

Phason-resolved finite-temperature topological diagnostics in quasiperiodic Chern-insulator approximants

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Finite-temperature topological diagnostics in quasiperiodic systems are subtle because quasiperiodicity removes the Bloch momentum torus on which mixed-state band indices are usually formulated. We study a quasiperiodic Qi-Wu-Zhang Chern insulator on finite Fibonacci approximants and use twisted boundary phases as an operational two-torus. A gap scan over $L = 21, 34, 55$ shows that the fixed-boundary spectral gap can remain finite while extended low-gap regions appear elsewhere on the twist torus. Projected-Uhlmann diagnostics reveal a sharper finite-size result: The stability of the projected winding is not determined by the scalar minimum twist gap alone. A high-resolution $L = 34, 25 \times 41$ audit gives operational stability scales 5×10^{-4} , 10^{-2} , and 3×10^{-2} for three cases whose minimum twist gaps are all of order 10^{-3} . The same ranking is obtained with projection bases on the $\theta_x = 0$ and exact $\theta_x = \pi$ slices. Direct phase-curve diagnostics show that collapse is associated with the loss or singularization of a 2π projected-Uhlmann phase winding, whereas stable cases preserve the winding despite small spectral gaps. Within the finite approximants studied here, these results identify phason-resolved boundary-phase geometry and projected-holonomy sensitivity, rather than the fixed-boundary gap or scalar minimum gap alone, as the mechanism controlling this finite-size projected diagnostic.

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I. INTRODUCTION

At zero temperature, topological phases are conventionally characterized by geometric invariants of occupied states [1–3]. For crystalline Chern insulators, the Chern number is defined on the Bloch momentum torus [1]. In disordered or quasiperiodic systems, translation symmetry is absent, and real-space or twisted-boundary formulations are required [4–9]. Quasiperiodic systems are especially interesting because topology can persist without ordinary crystalline translational symmetry and can be controlled by phason degrees of freedom [10–18].

The finite-temperature problem is more delicate. A thermal state is described by a density matrix rather than a pure occupied-band projector. The Uhlmann construction provides a geometric parallel transport for density matrices and has motivated finite-temperature topological diagnostics

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in band systems [19–26]. Recent ensemble-geometric-phase, Uhlmann-Chern, and mixed-state order-parameter constructions further illustrate that mixed-state topology depends on the chosen density-matrix geometry and physical observable [27–30]. For quasiperiodic Chern-insulator approximants, however, the absence of the Bloch torus raises a concrete question: Which finite-system manifold controls mixed-state diagnostics? Twisted boundary conditions provide an operational answer [31,32]. The phases (θ_x, θ_y) define a boundary-phase torus for each finite approximant, making it possible to compare the fixed-boundary Bott index, the full twist-space gap landscape, and projected-Uhlmann holonomy in the same calculation.

In this work, we show that the projected finite-temperature response in these approximants is controlled by a phason-resolved boundary-phase mechanism. The result is not simply that a small minimum gap destroys the diagnostic. Instead, the scalar minimum twist gap is an important warning sign but an insufficient predictor. Some phason sectors with small twist gaps lose the projected-Uhlmann winding at very low temperature, while other sectors with comparable small gaps retain a quantized projected winding over the full sampled temperature range. The discriminating information is contained in the geometry of the projected-Uhlmann phase winding on the boundary-phase torus: Phasons can preserve or destroy the winding even at similar scalar spectral bottlenecks.

Figure 1 gives the roadmap for the article. Panel (a) defines the finite real-space approximant, the two boundary twists, and the phason offsets that shift the quasiperiodic modulation. Panel (b) illustrates the central spectral observation: Two phason choices can have comparable minimum twist gaps but different finite-temperature responses. Panel (c) translates this comparison into projected-holonomy language, where a stable case winds around the origin in the complex plane of $\det W_{\text{occ}}$, whereas a collapse case loses or singularizes this winding. Panel (d) summarizes the logic followed in the results sections: First map the twist-gap terrain, then measure the projected-Uhlmann response, and finally confirm the mechanism through phase-winding diagnostics.

II. MODEL AND DIAGNOSTICS

A. Quasiperiodic Qi-Wu-Zhang approximants

To make this comparison concrete, we consider a real-space Qi-Wu-Zhang Chern insulator on an $L_x \times L_y$ square lattice with two internal orbitals per site [3,33]. At each lattice site $\mathbf{r} = (x, y)$, $\hat{x}$ and $\hat{y}$ denote the unit lattice vectors along the x and y directions, respectively, with the lattice constant set to unity. $c_{\mathbf{r}}^\dagger = (c_{\mathbf{r},1}^\dagger, c_{\mathbf{r},2}^\dagger)$ creates a two-component orbital spinor, and $\sigma_{x,y,z}$ and σ_0 denote Pauli matrices and the identity in this orbital space. The Hamiltonian is

$$H(\theta_x, \theta_y) = H_{\text{QWZ}}(\theta_x, \theta_y) + V_{\text{qp}}, \quad (1)$$

with

$$H_{\text{QWZ}} = \sum_{\mathbf{r}} c_{\mathbf{r}}^\dagger m \sigma_z c_{\mathbf{r}} + \sum_{\mathbf{r}} (c_{\mathbf{r}+\hat{x}}^\dagger T_x c_{\mathbf{r}} + \text{H.c.}) + \sum_{\mathbf{r}} (c_{\mathbf{r}+\hat{y}}^\dagger T_y c_{\mathbf{r}} + \text{H.c.}). \quad (2)$$

