

The proof is provided in Appendix C. We stress that Result 2 does not imply that all separable states in $\mathcal{S}_2^{(N/2+1)}$ live in $\tilde{\mathcal{S}}_2^{(N/2+1)}$, as mentioned in Fig. 1. As an example, for $N = 4$, the state $|\psi_{11}^{(3)}\rangle$ is separable, yet no symmetric separable state $|\Phi_{\text{sep}}\rangle \in \mathcal{S}_4^{(2)}$ satisfies $\mathcal{M}(|\Phi_{\text{sep}}\rangle) = |\psi_{11}^{(3)}\rangle$. Instead, $|\psi_{11}^{(3)}\rangle = \sqrt{2/3}\mathcal{M}(|D_4^2\rangle) + (1/\sqrt{3})|\hat{\psi}_4\rangle$, where $|\hat{\psi}_4\rangle = (1/\sqrt{3})|\psi_{11}^{(3)}\rangle - \sqrt{2/3}|\psi_{02}^{(3)}\rangle \notin \tilde{\mathcal{S}}_2^{(3)}$.

Remarkably, Result 2 can be generalized to mixed states. A mixed symmetric state $\varrho \in \mathcal{S}_N^{(2)}$ can be expressed as $\varrho = \sum_{i,j} \lambda_{ij} |\Psi_i\rangle\langle\Psi_j|$, with coefficients λ_{ij} and $|\Psi_i\rangle \in \mathcal{S}_N^{(2)}$. The corresponding mixed-state mapping is defined as $\mathcal{M}(\varrho) = \sum_{i,j} \lambda_{ij} \mathcal{M}(|\Psi_i\rangle)[\mathcal{M}(|\Psi_j\rangle)]^\dagger$. Since any mixed separable state can be written as a convex combination of pure product states, we can conclude that for any $\varrho_{\text{sep}} \in \mathcal{S}_N^{(2)}$, the mapped state $\mathcal{M}(\varrho_{\text{sep}}) \in \tilde{\mathcal{S}}_2^{(N/2+1)}$ is also separable. Conversely, if $\mathcal{M}(\varrho)$ is entangled, then ϱ is entangled. This means that the mapping allows for reducing the separability problem of multipartite symmetric states to the bipartite symmetric subspace. As an example, consider the mixed state $\varrho_p = p|W_N\rangle\langle W_N| + (1-p)\Pi_S/(N+1)$, with $|W_N\rangle \equiv |D_N^1\rangle$ and Π_S being the projector onto $\mathcal{S}_N^{(2)}$ and set $N = 6$. Applying the positive partial transpose (PPT) criterion [43] to $\mathcal{M}(\varrho_p)$, we find that it is entangled for $p > 0.034$, which is smaller than the value 0.042 presented in Ref. [44]. Finally, we note that there exists a mapped bipartite state that is entangled but cannot be detected by the PPT criterion. Specifically, by applying our mapping to the four-qubit state in Eq. (105) in Ref. [45] and to the four-qubit state ρ_{BE4} in Ref. [12], we obtain two-qutrit PPT-entangled states of the same form of the one reported in Ref. [12].

V. RANK OF A TENSOR

Here, we investigate the relation between the entanglement of a multipartite symmetric state and its mapped bipartite state. Let us recall that any pure bipartite state $|\psi\rangle \in \mathcal{H}_2^{(d)}$ admits, up to local unitaries, a Schmidt decomposition of the form $|\psi\rangle = \sum_{i=1}^r \gamma_i |a_i\rangle \otimes |b_i\rangle$, where $\gamma_i \in \mathbb{R}^+$ and $\langle a_i | a_j \rangle = \langle b_i | b_j \rangle = \delta_{ij}$. The minimal number of terms, r , defines the *Schmidt rank* of $|\psi\rangle$, denoted by $R(|\psi\rangle)$ [46,47], which corresponds to the matrix rank of the reduced state $\varrho_A = \text{tr}_B(|\psi\rangle\langle\psi|)$, and is thus analytically computable.

A generalization of the Schmidt rank for a multipartite state $|\Psi\rangle \in \mathcal{H}_N^{(d)}$ is given by the *tensor rank* [13–15], denoted by $T(|\Psi\rangle)$, which corresponds to the minimal r such that $|\Psi\rangle = \sum_{i=1}^r t_i |\Psi_i^{(1)}\rangle \otimes \cdots \otimes |\Psi_i^{(N)}\rangle$. Here, $t_i \in \mathbb{C}$ and the states $|\Psi_i^{(k)}\rangle$ are not necessarily orthogonal. Examples for three qubits are $T(|\text{GHZ}_3\rangle) = 2$ and $T(|W_3\rangle) = 3$. Unlike the Schmidt rank, computing the tensor rank is, in general, an NP-hard problem [17]. Another related concept is the *border rank* [48–52], denoted by $B(|\Psi\rangle)$, which corresponds to the minimum r such that $|\Psi\rangle$ can be approximated arbitrarily closely by a sequence $\{|\Psi_\epsilon\rangle\}$, with $T(|\Psi_\epsilon\rangle) = r$ for all $\epsilon \neq 0$. Clearly, $B(|\Psi\rangle) \leq T(|\Psi\rangle)$. A notable example is given by

$$|W_3\rangle = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} [(|0\rangle + \epsilon|1\rangle)^{\otimes 3} - |0\rangle^{\otimes 3}], \quad (7)$$

which shows that $B(|W_3\rangle) = 2$ (see Ref. [53]).

For a symmetric state $|\Psi\rangle \in \mathcal{S}_N^{(d)}$, one may look for symmetric decompositions and introduce alternative notions of ranks. The *symmetric tensor rank* [19,54], denoted by $\text{ST}(|\Psi\rangle)$, corresponds to the minimal r such that $|\Psi\rangle = \sum_{i=1}^r s_i |\Psi_i\rangle^{\otimes N}$ for $s_i \in \mathbb{C}$. Similarly, the *symmetric border rank*, denoted by $\text{SB}(|\Psi\rangle)$, is defined through approximated symmetric decompositions, satisfying $\text{SB}(|\Psi\rangle) \leq \text{ST}(|\Psi\rangle)$. By construction, one has $T(|\Psi\rangle) \leq \text{ST}(|\Psi\rangle)$ and $B(|\Psi\rangle) \leq \text{SB}(|\Psi\rangle)$.

