Solution of the Ising model with Brascamp-Kunz boundary conditions by the transfer matrix method

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The square lattice Ising model under the Brascamp-Kunz boundary conditions is a well-known exactly solvable lattice model. The exact solution of this system has been derived within the framework of Pfaffian-type method. In this paper, we provide a derivation for the solution by the Schultz-Mattis-Lieb method in the transfer matrix formalism. We set special interactions on the boundaries and take certain limit of these interactions, so that the system under the Brascamp-Kunz boundary conditions is transformed into another system under the toroidal boundary conditions. The Schultz-Mattis-Lieb method is applied to the mapping system and the partition function is exactly solved in the fermionic representation. The Fisher zeros are analytically calculated and the physical critical point is identified. We also discuss the difference between the transfer matrix approaches to the Brascamp-Kunz and to the toroidal boundary conditions. Our work introduces a member to the family of transfer-matrix-based studies for Ising model under various boundary conditions.

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I. INTRODUCTION

Ising model is one of the most fundamental systems in the field of statistical lattice models. First introduced by Lenz and Ising in the 1920s [1–3], it has a long history of more than one hundred years. In the early studies, the partition function of one-dimensional Ising model was exactly solved [1], and theoretical approximations for higher dimensions were proposed [4–6]. The first exact solution in two dimensions is Onsager's solution for the square lattice in the absence of a magnetic field [7]. Since then, the zero-field models on some other two-dimensional lattices, including the honeycomb [8,9], the triangular [10], the Kagomé [11,12], and the checkerboard lattices [13], have been exactly solved. Besides the real-valued field, Lee and Yang first found the exact solution for the square lattice Ising model in an imaginary field $i(\pi/2)k_BT$ [14], in their work on Lee-Yang zeros. The zero-field and imaginary-field cases correspond to the two points of intersection of the unit circle and the real axis in the complex $x = e^{-2\beta H_{\rm ex}}$ plane, where $H_{\rm ex}$ represents the field. The imaginary field case is unique as the solution of Ising model in a nonzero field is generally an unsolved problem. For the models on other two-dimensional lattices mentioned above, the imaginary field case has also been solved [15–19].

In this paper, we focus on the square lattice Ising model in a zero field. Onsager derived the exact solution of this model in the thermodynamic limit by the transfer matrix method [7]. This

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achievement is a milestone in the research of statistical physics of lattice systems, in particular, the exactly solvable models [20,21]. Kaufman simplified Onsager's method using the spinor representation theory, to give the finite lattice partition function under the toroidal (periodic in both directions) boundary conditions (BCs) [22]. In the framework of transfer matrix method, in addition to Kaufman's approach, there are also various notable simplications [23–26]. After Kaufman's solution was presented, a quite different type of approach, now known as the combinatorial formulation, was proposed and developed [27–34]. One important type of variant of the combinatorial formulation is the Pfaffian or Pfaffian-type method [35–39], which shows significant usefulness in solving statistical lattice models [40]. There are also solutions neither from the transfer matrix method nor from the combinatorial formalism, e.g., from the anticommuting variable integrals [41] and the Grassmann integrals [42]. Among all the derivations of the solution, the Schultz-Mattis-Lieb (SML) method [24] using the fermionic representation of the transfer matrix is of particular interest to us, as it has considerably simplified and clarified Onsager's original algebraic derivation.

This paper aims to employ the SML method to derive the partition function of a finite square lattice Ising model under the Brascamp-Kunz (B-K) BCs [43]. The introduction of B-K BCs is a breakthrough in the study of Fisher zeros of Ising model [44]. Kaufman [22] first showed that the finite lattice partition function under the toroidal BCs consists of the sum of four product terms, which makes the calculation of Fisher zeros analytically intractable. While under the B-K BCs, the finite lattice partition function can be expressed in a double product form, such that the Fisher zeros can be solved explicitly and found to lie precisely on well-defined loci. The use of B-K BCs enables a rigorous determination of the Fisher loci and the density distribution function of Fisher zeros [45] in the thermodynamic limit. Since the B-K BCs were introduced, the property of the partition function under these BCs, as well as the method of deriving it, has attracted significant attention. In the case of isotropic interactions, the partition function of the model was first obtained via its dual system [43], which had been exactly solved before [46]. The solution in the case of anisotropic interactions was also proposed [47–49]. A recent work by the authors of the present paper rediscovered the result with isotropic interactions using a mapping into the special BCs for the free-fermion model [50]. The existing derivations of the partition function under the B-K BCs are within the framework of Pfaffian-type method. However, we have found no transfer-matrix-based derivation. In this work, we contribute to this field by providing such a derivation using the SML method. In addition to the transfer matrix approach related to the toroidal BCs [22,24–26], there have been transfer-matrix-based studies under various alternative BCs [47,51–54]. Our work will introduce a member to this family.