Hoppings crossing the boundary acquire phases $e^{i\theta_x}$ or $e^{i\theta_y}$, where θ_x and θ_y are the boundary twists threaded through the two cycles of the finite torus. We use

$$T_x = \frac{1}{2}\sigma_z - \frac{i}{2}\sigma_x, \quad T_y = \frac{1}{2}\sigma_z - \frac{i}{2}\sigma_y, \quad (3)$$

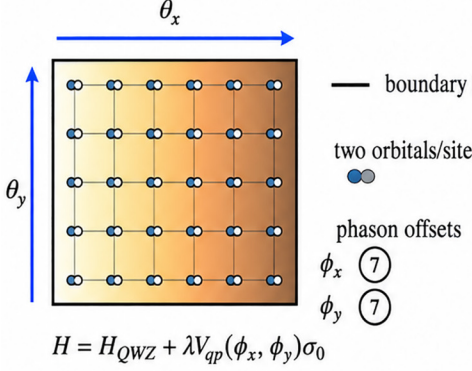
and set $m = -1$. The quasiperiodic on-site modulation is

$$V_{\text{qp}}(x, y) = \lambda[\cos(2\pi\alpha x + \phi_x) + \cos(2\pi\alpha y + \phi_y)]\sigma_0. \quad (4)$$

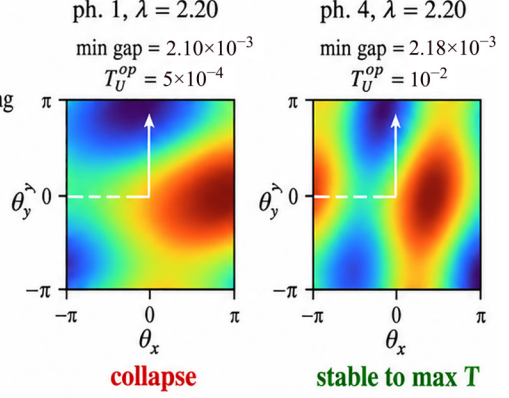
The phason offsets (ϕ_x, ϕ_y) shift the incommensurate modulation. We use Fibonacci approximants $L = q = 21, 34, 55$, with $\alpha = p/q = 13/21, 21/34, 34/55$. For the Stage-1 gap scan, the offsets are the first N_ϕ points of the deterministic two-dimensional Kronecker sequence $\phi_x^{(j)} = 2\pi\{j(\sqrt{5}-1)/2\}$, $\phi_y^{(j)} = 2\pi\{j(\sqrt{2}-1)\}$, with $j = 0, 1, 2, \dots, N_\phi - 1$ and $N_\phi = 24, 12, 6$

Phason-resolved boundary-phase holonomy mechanism

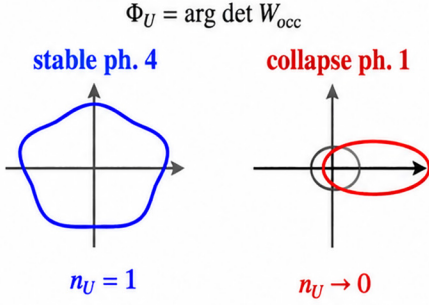
(a) Model, twists, and phasons



(b) Phason reshapes twist-gap terrain



(c) Projected holonomy distinguishes the cases



Comparable minimum gaps, distinct projected holonomy

(d) Evidence map and article flow

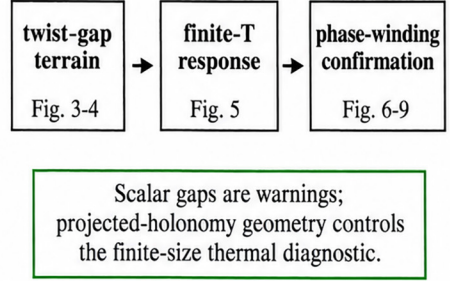


FIG. 1. Roadmap of the phason-resolved boundary-phase mechanism. (a) A finite quasiperiodic Qi-Wu-Zhang approximant is placed on a torus with boundary twists (θ_x, θ_y) ; phason offsets (ϕ_x, ϕ_y) shift the quasiperiodic on-site modulation. (b) The twist-gap terrain can be reshaped by the phason. The representative collapse and stable cases have comparable scalar minimum twist gaps but different operational stability scales. (c) The difference is resolved by the projected-Uhlmann holonomy: The stable case preserves the phase winding of $\det W_{occ}$, whereas the collapse case loses or singularizes it. (d) The article follows this logic by combining twist-gap scans, finite-temperature projected-Uhlmann response, and phase-winding confirmation.

