

Shift of the Bose-Einstein condensation transition in the presence of a second atomic species

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Atomic interactions play an important role in the properties of ultracold atomic gases. In single-component bosonic systems, its effect is already present at the critical point for the Bose-Einstein condensate phase transition by shifting it to lower temperatures as a consequence of effective repulsion between the atoms. When considering atomic bosonic mixtures, interesting effects arise from the competition between intra- and interspecies interactions such as the miscible-immiscible phase transition and the particular case of self-bounded quantum droplets. In such a scenario, it is natural to expect that these interactions will also affect the critical point of each species composing the mixture. In this paper, we obtain semianalytical expressions for the critical temperature shift of the phase transition to a Bose-Einstein condensate in the presence of a second species considering the mean-field framework. We treat differently the cases in which the second species is above or below its own critical temperature and apply the obtained relations to the case of a ^{23}Na - ^{39}K bosonic mixture that can be realized in current running experimental setups. Our findings can be easily extended to other atomic mixtures.

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I. INTRODUCTION

Over the past three decades, ultracold atom experiments have become versatile platforms to study phenomena ranging from the basis of quantum mechanics [1–3] to cosmology [4] and condensed matter physics [5]. Most of its success is related to the high level of control of the system's parameters and to the precision of their theoretical descriptions. In this sense, real ultracold atoms normally interact among themselves via a contact interaction governed by s -wave scattering, with control parameter given by the s -wave scattering length a . This interaction has strong impact on the system's properties as, for example, its own superfluid nature that is suppressed for an ideal gas even at absolute zero as a result of Landau's criterion for superfluidity [6]. It is due to interactions that one can explore the impurity problem in the context of Bose and Fermi polarons [7,8], produce ultracold gases made of molecules [9], or explore the phase diagram of ultracold atomic mixtures in different scenarios [10–15].

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Not only the properties of ultracold atomic systems are affected by interactions, but when considering phase transitions, their critical point will experience a shift. This has been well known since the early days of realizing atomic Bose-Einstein condensates (BECs). For an ideal Bose gas of N atoms trapped by a harmonic potential given by $V(\mathbf{r}) = \frac{1}{2}m(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2)$, where m is the atomic mass and $\omega_{x,y,z}$ are the angular frequencies along the $\hat{x}$, $\hat{y}$, and $\hat{z}$ axes, respectively, the critical temperature for BEC takes the form [16]

$$T_c^0 = \frac{\hbar\omega}{k_B} \left(\frac{N}{\zeta(3)} \right)^{1/3}, \quad (1)$$

where $\omega = (\omega_x\omega_y\omega_z)^{1/3}$, $\hbar$ is the reduced Planck's constant, k_B is the Boltzmann constant, and $\zeta(3) \sim 1.2$. For $T < T_c^0$, a BEC is formed and a simple relation between the condensed fraction of atoms N_{BEC}/N and the temperature of the system can be written as

$$\frac{N_{\text{BEC}}}{N} = 1 - \left(\frac{T}{T_c^0} \right)^3. \quad (2)$$

The correction to the critical temperature due to interactions, δT_c^{int} , is deduced considering the mean-field interaction potential $V_{\text{int}} = gn(\mathbf{r})$, where $g = 4\pi\hbar^2 a/m$ and $n(\mathbf{r})$ is the atomic density, and it reads [17]

$$\frac{\delta T_c^{\text{int}}}{T_c^0} = -1.33 \sqrt{\frac{m\omega}{\hbar}} a N^{1/6}. \quad (3)$$

An additional shift in the critical temperature due to finite size effects is also calculated in Ref. [17] and is given by

$$\frac{\delta T_c^{\text{size}}}{T_c^0} = -0.73 \frac{\bar{\omega}}{\omega} N^{-1/3}, \quad (4)$$

where $\bar{\omega} = \frac{\omega_x + \omega_y + \omega_z}{3}$ is the arithmetic average of the trap frequencies.