It has been conjectured that $T(|\Psi\rangle) = \text{ST}(|\Psi\rangle)$ holds for any symmetric state, widely known as Comon’s conjecture [19]. An analogous conjecture exists for the border rank, asserting $B(|\Psi\rangle) = \text{SB}(|\Psi\rangle)$ [55,56]. Despite several efforts in this area (see Ref. [57] and references therein), a conclusive result remains elusive. Here, we establish the following result:

Result 3. Let $|\Psi\rangle \in \mathcal{S}_N^{(2)}$ be a multipartite state. Then,

$$R[\mathcal{M}(|\Psi\rangle)] = \text{SB}(|\Psi\rangle). \quad (8)$$

The proof is given in Appendix D, where we used tools from algebraic geometry, exploiting the correspondence between symmetric states and homogeneous polynomials. Our result not only provides an analytical calculation of the symmetric border rank, but also establishes a connection to the Schmidt rank of the mapped state. Explicit computations for several states are given in Appendix E.

VI. GEOMETRIC MEASURE

Here, we show that the amount of multipartite symmetric entanglement can be analytically estimated via the mapping. Consider the geometric measure of entanglement for a state $|\Psi\rangle \in \mathcal{H}_N^{(d)}$ [20–22]:

$$E(|\Psi\rangle) = 1 - \Lambda^2(|\Psi\rangle), \quad (9)$$

where $\Lambda(|\Psi\rangle) \equiv \max_{|\Phi_{\text{sep}}\rangle \in \mathcal{H}_N^{(d)}} \sqrt{F(|\Phi_{\text{sep}}\rangle, |\Psi\rangle)}$ is called the spectral norm (see review in Ref. [53]), and the maximization is taken over all product states. For any bipartite state $|\psi\rangle \in \mathcal{H}_2^{(d)}$, it holds that $\Lambda^2(|\psi\rangle) = \max_i \gamma_i^2 \equiv \gamma_{\text{max}}^2$, where γ_i is the Schmidt coefficient of $|\psi\rangle$ [58]. For a symmetric state $|\Psi\rangle \in \mathcal{S}_N^{(d)}$, the geometric measure can be efficiently computed by restricting to symmetric product states [25], i.e.,

$$\Lambda(|\Psi\rangle) = \Lambda_{\text{sym}}(|\Psi\rangle), \quad (10)$$

where $\Lambda_{\text{sym}}(|\Psi\rangle)$ is defined by replacing $|\Phi_{\text{sep}}\rangle \in \mathcal{H}_N^{(d)}$ in $\Lambda(|\Psi\rangle)$ with $|\Phi_{\text{sep}}\rangle \in \mathcal{S}_N^{(d)}$.

Now we can present the following result:

Result 4. Let $|\Psi\rangle \in \mathcal{S}_N^{(2)}$ be a multipartite state. Then,

$$1 - \gamma_{\text{max}}^2 \leq E(|\Psi\rangle), \quad (11)$$

where γ_{max} is the maximal Schmidt coefficient of the mapped state $\mathcal{M}(|\Psi\rangle)$.

Proof. We note that $E[\mathcal{M}(|\Psi\rangle)] \leq E(|\Psi\rangle)$ is equivalent to $\Lambda_{\text{sym}}^2(|\Psi\rangle) \leq \Lambda_{\text{sym}}^2[\mathcal{M}(|\Psi\rangle)]$, according to Eqs. (9) and (10). This can be shown as

$$\Lambda_{\text{sym}}^2(|\Psi\rangle) = \max_{\mathcal{M}(|\Phi_{\text{sep}}\rangle) \in \mathcal{S}_2^{(N/2+1)}} F[\mathcal{M}(|\Phi_{\text{sep}}\rangle), \mathcal{M}(|\Psi\rangle)] \quad (12a)$$

$$\leq \max_{|\phi_{\text{sep}}\rangle \in \mathcal{S}_2^{(N/2+1)}} F[|\phi_{\text{sep}}\rangle, \mathcal{M}(|\Psi\rangle)] \quad (12b)$$

$$= \Lambda_{\text{sym}}^2[\mathcal{M}(|\Psi\rangle)], \quad (12c)$$

where in Eq. (12a) we used Eq. (6), and in Eq. (12b), we employed Eq. (5). Thus, the proof is completed. $\blacksquare$

We have several remarks. First, Result 4 provides an analytical estimate of the geometric measure for multipartite symmetric states. In particular, it can be computed exactly when the inequality (11) is saturated, i.e., when the closest product state lies in $\mathcal{S}_2^{(N/2+1)}$, such as $|0\rangle^{\otimes N}$. In fact, the GHZ state achieves this saturation.

Next, we collect some cases where, for a symmetric entangled state $|\Psi\rangle$ with large $E(|\Psi\rangle)$, its mapped state $\mathcal{M}(|\Psi\rangle)$ also retains a large value $E[\mathcal{M}(|\Psi\rangle)]$. In particular, for the maximally entangled symmetric (MES) states for $N = 4, 6$ [29,30] (also see Ref. [59]), given by

$$|\Psi_4^{\text{MES}}\rangle = \sqrt{\frac{1}{3}}|D_4^0\rangle + \sqrt{\frac{2}{3}}|D_4^3\rangle, \quad (13a)$$

$$|\Psi_6^{\text{MES}}\rangle = \sqrt{\frac{1}{2}}|D_6^1\rangle + \sqrt{\frac{1}{2}}|D_6^5\rangle, \quad (13b)$$

the mapped states $\mathcal{M}(|\Psi_4^{\text{MES}}\rangle)$ and $\mathcal{M}(|\Psi_6^{\text{MES}}\rangle)$ remain maximally entangled. On the other hand, for $N = 8$, the candidate MES state, $|\Psi_8^{\text{MES}}\rangle = \alpha|D_8^1\rangle + \beta|D_8^0\rangle$, with $\alpha \approx 0.672$ and $\beta \approx 0.741$, as numerically found in Ref. [29], does not become maximally entangled after the mapping, despite achieving a high value $E[\mathcal{M}(|\Psi_8^{\text{MES}}\rangle)] = 0.816$.