for $L = 21, 34, 55$, respectively. This is neither a Cartesian grid nor a random draw; no random seed or physical trajectory is involved. The highlighted projected-Uhlmann cases were fixed in a later scan-informed targeted step. In particular, the phason 4 and 7 offsets were manually specified after inspection of the Stage-1 results, not by a recorded automatic global-extremum rule, and are not claimed to represent the phason ensemble. The actual calculation offsets are listed in Table I and the complete Stage-1 list is given in the Supplemental Material [34]. All phases are in radians modulo 2π .

B. Fixed-half-filling Bott and twist-gap diagnostics

At fixed boundary phase, we compute the Bott index from the projector P onto the lowest N_{occ} single-particle eigenstates [5,7]. Here and below, the single-particle eigenvalues E_n are ordered increasingly. With real-space phase operators $U_x = e^{2\pi i X/L_x}$ and $U_y = e^{2\pi i Y/L_y}$, where X and Y are

TABLE I. Targeted phason offsets used in the projected-Uhlmann scans. Phasons 1 and 2 coincide with the first two Stage-1 sequence points; phasons 4 and 7 are manually specified scan-informed offsets. Rounded values are displayed here; the calculation values are listed in the Supplemental Material [34].

Label	ϕ_x	ϕ_y
Phason 1	0	0
Phason 2	3.8832	2.6026
Phason 4	5.1662	0.9224
Phason 7	4.1839	5.5258

diagonal position operators on the finite torus,

$$\tilde{U} = PU_xP + (1 - P), \quad \tilde{V} = PU_yP + (1 - P), \quad (5)$$

and

$$B = \frac{1}{2\pi} \text{Im Tr } \ln(\tilde{V}\tilde{U}\tilde{V}^\dagger\tilde{U}^\dagger). \quad (6)$$

To characterize the boundary-phase manifold, we compute the twist-space gap

$$\Delta(\theta_x, \theta_y) = E_{N_{\text{occ}}+1}(\theta_x, \theta_y) - E_{N_{\text{occ}}}(\theta_x, \theta_y) \quad (7)$$

and track the minimum twist-space gap together with the near-gapless twist fraction

$$\Delta_{\min}^{\text{twist}} = \min_{(\theta_x, \theta_y) \in \mathcal{G}_\theta} \Delta(\theta_x, \theta_y), \quad (8)$$

$$f_{\text{ng}}(\delta) = \frac{1}{N_\theta} \sum_{(\theta_x, \theta_y) \in \mathcal{G}_\theta} \mathbf{1}[\Delta(\theta_x, \theta_y) < \delta]. \quad (9)$$

Here, $\mathcal{G}_\theta$ is the discrete twist grid, $N_\theta = |\mathcal{G}_\theta|$, and $\mathbf{1}[\cdots]$ is the indicator function, equal to 1 when its argument is true and 0 otherwise. Unless otherwise stated, the near-gapless fraction is evaluated at $\delta = 10^{-2}$.

C. Zero-temperature consistency checks

Because the quasiperiodic scalar potential shifts individual finite-size spectra, it is useful to separate the fixed-half-filling reference subspace from the reference chemical potential used in thermal correlation matrices. After these definitions, Fig. 2 summarizes the zero-temperature checks used to validate the fixed-half-filling convention. The Bott index and the twist-Chern diagnostic agree in the Bott-one window for the representative trajectories. The occupancy-offset panel shows the deviation between the number of single-particle levels below $\mu = 0$ and N_{occ} ; this quantity is a diagnostic of the reference spectrum, not the definition of the occupied projector. The fixed-half-filling projector is always defined by the lowest $N_{\text{occ}} = L_x L_y$ eigenstates. The twist-to-fixed gap ratio further confirms that the weakest boundary-phase gap can occur away from the fixed boundary.

All zero-temperature projectors and gaps are defined at fixed half filling, $N_{\text{occ}} = L_x L_y$, i.e., one occupied single-particle state per lattice site before the two-orbital doubling. The symbol $\mu = 0$ below specifies the reference chemical potential used in the single-particle thermal occupation matrix; it should not be confused with a variable-particle-number definition of the occupied projector.