It is expected that in the context of atomic mixtures, the interspecies interaction will cause an additional shift to the critical temperature of a phase transition. For the particular case of Bose-Fermi mixtures, the shift of the BEC critical temperature has been calculated in Ref. [18]; however, the case of bosonic mixtures has remained open, although a recent study estimates it for the simpler situation of a symmetric and thermal mixture [19]. In this paper, we cover a broader scenario than only the symmetric mixture, by obtaining an analytical expression for the shift in the critical temperature of the BEC phase transition of a principal species in the presence of a secondary species that could be solved numerically for any bosonic mixture trapped by harmonic potentials fulfilling that $\frac{\omega_{x,1}}{\omega_{x,2}} = \frac{\omega_{y,1}}{\omega_{y,2}} = \frac{\omega_{z,1}}{\omega_{z,2}} = \frac{(\omega_{x,1}\omega_{y,1}\omega_{z,1})^{1/3}}{(\omega_{x,2}\omega_{y,2}\omega_{z,2})^{1/3}} \equiv \frac{\omega_1}{\omega_2}$, which is usually the case for pure magnetic and optical traps [20,21]. We apply our results to realistic experimental parameters by considering a ^{23}Na - ^{39}K mixture [22–24] and show that, by changing the atom number ratio between the species, it is possible to obtain a shift that is of the same order as the one resulting from intraspecies interactions and therefore measurable in current experimental setups.

II. SINGLE BOSONIC SPECIES

This section summarizes the calculations made in Ref. [17], which are later used in an analogous way for the calculation with two species. The starting point to describe a weakly interacting Bose gas is the Gross-Pitaevskii equation (GPE), given by

$$\left(\frac{-\hbar^2}{2m} \nabla^2 + V(\mathbf{r}) + g|\psi(\mathbf{r})|^2 \right) \psi(\mathbf{r}) = \mu \psi(\mathbf{r}), \quad (5)$$

where $\psi(\mathbf{r})$ is the wave function, $V(\mathbf{r}) = \frac{1}{2}m(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2)$, and μ is the chemical potential.

First, using a mean field approximation, the interacting density $n(\mathbf{r})$ can be rewritten in terms of the noninteracting density $n^0(\mathbf{r})$:

$$n(\mathbf{r}) = n^0(\mathbf{r}) - 2gn^0(\mathbf{r})\frac{\partial n^0}{\partial \mu}, \quad (6)$$

and the density for a noninteracting thermal cloud is well known as

$$n^0(\mathbf{r}) = (\lambda_T)^{-3} g_{3/2} \left(\exp \left[\frac{\mu - V(\mathbf{r})}{k_B T} \right] \right), \quad (7)$$

where $\lambda_T = \hbar(\frac{2\pi}{mk_B T})^{1/2}$ is the thermal wavelength. Using the normalization condition $N = \int n(\mathbf{r}) d\mathbf{r}$ and expanding it in first order around $T_c = T_c^0 + \delta T_c$ and $\mu = 0$, we find two different shifts:

$$\begin{aligned} \frac{\delta T_c^{\text{int}}}{T_c^0} &= -\frac{2g}{T_c^0} \frac{\int d\mathbf{r} \frac{\partial n^0}{\partial \mu} [n^0(\mathbf{r}=0) - n^0(\mathbf{r})]}{\int d\mathbf{r} \frac{\partial n^0}{\partial T}}, \\ \frac{\delta T_c^{\text{size}}}{T_c^0} &= -\frac{3\hbar\omega}{2T_c^0} \frac{\int d\mathbf{r} \frac{\partial n^0}{\partial \mu}}{\int d\mathbf{r} \frac{\partial n^0}{\partial T}}. \end{aligned} \quad (8)$$

These equations are easily solved and result in Eqs. (3) and (4), respectively. For the ^{23}Na parameters found in Ref. [23] with $N = 5 \times 10^6$ atoms, $a = 54 a_0$, and $\omega = 2\pi \times 119 \text{ Hz}$, $\bar{\omega} = 2\pi \times 120 \text{ Hz}$, the shift due to finite size is around 0.4%, while the shift due to interaction is around 2.6%.