Finally, we note that the mapping does not preserve the ordering of entangled states in general:

$$E(|\Psi_1\rangle) \geq E(|\Psi_2\rangle) \not\Rightarrow E[\mathcal{M}(|\Psi_1\rangle)] \geq E[\mathcal{M}(|\Psi_2\rangle)] \quad (14)$$

for symmetric states $|\Psi_1\rangle, |\Psi_2\rangle$. Examples are $|D_N^2\rangle$ and $|W_N\rangle$ for $N \geq 4$, satisfying $E(|D_N^2\rangle) = 1 - 2(N-1)(N-2)^{N-2}/N^{N-1} > E(|W_N\rangle) = 1 - [(N-1)/N]^{N-1}$ [21], but $E[\mathcal{M}(|D_N^2\rangle)] = 1 - N/[2(N-1)] < E[\mathcal{M}(|W_N\rangle)] = 1/2$. Beyond the geometric measure, finding multipartite entanglement measures that preserve the ordering for all two symmetric entangled states under the mapping is left for future work.

VII. ENTANGLEMENT FROM NUCLEAR NORM

Let us consider another entanglement measure for a state $|\Psi\rangle \in \mathcal{H}_N^{(d)}$ [60]:

$$H(|\Psi\rangle) = 1 - \frac{1}{\Xi^2(|\Psi\rangle)}, \quad (15)$$

where $\Xi(|\Psi\rangle)$ is the nuclear norm [61], defined as

$$\Xi(|\Psi\rangle) = \min \left\{ \sum_{i=1}^r |\xi_i|, \text{ s.t. } |\Psi\rangle = \sum_{i=1}^r \xi_i \bigotimes_{k=1}^N |\psi_i^{(k)}\rangle \right\}, \quad (16)$$

where $\xi_i \in \mathbb{C}$ and $|\psi_i^{(k)}\rangle$ are not necessarily orthogonal.

The nuclear norm is the dual norm of the spectral norm [62], i.e., $\Xi(|\Psi\rangle) = \max\{|\langle\Psi|X\rangle|, \text{ s.t. } \Lambda(|X\rangle) \leq 1\}$, which leads to the inequality $1 \leq \Xi(|\Psi\rangle)\Lambda(|\Psi\rangle)$. As a consequence, we have $E(|\Psi\rangle) \leq H(|\Psi\rangle)$. For any bipartite state $|\psi\rangle \in \mathcal{H}_2^{(d)}$, it holds that $\Xi(|\psi\rangle) = \sum_{i=1}^r \gamma_i$ for its Schmidt coefficient γ_i , which is connected to the negativity of $|\psi\rangle$ given by $\mathcal{N}(|\psi\rangle) = (1/2)[(\sum_{i=1}^r \gamma_i)^2 - 1]$ [63]. Using our mapping, we prove the analog of Result 4 for the entanglement measure $H(|\Psi\rangle)$ of a symmetric state:

Result 5. Let $|\Psi\rangle \in \mathcal{S}_N^{(2)}$ be a multipartite state. Then,

$$1 - \frac{1}{2\mathcal{N} + 1} \leq H(|\Psi\rangle), \quad (17)$$

where $\mathcal{N}$ is the negativity of the mapped state $\mathcal{M}(|\Psi\rangle)$.

Proof. We note that for a symmetric state $|\Psi\rangle \in \mathcal{S}_N^{(d)}$, the following relation holds [61]:

$$\Xi(|\Psi\rangle) = \Xi_{\text{sym}}(|\Psi\rangle), \quad (18)$$

where $\Xi_{\text{sym}}(|\Psi\rangle)$ is defined by replacing $\bigotimes_{k=1}^N |\psi_i^{(k)}\rangle$ in Eq. (16) with $|\psi_i\rangle^{\otimes N}$. Since this is the analog of Eq. (10) and the nuclear norm is the dual norm of the spectral norm, one can show that $\Xi_{\text{sym}}^2(|\Psi\rangle) \leq \Xi_{\text{sym}}^2[\mathcal{M}(|\Psi\rangle)]$, similarly to the proof of Result 4. Thus, the proof is completed. ■

Based on this, we can derive a necessary condition for a state to be an MES with respect to the geometric measure. For an MES state $|\Psi_N^{\text{MES}}\rangle$, it holds that $H[\mathcal{M}(|\Psi_N^{\text{MES}}\rangle)] \leq E(|\Psi_N^{\text{MES}}\rangle)$. Otherwise, the state is not MES. This can be shown by using the fact that $E(|\Psi_N^{\text{MES}}\rangle) = H(|\Psi_N^{\text{MES}}\rangle)$ holds for MES states $|\Psi_N^{\text{MES}}\rangle$ [60]. In fact, we can check that the candidate MES state $|\Psi_8^{\text{MES}}\rangle$ has $H[\mathcal{M}(|\Psi_8^{\text{MES}}\rangle)] = 0.793$ and $E(|\Psi_8^{\text{MES}}\rangle) = 0.816$.

VIII. ENTANGLED SYMMETRIC SUBSPACE

Finally, using our approach, we reveal the existence of an entangled subspace in the bipartite symmetric subspace. Let us recall that a subspace is called entangled if all pure states in it are entangled [64,65]. A famous example is the bipartite antisymmetric subspace, denoted as $\mathcal{A}_2^{(d)}$, with dimension $d(d-1)/2$ [64], where $\mathcal{H}_2^{(d)} = \mathcal{S}_2^{(d)} \oplus \mathcal{A}_2^{(d)}$ and $\mathcal{A}_2^{(d)}$ is spanned by the orthonormal basis $|\chi_{ij}^{(d)}\rangle = (1/\sqrt{2})(|ij\rangle_d - |ji\rangle_d)$. It is interesting to ask whether there is a nontrivial entangled subspace within $\mathcal{S}_2^{(d)}$.

We answer this question affirmatively as follows:

Result 6. Let $\mathcal{S}_2^{(d)}$ be the two-qudit symmetric subspace and $\tilde{\mathcal{S}}_2^{(d)}$ be its mapped subspace, where $d = N/2 + 1$ for any even N . Then, the complementary subspace $\hat{\mathcal{S}}_2^{(d)}$, such that $\mathcal{S}_2^{(d)} = \tilde{\mathcal{S}}_2^{(d)} \oplus \hat{\mathcal{S}}_2^{(d)}$, is entangled with dimension $(d-1)(d-2)/2$ (or, equivalently, $N(N-2)/8$).