D. Projected finite-temperature Uhlmann diagnostic

For finite temperature, we use the single-particle Fermi occupation matrix associated with the Gaussian thermal state. We set $k_B = 1$, so T is measured in the same energy units as the

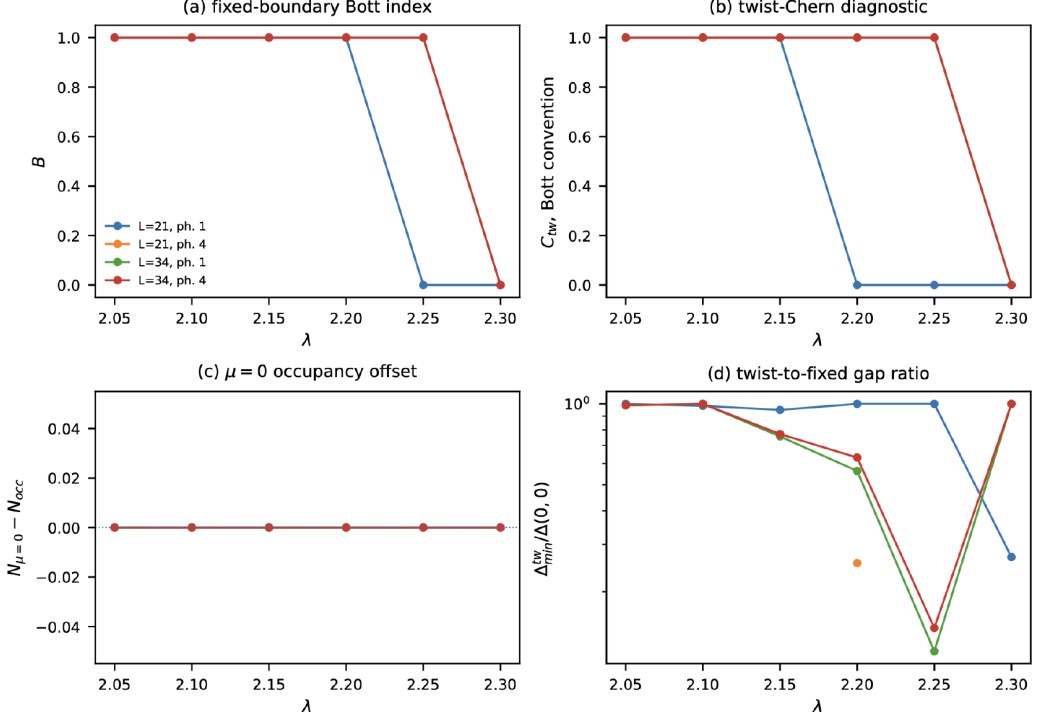


FIG. 2. Zero-temperature consistency checks for the fixed-half-filling convention. (a) Fixed-boundary Bott index B along representative fixed-phason scans. (b) Twist-Chern diagnostic computed on the boundary-phase torus, shown in the same Bott convention. (c) Occupancy offset at $\mu = 0$ relative to N_{occ} ; this is a spectrum diagnostic, not the definition of the occupied projector. (d) Ratio of the minimum twist-space gap to the fixed-boundary gap, showing when the weakest gap occurs away from $(\theta_x, \theta_y) = (0, 0)$.

Hamiltonian,

$$\rho(\theta_x, \theta_y; T) = \frac{1}{e^{\beta[H(\theta_x, \theta_y) - \mu]} + 1}, \quad \beta = 1/T, \quad \mu = 0. \quad (10)$$

This correlation-matrix object is not the full many-body Gibbs density operator. The projected construction below therefore tests the thermal stability of the fixed half-filled zero-temperature bundle rather than defining an independent many-body thermal invariant. To isolate this reference-bundle diagnostic, μ is kept fixed at zero in the calculations below; a fixed-density thermal ensemble with $\mu(T)$ adjusted to keep $\text{Tr } \rho = N_{occ}$ is a natural extension for future thermodynamic studies. Along a discretized loop in θ_x at fixed θ_y , we write $\rho_j = \rho(\theta_{x,j}, \theta_y; T)$. Neighboring square-root density matrices define polar unitaries

$$\sqrt{\rho_{j+1}}\sqrt{\rho_j} = Q_j|\sqrt{\rho_{j+1}}\sqrt{\rho_j}|, \quad (11)$$

where Q_j is the unitary polar factor and $|A| = (A^\dagger A)^{1/2}$. The loop holonomy is $\mathcal{U}_\gamma = \prod_j Q_j$. We then project this Uhlmann holonomy onto the zero-temperature occupied subspace at the loop base point. The matrix $V_{occ}(0, \theta_y)$ has size $2L_x L_y \times N_{occ}$, with orthonormal columns given by the lowest N_{occ} eigenvectors of $H(0, \theta_y)$:

$$\Phi_U(\theta_y) = \arg \det [V_{occ}^\dagger(0, \theta_y) \mathcal{U}_\gamma V_{occ}(0, \theta_y)]. \quad (12)$$

The bracketed $N_{\text{occ}} \times N_{\text{occ}}$ matrix in Eq. (12) is denoted as $W_{\text{occ}}(\theta_y; T)$, the occupied-subspace projected holonomy. We also monitor the projected-holonomy amplitude

$$D_U(T) = \min_{\theta_y} \left| \det \left[V_{\text{occ}}^\dagger(0, \theta_y) \mathcal{U}_{\gamma(\theta_y)} V_{\text{occ}}(0, \theta_y) \right] \right|. \quad (13)$$