III. TWO BOSONIC SPECIES

For the case of weakly interacting bosonic mixtures, the system is described by a pair of coupled GPEs

$$\begin{aligned} \left(\frac{-\hbar^2}{2m_1} \nabla^2 + V_1(\mathbf{r}) + g_1 |\psi_1(\mathbf{r})|^2 + g_{12} |\psi_2(\mathbf{r})|^2 \right) \psi_1(\mathbf{r}) &= \mu_1 \psi_1(\mathbf{r}), \\ \left(\frac{-\hbar^2}{2m_2} \nabla^2 + V_2(\mathbf{r}) + g_2 |\psi_2(\mathbf{r})|^2 + g_{12} |\psi_1(\mathbf{r})|^2 \right) \psi_2(\mathbf{r}) &= \mu_2 \psi_2(\mathbf{r}), \end{aligned} \quad (9)$$

where the subindex in each term is related to each one of the two species (e.g., $V_1(\mathbf{r}) = \frac{1}{2}m_1(\omega_{x,1}^2 x^2 + \omega_{y,1}^2 y^2 + \omega_{z,1}^2 z^2)$ is the trapping potential of species 1) and $g_{12} = 2\pi\hbar^2 a_{12}/m_R$ is the interspecies coupling constant, with $m_R = \frac{m_1 m_2}{m_1 + m_2}$ the reduced mass. We will consider only the case of repulsive intraspecies interactions, with $g_1, g_2 > 0$, for which the individual BECs are stable and we will keep $g_{12}^2 < g_1 g_2$, which is the miscible condition for the homogeneous system, such that we do not expect any phase separation to occur at the critical point of species 1 [25].

Let species 1 to be our main species for which we aim to calculate the shift in $T_{c,1}^0$ due to intra- and interspecies interactions, while species 2 is the secondary species that could be either thermal ($T_{c,1}^0 > T_{c,2}^0$) or condensed ($T_{c,1}^0 < T_{c,2}^0$). Following the same procedure done in Ref. [17] for the case of a single species, the interacting density of species 1 here is

$$n_1(\mathbf{r}) = n_1^0(\mathbf{r}) - 2g_1 n_1^0(\mathbf{r}) \frac{\partial n_1^0}{\partial \mu_1} - g_{12} n_2^0(\mathbf{r}) \frac{\partial n_1^0}{\partial \mu_1}, \quad (10)$$

where the factor of 2 is due to exchange effects: species 1 has a two-way interaction, while species 2 interact with species 1 in one way. Again, using the normalization condition $N_1 = \int n_1(\mathbf{r}) d\mathbf{r}$ and expanding in first order around $T_{c,1} = T_{c,1}^0 + (\delta T_{c,1})_{11} + (\delta T_{c,1})_{12}$ and $\mu_1 = 0$, we find that the shift

in the critical temperature of species 1 due to interaction effects of species 2 is

$$\left(\frac{\delta T_{c,1}}{T_{c,1}^0}\right)_{12} = -\frac{g_{12}}{T_{c,1}^0} \frac{\int d\mathbf{r} \frac{\partial n_1^0}{\partial \mu_1} [n_2^0(\mathbf{r}=0) - n_2^0(\mathbf{r})]}{\int d\mathbf{r} \frac{\partial n_1^0}{\partial T}}. \quad (11)$$

In order to solve this equation, one needs to choose the most proper density profile for species 2. This will depend on the regime this atomic cloud is found to be. In the following, we divide the two possible cases considering the second species being above or below its own critical point.