The remainder of this paper is organized as follows. In Sec. II, the square lattice Ising model under the B-K BCs is introduced. In Sec. III, we give the derivation using the SML method in detail and present the partition function of a finite lattice. To deal with the difference between the B-K and toroidal BCs, we adopt a technique of taking certain limit of the interactions on the boundaries. Section III includes three subsections. In Sec. III A, we express the transfer matrices in the Pauli representation. Then, in Sec. III B, we use the Jordan-Wigner transformation to introduce the fermionic representation and take a linear canonical transformation to construct a direct product decomposition for the transfer matrices. The partition function is exactly solved in Sec. III C based on the direct product form. Discussion and summary are given in Sec. IV.

II. MODEL

The system we study is an Ising model on a square lattice of M rows and $2N$ columns, with nearest-neighbor interactions and in the absence of a magnetic field. Each spin s_i of the model can take two values ± 1. The Hamiltonian is given by

$$H = -J_1 \sum_{\langle i,j\rangle_\parallel} s_i s_j - J_2 \sum_{\langle i,j\rangle_\perp} s_i s_j, \tag{1}$$

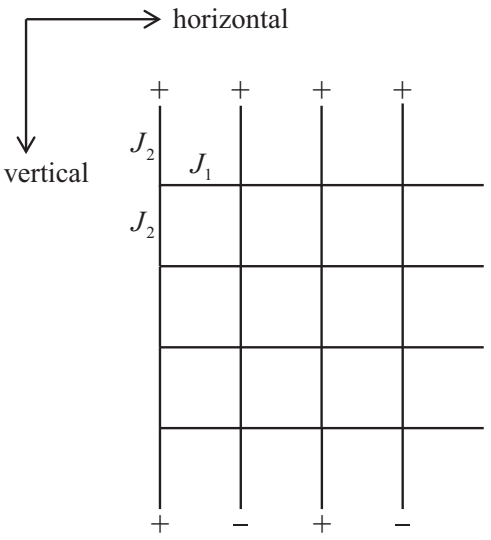


FIG. 1. The Ising model under the B-K BCs on the 4×4 lattice. The horizontal and vertical directions are marked. The interactions constants in the horizontal and vertical directions, i.e., J_1 and J_2 in Eq. (1), are presented.

where $\langle i, j\rangle_{\parallel}$ and $\langle i, j\rangle_{\perp}$ denote the nearest neighbors in the horizontal and vertical directions, respectively, and J_1 and J_2 are the corresponding interaction constants. The system is set under the B-K BCs [43], which were first proposed in 1974. Under the B-K BCs, the partition function of a finite lattice can be expressed in a double product form, and the Fisher zeros can be explicitly solved. The nice property of the B-K BCs is of significant interest in the study of Ising model. The B-K BCs can be described as follows: periodic BCs in the N direction, $2N$ "+" spins on the upper edge (the zeroth row), and $2N$ alternating spins "$+ - \cdots + -$" on the lower edge [the $(M+1)$th row]. That is, the system is arranged on a cylindrical surface with fixed upper and lower boundaries. Figure 1 shows an example on the 4×4 lattice.

The partition function of the system is defined as the sum of the Boltzmann factors over all possible spin configurations:

$$Z = \sum_{\{s_i\}=\pm 1} e^{-\beta H(\{s_i\})}, \tag{2}$$

with $\beta = 1/k_B T$. We aim to find the mathematically exact solution of the partition function under the B-K BCs, using an approach different from those in the previous studies. As mentioned in Sec. I, the existing derivations are within the framework of Pfaffian-type method. Since it is well known that the first exact solution of two-dimensional Ising model was found using the transfer matrix method, it is of mathematical interest to consider rederiving the result under the B-K BCs in the transfer matrix formalism. Furthermore, we believe that the transfer matrix method can provide new insights in the physical point of view, by introducing the fermionic representation. Thus, we employ the very simplified SML method, which establishes the mapping from a two-dimensional classical Ising model into a one-dimensional fermion system. In Sec. III, we give the detailed derivation.