The dotted line at 10^{-8} in Fig. 6 is an operational numerical sensitivity flag, not machine epsilon, a polar-decomposition condition-number cutoff, a noise-injection scale, or a physical threshold. It is chosen conservatively about eight orders of magnitude above double-precision machine epsilon and many orders below the determinant amplitudes in the stable intervals; its role is tested explicitly by repeating the audit at 10^{-6} and 10^{-10} . The determinant amplitude and phase are retained at every sampled θ_y . For a chosen threshold τ , if any point on the original closed θ_y cycle satisfies $|\det W_{\text{occ}}| < \tau$, the formal winding at that temperature is reported as undefined; flagged points are not deleted and non-neighboring points are not reconnected. A 25×41 audit for $\tau = 10^{-6}, 10^{-8}, 10^{-10}$ is given in the Supplemental Material [34]. The projected finite-temperature winding is

$$n_U = -\frac{1}{2\pi} [\text{unwrap } \Phi_U(2\pi) - \text{unwrap } \Phi_U(0)], \quad (14)$$

where the sign fixes the orientation convention so that low-temperature n_U agrees with the positive Bott sector. For a Bott-one case, the operational Uhlmann stability scale T_U^{op} is the last temperature in the contiguous stable prefix beginning at the lowest sampled temperature, for which the formal winding is defined, $|n_U - 1| < 0.5$, $|\mathcal{W}_\Phi - 1| < 0.25$, and no local determinant is below 10^{-8} , where $\mathcal{W}_\Phi$ is defined in Eq. (16). It does not count a possible reentry after an intervening failure. This is a finite-size, finite-grid projected diagnostic, not a universal many-body mixed-state invariant or a finite-temperature Hall conductance. When the winding is lost between adjacent sampled temperatures, T_U^{op} is a grid-resolved stability scale rather than a continuous critical temperature; a case stable at $T = 0.03$ is stable only up to the largest sampled temperature.

III. BOUNDARY-PHASE GAP BOTTLENECKS

We first perform a gap scan over the twist torus. The scan uses $L = 21, 34, 55$, $\lambda = 2.05, 2.10, 2.125, 2.15, 2.175, 2.20, 2.225, 2.25, 2.30$, and 24, 12, and 6 phason offsets for the three sizes, respectively. The twist grids are 31×31 , 21×21 , and 11×11 . The $L = 55$ data, obtained from only an 11×11 grid and six phason offsets, are used solely as a qualitative organization check. Their absolute values are not interpreted as converged finite-size scaling, and the quantitative projected-Uhlmann conclusions below are based on the denser $L = 21$ and $L = 34$ calculations. For any phason-dependent quantity $A(\phi_x, \phi_y)$, we denote the sample median over the N_ϕ phason offsets used at fixed L and λ by

$$\text{med}_\phi A = \text{median} \{ A(\phi_x^{(a)}, \phi_y^{(a)}) : a = 1, \dots, N_\phi \}. \quad (15)$$

This notation is used in Figs. 3 and 4; for example, $\text{med}_\phi \Delta_{\min}^{\text{twist}}$ is the median of the minimum twist gap over the sampled phason offsets.

We now follow the first step of the roadmap in Fig. 1(d) and map the boundary-phase spectral terrain. A fixed-boundary calculation can show a quantized Bott index and an apparently finite gap at $(\theta_x, \theta_y) = (0, 0)$, while the full twist torus contains near-gapless regions. Figure 3 shows this directly: Panel (a) tracks the phason-median minimum twist gap, panel (b) compares it with the fixed-boundary gap, panel (c) plots their ratio, and panel (d) measures how much of the twist grid is near gapless. In the bottleneck window, $\Delta_{\min}^{\text{twist}}$ is strongly suppressed and $f_{\text{ng}}(10^{-2})$ becomes appreciable, even when the fixed-boundary gap is not the smallest spectral scale. The qualitative organization of the bottleneck remains visible across the accessible Fibonacci sizes,