A. Not condensed second species

Consider first the case for which the system achieves a temperature $T = T_{c,1}^0$, the critical temperature for species 1, for which species 2 is still above its own critical point with $T_{c,2}^0 < T_{c,1}^0$. For this case, the density of the second species has the same shape of Eq. (7) and can be written as

$$n_2^0(\mathbf{r}) = (\lambda_{T_{c,1}^0,2})^{-3} g_{3/2} \left(\exp \left[\frac{\mu_2 - V_2(\mathbf{r})}{k_B T_{c,1}^0} \right] \right). \quad (12)$$

By replacing $n_2^0(\mathbf{r})$ in Eq. (11), the critical temperature shift of species 1 due to its interaction with species 2 is obtained by evaluating

$$\begin{aligned} \left(\frac{\delta T_{c,1}}{T_{c,1}^0}\right)_{12} &= -\frac{g_{12}}{3N_1} \frac{(\lambda_{T_{c,1}^0,1})^{-3} (\lambda_{T_{c,1}^0,2})^{-3}}{k_B T_{c,1}^0} \int d\mathbf{r} g_{1/2} \left(\exp \left[\frac{-V_1(\mathbf{r})}{k_B T_{c,1}^0} \right] \right) \\ &\quad \times \left[g_{3/2} \left(\exp \left[\frac{\mu_2}{k_B T_{c,1}^0} \right] \right) - g_{3/2} \left(\exp \left[\frac{\mu_2 - V_2(\mathbf{r})}{k_B T_{c,1}^0} \right] \right) \right]. \end{aligned} \quad (13)$$

To start solving the previous integral, we will apply the following transformation:

$$\begin{aligned} x'^2 &= \frac{m_1 \omega_{x,1}^2}{2k_B T_{c,1}^0} x^2, \\ y'^2 &= \frac{m_1 \omega_{y,1}^2}{2k_B T_{c,1}^0} y^2 \implies d\mathbf{r} = \left(\frac{2k_B T_{c,1}^0}{m_1 \omega_1^2} \right)^{3/2} d\mathbf{r}', \\ z'^2 &= \frac{m_1 \omega_{z,1}^2}{2k_B T_{c,1}^0} z^2 \end{aligned} \quad (14)$$

once it simplifies the trapping potentials of each species to $\frac{V_1(\mathbf{r}')}{k_B T_{c,1}^0} = \alpha(x'^2 + y'^2 + z'^2)$ and $\frac{V_2(\mathbf{r}')}{k_B T_{c,1}^0} = x'^2 + y'^2 + z'^2$, with $\alpha = \frac{m_1 \omega_1^2}{m_2 \omega_2^2}$, considering only that $\frac{\omega_{x,1}}{\omega_{x,2}} = \frac{\omega_{y,1}}{\omega_{y,2}} = \frac{\omega_{z,1}}{\omega_{z,2}} = \frac{\omega_1}{\omega_2}$. Then, changing to spherical coordinates with $r' = \sqrt{x'^2 + y'^2 + z'^2}$, θ , and ϕ , we obtain the shift of $T_{c,1}$ in terms of μ_2 :

$$\left(\frac{\delta T_{c,1}}{T_{c,1}^0}\right)_{12} = \frac{-g_{12} (\lambda_{T_{c,1}^0,1})^{-3}}{3N_1 k_B T_{c,1}^0} \left(\frac{k_B T_{c,1}^0}{\hbar \omega_2} \right)^3 \left[\sum_{n,n'=1}^{\infty} \frac{\exp(n' \frac{\mu_2}{k_B T_{c,1}^0})}{n^{1/2} n'^{3/2}} \left(\frac{1}{(\alpha n)^{3/2}} - \frac{1}{(\alpha n + n')^{3/2}} \right) \right]. \quad (15)$$

The chemical potential of the species 2 is obtained using the normalization condition $N_2 = \int d\mathbf{r} n_2^0(\mathbf{r})$, resulting in the relation

$$\sum_{n=1}^{\infty} \frac{\exp(n \frac{\mu_2}{k_B T_{c,1}^0})}{n^3} = N_2 \left(\frac{\hbar \omega_2}{k_B T_{c,1}^0} \right)^3. \quad (16)$$