The proof is given in Appendix F, and the subspace $\hat{\mathcal{S}}_2^{(d)}$ corresponds to the darkest orange-colored region in Fig. 1. There we will show that for any $|\psi_{\mathcal{S}}\rangle \in \hat{\mathcal{S}}_2^{(d)}$, the geometric measure is strictly greater than zero. Result 6 provides a fundamental consequence: In the two-qudit Hilbert space, $\mathcal{H}_2^{(d)} = \tilde{\mathcal{S}}_2^{(d)} \oplus \hat{\mathcal{S}}_2^{(d)} \oplus \mathcal{A}_2^{(d)}$, all states in $\tilde{\mathcal{S}}_2^{(d)}$ and $\mathcal{A}_2^{(d)}$ are entangled for any d . The simplest case is $d = 3$ (i.e., $N = 4$), when $\hat{\mathcal{S}}_2^{(3)}$ is a one-dimensional subspace spanned by $|\hat{\psi}_4\rangle = (1/\sqrt{3})|\psi_{11}^{(3)}\rangle - \sqrt{2/3}|\psi_{02}^{(3)}\rangle$.

IX. CONCLUSION

We presented a systematic approach to map multipartite symmetric states onto bipartite symmetric states of higher local dimension. This mapping allows for the analysis of the original multipartite entangled state via its mapped counterpart. Finally, we uncovered the existence of nontrivial entangled symmetric subspaces, whose states are not reached by the mapping.

Our findings open several avenues for further research. First, it would be worthwhile to relate the entanglement criterion discussed after Result 2 to the PPT criterion [43]. In particular, it would be interesting to investigate whether the mapping preserves the PPT condition. Second, connecting our results to the Majorana representation [29,30] may provide further insights into the structure of MES states. Also, it would be interesting to use the Schmidt rank to explore the entanglement dimensionality in the complementary subspace $\hat{\mathcal{S}}_2^{(d)}$ [66]. More broadly, our results may encourage advances in Bell nonlocality, alternative entanglement measures, and might be extended beyond the symmetric subspace. Finally, an experimental implementation of the proposed mapping would be of great interest. To this end, platforms based on atomic ensembles—such as those exploited in Refs. [67–69]—appear particularly promising for future experiments.

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX A: PROOF OF RESULT 1

Here, we give the proof of Result 1 presented in the main text by showing that

$$\mathcal{M}(|D_N^k\rangle) = \sum_{i \leq j=0}^{N/2} \delta_{k,i+j} \mu_{ij} |\psi_{ij}^{(N/2+1)}\rangle, \quad \mu_{ii} = \frac{\binom{N/2}{i}}{\sqrt{\binom{N}{2i}}}, \quad \mu_{ij} = \sqrt{2 \frac{\binom{N/2}{i} \binom{N/2}{j}}{\binom{N}{i+j}}}, \quad (\text{A1})$$

where $\delta_{a,b}$ is the Kronecker-delta symbol and $|\psi_{ij}^{(N/2+1)}\rangle \in \mathcal{S}_2^{(N/2+1)}$.

Proof. We start by recalling that any N -qubit Dicke state can be decomposed as [70–72]

$$|D_N^k\rangle = \sum_{i=\max\{0, k-N/2\}}^{\min\{k, N/2\}} \sqrt{\frac{\binom{N/2}{i} \binom{N/2}{k-i}}{\binom{N}{k}}} |D_{N/2}^i\rangle |D_{N/2}^{k-i}\rangle. \quad (\text{A2})$$

Assuming $k \geq N/2$ and using the fact that $\binom{N/2}{k-i} = 0$ for $k-i > N/2$, Eq. (A2) takes the form

$$|D_N^k\rangle = \sum_{i=0}^{N/2} \sqrt{\frac{\binom{N/2}{i} \binom{N/2}{k-i}}{\binom{N}{k}}} |D_{N/2}^i\rangle |D_{N/2}^{k-i}\rangle. \quad (\text{A3})$$

Recalling that $\mathcal{S}_N^{(2)} \cong \mathbb{C}^{N+1}$, we can set $|D_{N/2}^i\rangle \equiv |i\rangle_{N/2+1}$. Hence, Eq. (A3) can be rewritten as

$$\mathcal{M}(|D_N^k\rangle) = \sum_{i,j=0}^{N/2} \delta_{k,i+j} \sqrt{\frac{\binom{N/2}{i} \binom{N/2}{j}}{\binom{N}{k}}} |i\rangle_{N/2+1} |j\rangle_{N/2+1}. \quad (\text{A4})$$

Splitting the sum into $i = j$ and $i \neq j$ allows to identify μ_{ij} as in Eq. (A1), thus leading to the conclusion. An analogous reasoning yields the same result when $k < N/2$. ■

For the sake of clarity, we here provide an explicit example of the mapping by considering the four-qubit Dicke state $|D_4^1\rangle = (1/2)(|1000\rangle + |0100\rangle + |0010\rangle + |0001\rangle)$. The first step is to split it into two parts:

$$|D_4^1\rangle = \frac{1}{\sqrt{2}} \left[\left(\frac{|10\rangle + |01\rangle}{\sqrt{2}} \right) |00\rangle + |00\rangle \left(\frac{|10\rangle + |01\rangle}{\sqrt{2}} \right) \right]. \quad (\text{A5})$$