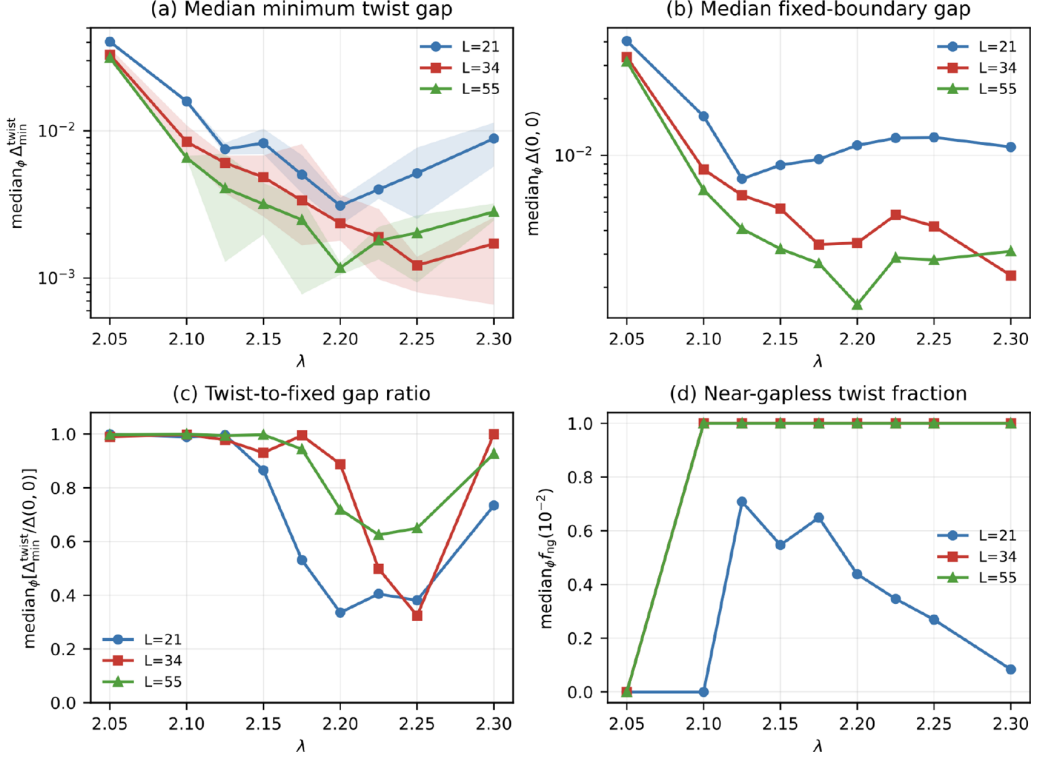


FIG. 3. Boundary-phase gap scan of the spectral bottleneck. Curves show phason medians for $L = 21, 34, 55$. (a) The median minimum twist gap is suppressed near $\lambda \simeq 2.15$ – 2.25 . (b) The fixed-boundary gap remains comparatively larger in parts of the same window. (c) The twist-to-fixed gap ratio falls well below unity, showing that the weakest spectral gap often occurs away from $(\theta_x, \theta_y) = (0, 0)$. (d) The near-gapless fraction $f_{\text{ng}} (10^{-2})$ shows that low-gap regions occupy extended portions of the twist torus.

as shown in Fig. 4; no quantitative scaling conclusion is drawn from the coarse $L = 55$ entry. Thus, the boundary-phase torus, not one boundary condition, is the correct stage on which to test finite-temperature diagnostic stability.

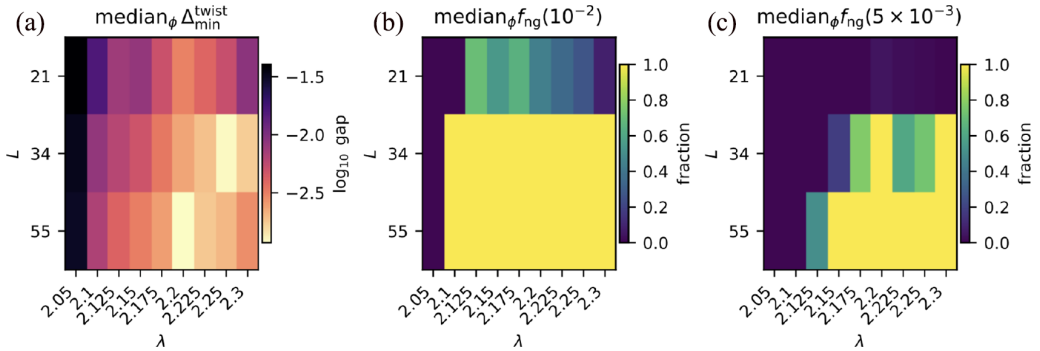


FIG. 4. Finite-size organization of the twist-space gap bottleneck for $L = 21, 34, 55$. (a) Phason-median minimum twist gap, plotted on a logarithmic scale. (b) Near-gapless fraction $f_{\text{ng}} (10^{-2})$. (c) Stricter near-gapless fraction $f_{\text{ng}} (5 \times 10^{-3})$. The $L = 55$ data use only an 11×11 twist grid and six phason offsets and are qualitative only; their absolute values are not evidence of converged finite-size scaling.

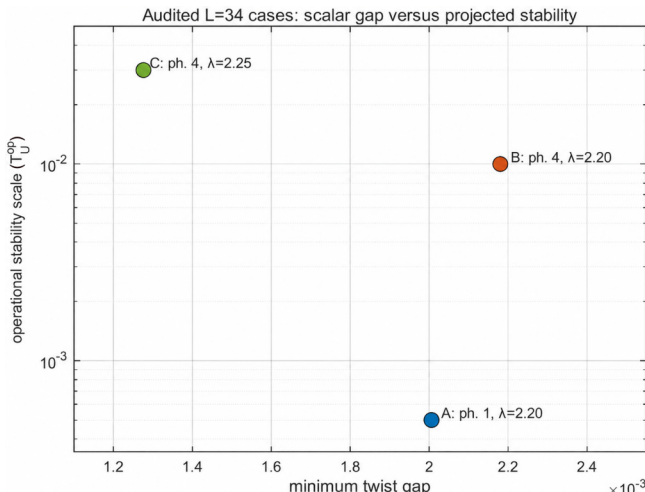


FIG. 5. Audited $L = 34$ comparison of T_U^{op} and $\Delta_{\text{min}}^{\text{twist}}$ for the three decisive cases. Similar minimum gaps correspond to stability scales spanning nearly two orders of magnitude.