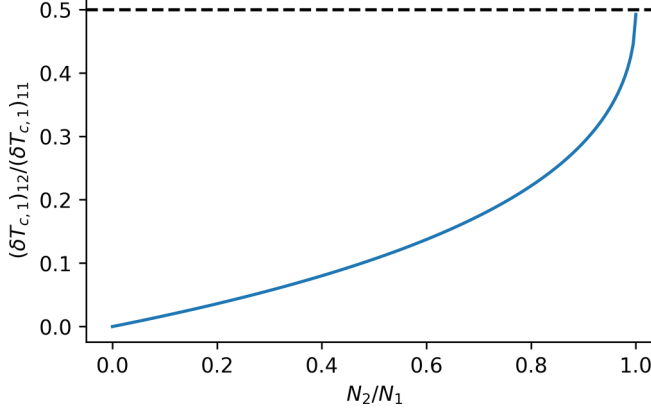


FIG. 1. Comparing the shift due to single species and due to the second species for species 1 equal to species 2 when varying N_2 . The curve reaches exactly 0.5 at $N_2/N_1 = 1$ as expected thanks to the absence in exchange effects between species 1 and 2 (see the main text).

This normalization condition describes a generic thermal chemical potential that is negative and much larger than the thermal energy $k_B T_{c,1}^0$ when the second species is far from its critical temperature and goes to zero when it is close to $T_{c,2}^0$.

In order to find the shift in the critical temperature of species 1 for a given number of atoms of species 2, N_2 , one needs to combine Eqs. (15) and (16), solving them numerically, for example, using the bisection method to find the value of μ_2 , which is then used to evaluate the shift. To gain some physical intuition on the values we expect for $(\delta T_{c,1})_{12}$ and check our results, we can consider the case of identical species, i.e., $m_1 = m_2$, $a_1 = a_2 = a_{12}$, and $V_1(\mathbf{r}) = V_2(\mathbf{r})$. In this case, we should expect that for $N_1 = N_2$, $T_{c,1}^0 = T_{c,2}^0$ and $(\delta T_{c,1})_{12} = 0.5(\delta T_{c,1})_{11}$, with $(\delta T_{c,1})_{11}$ given by Eq. (3). The factor of half in the previous relation is due to the factor of 1 in Eq. (10) for interspecies interactions instead of the factor of 2 for intraspecies interactions related to exchange effects, as already discussed. In Fig. 1, we show the numerical result of Eq. (15) normalized by the intraspecies shift when considering identical species and varying the number of atoms of species 2 as $0 \leq N_2 \leq N_{c,2} = \zeta(3) \left(\frac{k_B T_{c,1}^0}{\hbar \omega_2} \right)^3$. The largest limit of N_2 represents the critical number for which one saturates the thermal state, meaning that any additional atom would populate the condensed state, and it coincides with the condition of $N_2 = N_1$ considered above. The dashed line marks the half condition that is achieved exactly for $N_2 = N_1$, as expected.

B. Condensed second species

Now, let us consider the case where $T_{c,2}^0 > T_{c,1}^0$. In this region, the density of species 2 is composed of a thermal and a condensed fraction. In our simple model, the thermal fraction is represented by Eq. (12) with $\mu_2 = 0$ (i.e., related to the critical number), while the condensed fraction is given by the Thomas-Fermi limit [21]—in the case of a harmonic oscillator, an inverted parabola. The total density can be written as the sum of these two contributions and reads

$$n_2(\mathbf{r}) = (\lambda_{T_{c,1}^0,2})^{-3} g_{3/2} \left(\exp \left[\frac{-V_2(\mathbf{r})}{k_B T_{c,1}^0} \right] \right) + \max \left[\frac{\mu_2}{g_2} \left(1 - \frac{x^2}{R_{x,2}^2} - \frac{y^2}{R_{y,2}^2} - \frac{z^2}{R_{z,2}^2} \right), 0 \right], \quad (17)$$

where $\mu_2 = \frac{15^{2/5}}{2} (N_{\text{BEC},2} a_2 \sqrt{\frac{m_2 \omega_2}{\hbar}})^{2/5} \hbar \omega_2$ is the chemical potential of the condensed cloud [26], with $N_{\text{BEC},2}/N_2 = 1 - (T_{c,1}^0/T_{c,2}^0)^3$ the condensed fraction of species 2 and $R_{i,2} = \sqrt{\frac{2\mu_2}{m_2 \omega_{i,2}^2}}$ the Thomas-Fermi radius of species 2 in the $\hat{i}$ direction.