Next, Eq. (A5) can be rewritten in terms of the two-qubit states $|D_2^k\rangle$, i.e.,

$$|D_4^1\rangle = \frac{1}{\sqrt{2}} (|D_2^1\rangle |D_2^0\rangle + |D_2^0\rangle |D_2^1\rangle), \quad (\text{A6})$$

where $|D_2^0\rangle = |00\rangle$, $|D_2^1\rangle = (1/\sqrt{2})(|10\rangle + |01\rangle)$, and $|D_2^2\rangle = |11\rangle$. Finally, notice that Eq. (A6) can be cast as

$$|D_4^1\rangle = \frac{1}{\sqrt{2}} (|1\rangle_3 |0\rangle_3 + |0\rangle_3 |1\rangle_3) \equiv |\psi_{01}^{(3)}\rangle, \quad (\text{A7})$$

where we have set $|D_2^k\rangle = |k\rangle_3$ for $k = 0, 1, 2$, using the fact that $\mathcal{S}_2^{(2)}$ has dimension 3, i.e., $\mathcal{S}_2^{(2)} \cong \mathbb{C}^3$. Consequently, any Dicke state $|D_4^k\rangle$ can be expressed as a linear combination of the two-qutrit states $|\psi_{ij}^{(3)}\rangle$ as follows:

$$\begin{aligned} |D_4^0\rangle &= |\psi_{00}^{(3)}\rangle, & |D_4^1\rangle &= |\psi_{01}^{(3)}\rangle, \\ |D_4^2\rangle &= \frac{1}{\sqrt{3}} (|\psi_{02}^{(3)}\rangle + \sqrt{2} |\psi_{11}^{(3)}\rangle), \\ |D_4^3\rangle &= |\psi_{12}^{(3)}\rangle, & |D_4^4\rangle &= |\psi_{22}^{(3)}\rangle. \end{aligned} \quad (\text{A8})$$

APPENDIX B: PROOF OF EQ. (6)

Here, we give the proof of Eq. (6) by showing that for any $|\psi_1\rangle, |\psi_2\rangle \in S_N^{(2)}$, it holds that

$$F(|\psi_1\rangle, |\psi_2\rangle) = F[\mathcal{M}(|\psi_1\rangle), \mathcal{M}(|\psi_2\rangle)], \quad (\text{B1})$$

where $F(|\psi_1\rangle, |\psi_2\rangle) \equiv |\langle\psi_1|\psi_2\rangle|^2$.

Proof. We begin by writing $|\psi_1\rangle = \sum_k a_k |D_N^k\rangle$ and $|\psi_2\rangle = \sum_k b_k |D_N^k\rangle$ with coefficients $a_k, b_k \in \mathbb{C}$. Using Result 1, we have that

$$\mathcal{M}(|\psi_1\rangle) = \sum_k \sum_{i \leq j=0}^{N/2} a_k \delta_{k,i+j} \mu_{ij} |\psi_{ij}^{(N/2+1)}\rangle, \quad \mathcal{M}(|\psi_2\rangle) = \sum_k \sum_{i \leq j=0}^{N/2} b_k \delta_{k,i+j} \mu_{ij} |\psi_{ij}^{(N/2+1)}\rangle. \quad (\text{B2})$$

Therefore, $F[\mathcal{M}(|\psi_1\rangle), \mathcal{M}(|\psi_2\rangle)]$ can be given by

$$F[\mathcal{M}(|\psi_1\rangle), \mathcal{M}(|\psi_2\rangle)] = \left| \sum_{k,k'} \sum_{i \leq j=0}^{N/2} \sum_{i' \leq j'=0}^{N/2} a_k^* b_{k'} \delta_{k,i+j} \delta_{k',i'+j'} \mu_{ij} \mu_{i'j'} \langle \psi_{ij}^{(N/2+1)} | \psi_{i'j'}^{(N/2+1)} \rangle \right|^2 \quad (\text{B3a})$$

$$= \left| \sum_{k,k'} \sum_{i \leq j=0}^{N/2} \sum_{i' \leq j'=0}^{N/2} a_k^* b_{k'} \delta_{k,i+j} \delta_{k',i'+j'} \mu_{ij} \mu_{i'j'} \delta_{ii'} \delta_{jj'} \right|^2 \quad (\text{B3b})$$

$$= \left| \sum_k \sum_{i \leq j=0}^{N/2} a_k^* b_k \delta_{k,i+j} \mu_{ij}^2 \right|^2 \quad (\text{B3c})$$

$$= \left| \sum_k a_k^* b_k f(k, N) \right|^2, \quad f(k, N) \equiv \sum_{i \leq j=0}^{N/2} \delta_{k,i+j} \mu_{ij}^2. \quad (\text{B3d})$$

Then, it is sufficient to show $f(k, N) = 1$ for any $0 \leq k \leq N$. Splitting the sum in $f(k, N)$ into $i = j$ and $i \neq j$ yields

$$f(k, N) = \sum_{i=0}^{N/2} \delta_{k,2i} \mu_{ii}^2 + \frac{1}{2} \sum_{i,j=0, i \neq j}^{N/2} \delta_{k,i+j} \mu_{ij}^2 \quad (\text{B4a})$$

$$= \sum_{i=0}^{N/2} \delta_{k,2i} \left(\frac{\binom{N/2}{i}}{\sqrt{\binom{N}{2i}}} \right)^2 + \sum_{i,j=0, i \neq j}^{N/2} \delta_{k,i+j} \frac{\binom{N/2}{i} \binom{N/2}{j}}{\binom{N}{i+j}} \quad (\text{B4b})$$

$$= \sum_{i,j=0}^{N/2} \delta_{k,i+j} \frac{\binom{N/2}{i} \binom{N/2}{j}}{\binom{N}{i+j}} \quad (\text{B4c})$$

$$= \frac{1}{\binom{N}{k}} \sum_{i=0}^{N/2} \binom{N/2}{i} \binom{N/2}{k-i} = \frac{1}{\binom{N}{k}} \sum_{i=0}^k \binom{N/2}{i} \binom{N/2}{k-i} = 1, \quad (\text{B4d})$$

where in the last line we have used the fact that $\binom{N/2}{k-i} = 0$ for any $i > k$, and the Chu-Vandermonde identity, i.e.,

$$\sum_{i=0}^k \binom{N/2}{i} \binom{N/2}{k-i} = \binom{N}{k}. \quad (\text{B5})$$

Hence, we complete the proof. ■

APPENDIX C: PROOF OF RESULT 2

Here, we give the proof of Result 2 by showing that for an N -qubit separable symmetric state $|\Phi_{\text{sep}}\rangle \in \mathcal{S}_N^{(2)}$, the mapped bipartite state $\mathcal{M}(|\Phi_{\text{sep}}\rangle) \in \tilde{\mathcal{S}}_2^{(N/2+1)}$ is separable.