IV. PROJECTED-UHLMANN RESPONSE AND PHASE-WINDING MECHANISM

We next apply the projected-Uhlmann diagnostic to three $L = 34$ cases selected from the gap-scan bottleneck window and recompute them on a 25×41 twist grid. Their minimum twist gaps are 2.01×10^{-3} , 2.18×10^{-3} , and 1.28×10^{-3} , while their operational stability scales are 5×10^{-4} , 10^{-2} , and 3×10^{-2} , respectively. Thus, even gaps of the same order do not fix the stability scale. The finite-size projected response must also know where and how the low-gap regions couple to projected-Uhlmann holonomy.

To resolve the representative cases, we examine the projected-Uhlmann phase curves, unwrapped windings, phase total variation, maximum phase jump, and determinant amplitude of the projected holonomy. The exploratory scan contains eight representative cases; the new sensitivity audit focuses on the three decisive $L = 34$ cases. To make the holonomy criterion explicit, we use the normalized phase-winding strength

$$\mathcal{W}_\Phi(T) = \frac{1}{2\pi} \text{TV}_{\theta_y}[\Phi_U(\theta_y; T)], \quad (16)$$

where TV denotes total variation of the unwrapped phase curve. We use D_U as a determinant monitor: An unambiguous robust plateau requires $n_U \simeq 1$, $\mathcal{W}_\Phi \simeq 1$, and no flagged projected determinant. To test the projection convention, we repeat the three audited cases with V_{occ} defined on the exact $\theta_x = \pi$ slice. Because the uniform 25-node θ_x grid does not contain π , that point is diagonalized separately and the crossed edge is split into two transport steps; the π -based loop therefore starts and closes in its own exact local frame. All other conventions are unchanged.

Figure 5 shows the audited representative-case result. The three $L = 34$ minimum twist gaps are all of order 10^{-3} , whereas T_U^{op} ranges from 5×10^{-4} to 3×10^{-2} . In particular, phasons 1 and 4 at $\lambda = 2.20$ have nearly equal gaps, 2.01×10^{-3} and 2.18×10^{-3} , but stability scales 5×10^{-4} and 10^{-2} . The scalar gap alone therefore cannot determine the thermal response.

The distinction appears directly in Fig. 6, which reports the new $L = 34$, 25×41 audit. Case A, phason 1 at $\lambda = 2.20$, is stable through 5×10^{-4} and loses its winding at 10^{-3} . Case B, phason 4 at $\lambda = 2.20$, retains $n_U = \mathcal{W}_\Phi = 1$ through 10^{-2} , but at $T = 0.03$ one determinant amplitude is 3.38×10^{-16} , so the formal winding is undefined. Case C, the same phason at $\lambda = 2.25$, remains defined with $n_U = \mathcal{W}_\Phi = 1$ through the largest sampled temperature 0.03. Moving the projection base point from $\theta_x = 0$ to the exact $\theta_x = \pi$ slice gives the same three operational stability scales and ranking. The determinant amplitudes can differ pointwise between the two bases; the result