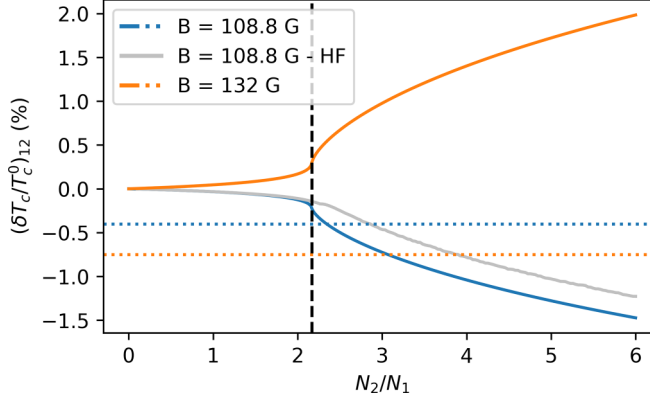


FIG. 2. Shift in the critical temperature of species 1 in terms of number of atoms of species 2 for a ^{23}Na - ^{39}K mixture [27] considering $B = 108.8$ G (in blue) and $B = 132$ G (in orange). The lines represents Eqs. (15) (before dashed line) and (18) (after dashed line), while the dotted lines represents Eq. (3) for each magnetic field. The curve in gray represents a full HF simulation of the shift (see the Supplemental Material [28]). The vertical dashed line represents the BEC transition point for species 2, with $T_{c,1}^0 = T_{c,2}^0$.

Using Eq. (17) in Eq. (11) allows us to find the shift in the critical temperature of species 1 when species 2 has some nonvanishing condensed fraction. The contribution of the thermal fraction in this shift continues to be given by Eq. (15) for $\mu_2 = 0$. Now, our task resumes in obtaining the contribution related to the condensed part. For that, it is convenient to normalize the coordinates in the integral in terms of the Thomas-Fermi radius ($i \rightarrow i/R_{i,2}$) together with the use of spherical coordinates given in Eq. (14). Then, the shift in the critical temperature due to the interaction with the second species, when $T_{c,2}^0 < T_{c,1}^0$, is

$$\left(\frac{\delta T_{c,1}}{T_{c,1}^0}\right)_{12} = -\frac{g_{12}(\lambda_{T_{c,1}^0}^0)^{-3}}{3N_1 k_B T_{c,1}^0} \left(\frac{k_B T_{c,1}^0}{\hbar \omega_2}\right)^3 \left[\sum_{n,n'=1}^{\infty} \frac{1}{n^{1/2} n'^{3/2}} \left(\frac{1}{(\alpha n)^{3/2}} - \frac{1}{(\alpha n + n')^{3/2}} \right) \right] - \frac{4\pi g_{12} R_x R_y R_z \mu_2}{3g_2 \lambda_{T_{c,1}^0}^3 N_1 k_B T_{c,1}^0} \sum_{n=1}^{\infty} \frac{1}{n^{1/2}} \int_0^1 dr r^4 \exp\left(-\frac{n\alpha\mu_2}{k_B T_{c,1}^0} r^2\right). \quad (18)$$