Proof. Let us begin by noticing that any N -qubit symmetric separable state can be cast as

$$|\Phi_{\text{sep}}\rangle = |\Phi\rangle^{\otimes N} = \sum_{k=0}^N \sqrt{\binom{N}{k}} \Phi_0^{N-k} \Phi_1^k |D_N^k\rangle, \quad (\text{C1})$$

where $|\Phi\rangle = \sum_{i=0}^1 \Phi_i |i\rangle \in \mathbb{C}^2$. Applying the mapping of Eq. (3) yields

$$\mathcal{M}(|\Phi_{\text{sep}}\rangle) = \sum_{i \leq j=0}^{N/2} \sqrt{\binom{N}{i+j}} \Phi_0^{N-i-j} \Phi_1^{i+j} \mu_{ij} |\psi_{ij}^{(N/2+1)}\rangle. \quad (\text{C2})$$

Recall that a two-qudit symmetric separable state can be written as

$$|\phi_{\text{sep}}\rangle = |\phi\rangle^{\otimes 2} = \sum_{i=0}^{d-1} \phi_i^2 |\psi_{ii}^{(d)}\rangle + \sum_{i < j=0}^{d-1} \sqrt{2} \phi_i \phi_j |\psi_{ij}^{(d)}\rangle, \quad (\text{C3})$$

with $|\phi\rangle = \sum_{i=0}^{d-1} \phi_i |i\rangle \in \mathbb{C}^d$. Setting $\phi_i = \sqrt{\binom{N/2}{i}} \Phi_0^{N/2-i} \Phi_1^i$, an explicit calculation yields $\mathcal{M}(|\Phi_{\text{sep}}\rangle) = |\phi_{\text{sep}}\rangle$, thus concluding the proof. $\blacksquare$

APPENDIX D: PROOF OF RESULT 3

Here, we prove Result 3 by showing that $R[\mathcal{M}(|\Psi\rangle)] = \text{SB}(|\Psi\rangle)$.

Proof. In what follows, we will make use of some tools from algebraic geometry, where a key role is played by multivariate polynomials. Similarly to the case of quantum states, also for polynomials several ranks can be introduced. For instance, the *polynomial rank* of $p(x_0, x_1, \dots, x_{d-1})$, denoted by $\text{PR}(p)$, is defined as the minimum number r such that there exists a decomposition of the form $p(x_0, x_1, \dots, x_{d-1}) = \sum_{i=1}^r (x_0 \beta_{0,i} + \dots + x_{d-1} \beta_{d-1,i})^N$ [18]. Similarly, the symmetric border rank $\text{SB}(p)$ of a polynomial is defined as the minimum integer r such that there exists a sequence of homogeneous polynomials $\{p_\epsilon\}$ with polynomial rank $\text{PR}(p) = r$ and such that $\lim_{\epsilon \rightarrow 0} p_\epsilon = p$. Remarkably, there exists an isomorphism between a state $|\Psi\rangle \in \mathcal{S}_N^{(d)}$ and a homogeneous polynomial $p(x_0, x_1, \dots, x_{d-1})$ of degree N in d variables (see, e.g., Refs. [18, 73]). In particular, a monomial $x_0^{k_0} x_1^{k_1} \dots x_{d-1}^{k_{d-1}}$ is associated with a quantum state with k_0 systems in $|0\rangle$, k_1 systems in $|1\rangle$, and so on. As an example, focusing on the qubit case (i.e., $d = 2$), the four-qubit state $|0101\rangle$ corresponds to $x_0^2 x_1^2$. As a consequence, a state $|\Psi\rangle = \sum_{k=0}^N \alpha_k |D_N^k\rangle$ can be associated with a polynomial p_Ψ of the form

$$p_\Psi = \sum_{k=0}^N \binom{N}{k} \frac{\alpha_k}{\sqrt{\binom{N}{k}}} x_0^{N-k} x_1^k. \quad (\text{D1})$$

Leveraging on this correspondence, the symmetric tensor rank of a quantum state can be identified with the polynomial rank of its associated polynomial, i.e., $\text{ST}(|\Psi\rangle) \equiv \text{PR}(p_\Psi)$ [18]. Analogously, we have $\text{SB}(|\Psi\rangle) \equiv \text{SB}(p_\Psi)$ [73].

A celebrated result in algebraic geometry is the Sylvester's algorithm [74] (see also Refs. [75, 76] for some modern versions), which allows to compute the polynomial rank of a bivariate polynomial $p(x_0, x_1)$. In combination with a theorem by Comas and Seiguer [76], a simplified algorithm can be obtained, which provides also its symmetric border rank (see Algorithm 2.62 in Ref. [77]). In order to illustrate this result, let us introduce the n th *catalecticant matrix* $C_{N-n,n}(p)$ [78] of a polynomial $p(x_0, x_1) = \sum_{k=0}^N \binom{N}{k} a_k x_0^{N-k} x_1^k$. This is defined as the $(N - n + 1) \times (n + 1)$ matrix with elements

$[C_{N-n,n}(p)]_{ij} = a_{i+j}$, with $i = 0, \dots, N-n$ and $j = 0, \dots, n$. Hence, we have the following [77]:

$$\text{rank}(C_{N/2,N/2}(p)) = \text{SB}(p). \quad (\text{D2})$$

We want to prove that

$$\text{rank}(C_\Psi) = R[\mathcal{M}(|\Psi\rangle)], \quad (\text{D3})$$

where we have denoted by $C_\Psi \equiv C_{N/2,N/2}(p_\Psi)$ the $N/2$ -catalecticant matrix associated with a state $|\Psi\rangle \in \mathcal{S}_N^{(2)}$.

Let us now consider a state $|\Psi\rangle = \sum_{k=0}^N \alpha_k |D_N^k\rangle$. As a consequence of Eq. (D1), the catalecticant matrix C_Ψ has elements $[C_\Psi]_{ij} = \alpha_{i+j} / \sqrt{\binom{N}{i+j}}$.