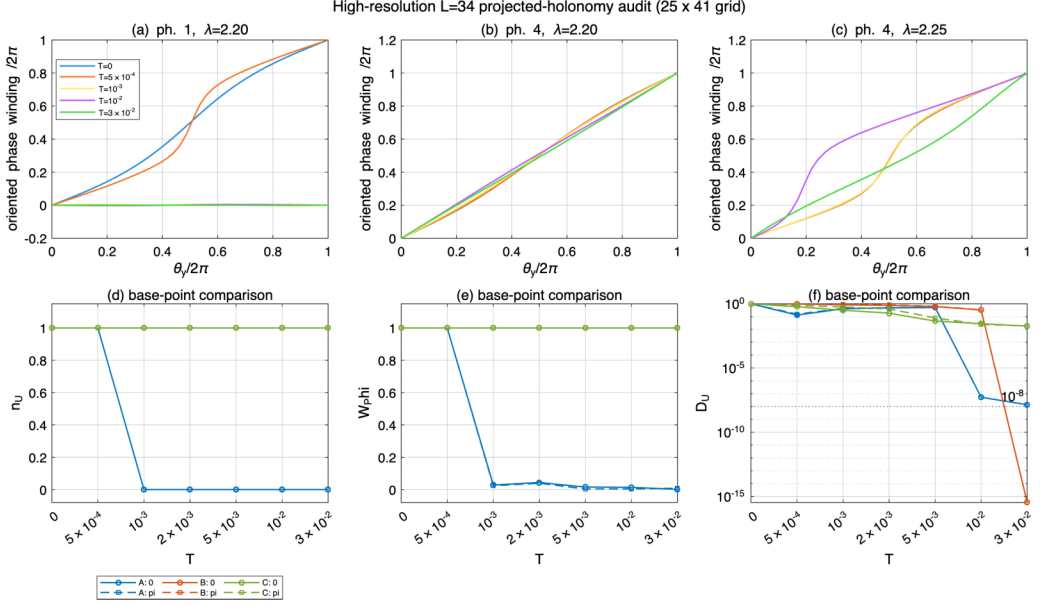


FIG. 6. High-resolution $L = 34$, 25×41 audit. (a)–(c) Oriented phase curves for cases A (34, 1, 2.20), B (34, 4, 2.20), and C (34, 4, 2.25) at selected temperatures, using the $\theta_x = 0$ base. (d)–(f) Comparison of n_U , W_Φ , and D_U for projection bases at $\theta_x = 0$ and exact $\theta_x = \pi$. The resulting stability scales are 5×10^{-4} , 10^{-2} , and 3×10^{-2} for A, B, and C, respectively, at either base point. At $T = 0.03$, B has one determinant amplitude below 10^{-8} , so its formal winding is undefined. The dotted line in panel (f) is the operational 10^{-8} flag.

is that, under the adopted 10^{-8} criterion, the stable prefixes and the resulting defined/undefined classifications of the three audited cases are unchanged. Thus, the survival or loss of the winding is not encoded in the scalar gap alone and is not sensitive to the tested base-point convention.

The three-case comparison in Fig. 6 is the final high-resolution diagnostic used for the revised numerical conclusions. The Supplemental Material [34] gives the complete threshold counts and base-point stability table. Together they show that a robust finite-grid signature requires preservation of the phase winding together with a nonsingular projected determinant, rather than a scalar minimum gap alone.

V. DISCUSSION

These results support a phason-resolved mechanism for the finite-size projected diagnostic. The gap scan establishes that finite Fibonacci approximants develop boundary-phase low-gap bottlenecks. The high-resolution three-case audit shows that similar-order scalar gaps coexist with sharply different stability scales. The phase analysis resolves this difference by showing that the survival or collapse of n_U is tied to whether the projected-Uhlmann phase retains a 2π winding over the θ_y cycle while avoiding a flagged projected determinant.

The important scientific point is the separation between spectral and holonomy diagnostics, which also distinguishes the present projected finite-size diagnostic from recent mixed-state constructions based on ensemble phases or density-matrix Chern characters [27,29,30]. A small $\Delta_{\min}^{\text{twist}}$ is a warning sign because it identifies where the boundary-phase bundle becomes fragile, but it is not a sufficient predictor of the projected finite-temperature response. The phason offset instead changes how low-gap regions couple to Uhlmann parallel transport, converting the phason from a passive origin shift into an active control of projected mixed-state stability in finite approximants.

There are also limitations. The calculations are exact diagonalizations of finite approximants, not a thermodynamic proof. The projected-Uhlmann winding is a finite-size diagnostic of the

zero-temperature occupied bundle under single-particle thermal correlation-matrix transport. Some determinant amplitudes become very small; these temperatures are reported as holonomy-sensitive and their formal winding is undefined under the operational flag. The 25×41 audit in Fig. 6 checks the decisive cases and the projection base point, while broader phason sampling, larger Fibonacci sizes, and a comparison with fixed-density thermal occupations would still be needed for a thermodynamic statement. The $L = 55$ results remain qualitative because of their coarse grid.

VI. CONCLUSIONS

In summary, the finite-size projected-Uhlmann diagnostic in quasiperiodic Chern-insulator approximants is governed by phason-resolved boundary-phase geometry. The full twist-torus gap landscape exposes low-gap bottlenecks invisible to a fixed-boundary view. Projected-Uhlmann diagnostics then show that such bottlenecks suppress the sampled thermal response only when they destroy or singularize the 2π boundary-phase winding of the projected-Uhlmann phase. Within finite Fibonacci approximants and for the projected-Uhlmann diagnostic studied here, the scalar minimum twist gap is therefore an incomplete diagnostic; the decisive object is the phason-dependent geometry of projected holonomy on the boundary-phase torus.

ACKNOWLEDGMENTS

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DATA AVAILABILITY

The numerical data supporting the figures in this article, together with the scripts used to generate the figures and projected-Uhlmann diagnostics, are available from the corresponding author upon reasonable request.

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- [34] See Supplemental Material at <http://link.aps.org/supplemental/10.1103/4ssv-5mvp> for the complete Stage-I phason sampling list, targeted offsets, determinant-threshold audit, and projection-base-point sensitivity analysis.