The previous expression can also be solved numerically considering a specific set of parameters defining the Bose-Bose mixture. Relevant to our purposes, we considered a mixture of sodium and potassium based on parameters found in Refs. [23,24], where potassium is defined as species 1 and sodium as species 2, with related scattering length $a_2 = 54 a_0$. The interest in such a mixture is mostly due to the ^{39}K and ^{39}K - ^{23}Na Feshbach resonances existing at moderate magnetic fields ($B < 300$ G) that allow to change the sign of the interspecies scattering length as well as the miscibility of the system. In the Brazilian ^{23}Na - ^{39}K experiment [27], the atoms are trapped by an optical dipole trap, with frequencies $\omega_1 = 2\pi \times 154$ Hz and $\omega_2 = 2\pi \times 119$ Hz near the critical point for the BEC transition of species 1. In our calculations, we fixed the number of atoms of species 1, $N_1 = 8 \times 10^5$ atoms, and varied the number of atoms of species 2 from zero to $6N_1$ considering two different magnetic fields lying between the first two resonances of ^{39}K . For $B = 108.8$ G, $a_1 = 7.7a_0$ and $a_{12} = 19.3a_0$ are both positive giving rise to negative shifts $\delta T_{c,1}$. For $B = 132$ G, $a_1 = 14.34a_0$ and $a_{12} = -26a_0$ and the interspecies shift becomes positive making the BEC at species 1 to appear earlier than for the single species case. Figure 2 shows the complete curves, Eqs. (15) and (18), for the shift in the critical temperature of species 1 due to the interaction with species 2 (blue line for $B = 108.8$ G and orange line for $B = 132$ G) in comparison with the intraspecies shift due to Eq. (3) for each case (dotted lines). The vertical dashed line indicates the condition for which $T_{c,2}^0 = T_{c,1}^0$.

representing the critical point of species 2. We compared our results with a full self-consistent Hartree-Fock (HF) model [6] for which the thermal cloud of species 2 suffers a central depletion as a result of the interaction with its BEC fraction [28]. The correction to the data of Fig. 2 is small at the critical point and it tends to the value of the shift due to the thermal part of Eq. (18) as the BEC increases and the thermal cloud gets more and more depleted. So, for the most interesting case where species 2 exhibit a large condensed fraction, our simple equations may be corrected *a posteriori*.

Then, by abruptly changing N_2 in the vicinity of the critical point for species 1 while still in the thermal regime, let us say, by simply removing the sodium atoms when a_{12} is positive, the remaining atomic cloud will instantaneously cross its critical point in such a way that the control parameter is no longer the temperature of the system. The possibility of studying the phase transition induced by a secondary species can be an interesting outcome of the results presented in Fig. 2, where the magnitude and direction of the shift could be controlled by the strength and sign of the interspecies interaction.

IV. CONCLUSIONS

In ultracold gases, understanding how the critical temperature varies with the interaction strength, whether in single-species systems or mixtures, is extremely useful for designing experiments and for investigations near the phase transition's point. In this paper, we discussed the two possible scenarios when considering a Bose-Bose mixture, with the secondary species above or below its own critical temperature, and obtained expressions for the shift in the critical temperature of a principal species in each case, which could be easily solved numerically. The expressions obtained here can be extended to any Bose-Bose mixture in all trapping geometries. For the single-species systems, the shift at T_c^0 has been measured in different scenarios using standard imaging techniques and thermometry methods [29,30]. The possibility of having interspecies shifts of the same order of magnitude or larger than the intraspecies one, as demonstrated in Fig. 2, allows it to be measured in current running experiments without the need of specific and more precise techniques.

The ability to manipulate the transition temperature by controlling the atom number ratio and/or the interspecies scattering length offers the possibility of inducing a phase transition by changing any of these two parameters and creating a complex phase diagram. By varying the concentration of each species, going from 0% to 100% for species 1 or 2, it is possible to identify regions where the system contains a single condensate, two condensates, or simply a mixture of thermal gases. This type of phase diagram, reminiscent of binary alloys, may reveal a rich structure of domains [31], which remains largely unexplored in the context of ultracold atomic gases. One may view a mixture of condensates as a binary superfluid system, where the balance between inter- and intraspecies interactions determines domains of varying miscibility, giving rise to a diverse landscape of textures and structural features within the system.

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P.M.G. provided the analytical and numerical results; all authors participated in the discussions and in the writing of the manuscript; and P.C.M.C. supervised the project.

DATA AVAILABILITY

The data that support the findings of this article are not publicly available upon publication because it is not technically feasible and/or the cost of preparing, depositing, and hosting the data

would be prohibitive within the terms of this research project. The data are available from the authors upon reasonable request.

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