Turning to the mapped state $\mathcal{M}(|\Psi\rangle)$, its Schmidt rank corresponds to the matrix rank of the reduced density matrix $\varrho_{\mathcal{M}(|\Psi\rangle)} = \text{tr}_B[\mathcal{M}(|\Psi\rangle)\mathcal{M}(|\Psi\rangle)^\dagger]$. First, note that we can write

$$\varrho_{\mathcal{M}(|\Psi\rangle)} = \sum_{i,j,k=0}^{N/2} \alpha_{i+j} \alpha_{k+j} \lambda_{ij} \lambda_{jk} |i\rangle\langle k| \quad (\text{D4})$$

$$= \sum_{i,k=0}^{N/2} \sum_{j=0}^{N/2} A_{ij} A_{jk} |i\rangle\langle k| = AA^T, \quad (\text{D5})$$

where we have introduced a matrix A such that

$$A_{ij} = \alpha_{i+j} \lambda_{ij}, \quad \lambda_{ij} = \sqrt{\frac{\binom{N/2}{i} \binom{N/2}{j}}{\binom{N}{i+j}}}. \quad (\text{D6})$$

As a consequence,

$$R[\mathcal{M}(|\Psi\rangle)] = \text{rank}(\varrho_{\mathcal{M}(|\Psi\rangle)}) = \text{rank}(AA^T) = \text{rank}(A). \quad (\text{D7})$$

A direct calculation shows that we can write

$$A = DC_\Psi D^T, \quad (\text{D8})$$

where D is a diagonal matrix with elements $D_{ii} = \sqrt{\binom{N/2}{i}}$. Since D is a full-rank matrix, it follows that

$$\text{rank}(A) = \text{rank}(DC_\Psi D^T) = \text{rank}(C_\Psi), \quad (\text{D9})$$

from which we find Eq. (D3). Then, using Eq. (D2), the proof is completed. $\blacksquare$

APPENDIX E: COMPUTATION OF THE SYMMETRIC BORDER RANK OF SOME MULTIPARTITE SYMMETRIC STATES

Here, we compute explicitly the border rank of some symmetric states of N qubits. We start by showing that $\text{SB}(|D_N^k\rangle)$ is given by $k+1$ for $k \leq N/2$ and $N-k+1$ for $k > N/2$.

Proof. Using Result 3, to compute the border rank of $|D_N^k\rangle$ is enough to calculate the Schmidt rank of the state $\mathcal{M}(|D_N^k\rangle)$. This is equivalent to compute the matrix rank of the reduced state $\varrho_{|D_N^k\rangle} = \text{tr}_B[\mathcal{M}(|D_N^k\rangle)\mathcal{M}(|D_N^k\rangle)^\dagger]$, where tr_B denotes the partial trace over the second party. Note that, due to the permutational invariance, the expression of $\varrho_{|D_N^k\rangle}$ does not depend on the choice of the party. Using Eq. (A4), we can write

$$\varrho_{|D_N^k\rangle} = \sum_{i,j=0}^{N/2} \sum_{m,n=0}^{N/2} \delta_{i+j,k} \delta_{m+n,k} \sqrt{\frac{\binom{N/2}{i} \binom{N/2}{j}}{\binom{N}{k}}} \sqrt{\frac{\binom{N/2}{m} \binom{N/2}{n}}{\binom{N}{k}}} \text{tr}_B[|ij\rangle\langle mn|] \quad (\text{E1})$$

TABLE I. Symmetric border rank of some N -qubit symmetric states.

$ \Psi\rangle$	$ W_N\rangle$	$ \text{GHZ}_N\rangle$	$ D_N^k\rangle$
$\text{SB}(\Psi\rangle)$	2	2	$k + 1, k \leq N/2$ $N - k + 1, k > N/2$

$$= \sum_{i,j,n=0}^{N/2} \delta_{i+j,k} \delta_{m+j,k} \sqrt{\binom{N/2}{i} \binom{N/2}{m}} \frac{\binom{N/2}{j}}{\binom{N}{k}} |i\rangle \langle m| \quad (\text{E2})$$

$$= \sum_{i,j=0}^{N/2} \delta_{i+j,k} \frac{\binom{N/2}{i} \binom{N/2}{j}}{\binom{N}{k}} |i\rangle \langle i| \quad (\text{E3})$$

$$= \sum_{i=0}^{N/2} \frac{\binom{N/2}{i} \binom{N/2}{k-i}}{\binom{N}{k}} |i\rangle \langle i|. \quad (\text{E4})$$

Since ϱ_k is diagonal in the computational basis, its matrix rank corresponds to the number of nonzero elements. This can be easily computed observing that the last term in Eq. (E4) can be cast as [see also Eq. (A2)]

$$\varrho_{|D_N^k\rangle} = \sum_{i=\max\{0, k-N/2\}}^{\min\{k, N/2\}} \frac{\binom{N/2}{i} \binom{N/2}{k-i}}{\binom{N}{k}} |i\rangle \langle i|. \quad (\text{E5})$$

Hence, when $k \leq N/2$, the matrix rank of $\varrho_{|D_N^k\rangle}$ is found to be $k + 1$. Analogously, when $k > N/2$, the matrix rank is $N - k + 1$. Hence, the proof is complete. $\blacksquare$

As a consequence of this result, since the N -qubit W and GHZ states can be expressed as linear combinations of Dicke states, it is immediate to compute their border rank for any even N . Indeed, we have $|W_N\rangle \equiv |D_N^1\rangle$, from which it follows $\text{SB}(|W_N\rangle) = 2$.

Similarly, considering the GHZ state, we can set $|\text{GHZ}_N\rangle \equiv (|D_N^0\rangle + |D_N^N\rangle)/\sqrt{2}$, from which we have $\text{SB}(|\text{GHZ}_N\rangle) = 2$. The above results can be summarized in the Table I.

APPENDIX F: PROOF OF RESULT 6

Here, we give the proof of Result 6 by showing that for any state $|\psi_{\mathcal{S}}\rangle \in \hat{\mathcal{S}}_2^{(d)}$, the geometric measure is nonzero, i.e., $E(|\psi_{\mathcal{S}}\rangle) > 0$.

Proof. We begin by denoting $\Pi_{\mathcal{S}}$, $\Pi_{\mathcal{S}}$, and $\Pi_{\mathcal{S}}$ as the projectors onto the subspaces $\mathcal{S}_2^{(d)}$, $\tilde{\mathcal{S}}_2^{(d)}$, and $\hat{\mathcal{S}}_2^{(d)}$, respectively. Note that $\Pi_{\mathcal{S}} = \Pi_{\mathcal{S}} + \Pi_{\mathcal{S}} = (1/2)(\mathbb{1} + \mathbb{S})$, where $\mathbb{S}$ denotes the swap operator such that $\mathbb{S}|ab\rangle = |ba\rangle$. Recalling that $E(|\psi\rangle) = 1 - \max_{|ab\rangle} |\langle ab|\psi\rangle|^2$ and $\hat{\mathcal{S}}_2^{(d)}$ is the symmetric subspace, the relation in Eq. (10) reduces our task to show that $\max_{|a\rangle} |\langle aa|\psi_{\mathcal{S}}\rangle|^2 < 1$ for $|\psi_{\mathcal{S}}\rangle \in \hat{\mathcal{S}}_2^{(d)}$.

Using Eq. (40) in Ref. [79], we have

$$\max_{|a\rangle} |\langle aa|\psi_{\mathcal{S}}\rangle|^2 \leq \max_{|a\rangle} \langle aa|\Pi_{\mathcal{S}}|aa\rangle = 1 - \min_{|a\rangle} \langle aa|\Pi_{\mathcal{S}}|aa\rangle, \quad (\text{F1})$$

where in the second equality we used the relation that $\Pi_{\mathcal{S}} = \Pi_{\mathcal{S}} - \Pi_{\mathcal{S}}$. This yields

$$E(|\psi_{\mathcal{S}}\rangle) \geq \min_{|a\rangle} \langle aa|\Pi_{\mathcal{S}}|aa\rangle. \quad (\text{F2})$$

Then, our task is further reduced to show that $\min_{|a\rangle} \langle aa | \Pi_{\mathcal{S}} | aa \rangle > 0$. Letting $|a\rangle = \sum_{i=0}^{d-1} a_i |i\rangle$, we write

$$|a\rangle^{\otimes 2} = \left(\sum_{i=0}^{d-1} a_i |i\rangle \right)^{\otimes 2} = \sum_i a_i^2 |\psi_{ii}^{(d)}\rangle + \frac{1}{2} \sum_{i \neq j} \sqrt{2} a_i a_j |\psi_{ij}^{(d)}\rangle. \quad (\text{F3})$$

Inserting this form into $\langle aa | \Pi_{\mathcal{S}} | aa \rangle$ and using $\Pi_{\mathcal{S}} = \sum_{k=0}^N |\phi_k\rangle \langle \phi_k|$, with $|\phi_k\rangle = \sum_{i \leq j=0}^{d-1} \delta_{k,i+j} \mu_{ij} |\psi_{ij}^{(d)}\rangle$, we obtain

$$\langle aa | \Pi_{\mathcal{S}} | aa \rangle = \sum_{k=0}^N |\langle aa | \phi_k \rangle|^2 \quad (\text{F4})$$

$$= \sum_{k=0}^N \left| \sum_{i=0}^{d-1} \delta_{k,2i} \mu_{ii} a_i^2 + \frac{\sqrt{2}}{4} \sum_{i \neq j} \sum_{l \neq m} \delta_{k,l+m} \mu_{lm} a_i a_j (\delta_{il} \delta_{jm} + \delta_{im} \delta_{jl}) \right|^2 \quad (\text{F5})$$

$$= \sum_{k=0}^N \left| \sum_{i=0}^{d-1} \delta_{k,2i} \mu_{ii} a_i^2 + \frac{\sqrt{2}}{2} \sum_{i \neq j} \delta_{k,i+j} \mu_{ij} a_i a_j \right|^2 \quad (\text{F6})$$

$$\equiv \sum_{k=0}^N \left| \sum_{i,j=0}^{d-1} a_i M_{ij}^{(k)} a_j \right|^2, \quad (\text{F7})$$

where we denoted the matrix $M^{(k)} = (M_{ij}^{(k)})$ with elements

$$M_{ii}^{(k)} = \delta_{k,2i} \mu_{ii}, \quad M_{ij}^{(k)} = \frac{\sqrt{2}}{2} \delta_{k,i+j} \mu_{ij}. \quad (\text{F8})$$

Furthermore, letting $\mathbf{a} = (a_0, \dots, a_{d-1})^\top \in \mathbb{C}^d$, we have

$$\sum_{k=0}^N \left| \sum_{i,j=0}^{d-1} a_i M_{ij}^{(k)} a_j \right|^2 = \sum_{k=0}^N |\mathbf{a}^\top M^{(k)} \mathbf{a}|^2. \quad (\text{F9})$$

Thus, our task is reduced as follows:

$$\begin{aligned} E(|\psi_{\mathcal{S}}\rangle) &\geq \min_{|a\rangle} \langle aa | \Pi_{\mathcal{S}} | aa \rangle = \min_{\mathbf{a}} \left\{ \sum_{k=0}^N |\mathbf{a}^\top M^{(k)} \mathbf{a}|^2 \right\} \\ &\geq \sum_{k=0}^N \{ \min_{\mathbf{a}} |\mathbf{a}^\top M^{(k)} \mathbf{a}|^2 \} = \sum_{k=0}^N \{ \sigma_{\min}[M^{(k)}] \}^2, \end{aligned} \quad (\text{F10})$$

where $\sigma_{\min}[M^{(k)}]$ denotes the minimal absolute eigenvalue of $M^{(k)}$. From the definition of the matrix $M^{(k)}$ in Eq. (F8), we notice that for $k < N/2$ or $k > N/2$, it can be always written as the following block form:

$$M^{(k < N/2)} = \begin{bmatrix} A_k & 0 \\ 0 & 0 \end{bmatrix}, \quad M^{(k > N/2)} = \begin{bmatrix} 0 & 0 \\ 0 & A_k \end{bmatrix} \quad (\text{F11})$$

for some matrices A_k . In this case, the matrix $M^{(k)}$ is not full rank, i.e., $\sigma_{\min}[M^{(k)}] = 0$. On the other hand, for $k = N/2$, the matrix $M^{(k=N/2)}$ can be written as the antidiagonal form, where $M_{ij}^{(k=N/2)} \neq 0$ for $i + j = N/2$, but $M_{ij}^{(k=N/2)} = 0$ for all the other i, j . Then, the matrix $M^{(k=N/2)}$ is full rank, i.e., $\sigma_{\min}[M^{(k=N/2)}] > 0$. Hence, we can complete the proof. $\blacksquare$

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