III. DERIVATION AND RESULT

The Ising model under the toroidal BCs is the most commonly studied case. The SML method using the row-to-row transfer matrix is originally applied to the case of toroidal BCs [24]. To employ the SML method to derive the solution under the B-K BCs, we use a technique of taking certain limit

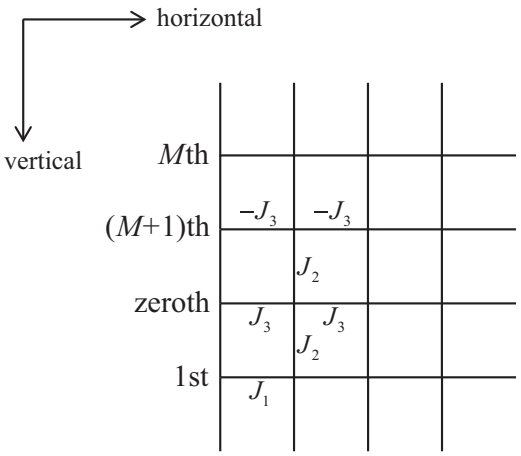


FIG. 2. The new system under the toroidal BCs, with the interactions presented. The horizontal and vertical directions are marked. The interaction constants in the horizontal and vertical directions are given in Eq. (1), except that the horizontal interactions in the zeroth row and $(M+1)$th row are J_3 and $-J_3$, respectively.

of the interactions on the upper and lower boundaries. Consider a system on the $(M+2)\times 2N$ lattice under the toroidal BCs, with the same interaction constants as given in Eq. (1) except for those in the zeroth and $(M+1)$th rows. Set the interaction constant in the zeroth row as J_3 and that in the $(M+1)$th row as $-J_3$, as shown in Fig. 2. Note that the spins in the zeroth and $(M+1)$th rows now are not fixed. Denote the partition function of this system by $Z(J_3)$. We want to show that the partition function under the B-K BCs can be achieved by taking the limit $J_3\to+\infty$. The detailed analysis is given below.

Consider the spin arrangements in the zeroth and $(M+1)$th rows in the limit $\lim_{J_3\to+\infty}\frac{1}{e^{4N\beta J_3}}$. Notice that

$$\lim_{J_3\to+\infty}\frac{1}{e^{\beta J_3}}e^{\beta J_3 s_1 s_2}=\begin{cases}1, & s_1s_2=1,\\ 0, & s_1s_2=-1,\end{cases}$$

$$\lim_{J_3\to+\infty}\frac{1}{e^{\beta J_3}}e^{-\beta J_3 s_1 s_2}=\begin{cases}0, & s_1s_2=1,\\ 1, & s_1s_2=-1.\end{cases}$$

We can verify that only two choices for the spin configurations in the zeroth row—"$++\cdots++$" and "$--\cdots--$"—can make a nonzero contribution to the partition function in this limit. Similarly, we have only two choices for the spin configurations in the $(M+1)$th row—"$+-\cdots+-$" and "$-+\cdots-+$". Therefore, we obtain four sets of BCs surviving in the limit $J_3\to+\infty$:

(i) Zeroth: $++\cdots++$; $(M+1)$th: $+-\cdots+-$ (B-K BCs).
(ii) Zeroth: $++\cdots++$; $(M+1)$th: $-+\cdots-+$.
(iii) Zeroth: $--\cdots--$; $(M+1)$th: $+-\cdots+-$.
(iv) Zeroth: $--\cdots--$; $(M+1)$th: $-+\cdots-+$.

Now we see that the sum of partition functions under these four sets of BCs is

$$Z_{(\mathrm{i})}+Z_{(\mathrm{ii})}+Z_{(\mathrm{iii})}+Z_{(\mathrm{iv})}=\lim_{J_3\to+\infty}\frac{1}{e^{4N\beta J_3}}Z(J_3). \tag{3}$$

We demonstrate that the partition functions under these four sets of BCs are identical. First, it is straightforward to verify $Z_{(\mathrm{i})}=Z_{(\mathrm{ii})}$ and $Z_{(\mathrm{iii})}=Z_{(\mathrm{iv})}$ by noticing that there are periodic BCs along the N direction. Second, since the Hamiltonian is conserved when all spins are reversed, we have $Z_{(\mathrm{i})}=Z_{(\mathrm{iv})}$ and $Z_{(\mathrm{ii})}=Z_{(\mathrm{iii})}$. Now we can see that the partition function under the B-K BCs is

transformed into a quarter of the partition function under the toroidal BCs in a certain limit, i.e.,

$$Z_{\text{B-K}} = \frac{1}{4} \lim_{J_3 \to +\infty} \frac{1}{e^{4N\beta J_3}} Z(J_3). \tag{4}$$

It is natural to express $Z(J_3)$ in the transfer matrix formalism, and the SML method can then be employed.

A. Transfer matrix

The row-to-row transfer matrix has been defined and studied for a long time. The Ising problem can be conveniently formulated using the language of transfer matrix. To do this, we should first use a row on the lattice as a transfer unit to calculate the partition function. Denote the spin configurations in the jth row by $\{s_i\}_j$, which consist of the states of $2N$ spins $\{(s_1)_j, \ldots, (s_{2N})_j\}$. Define three $2^{2N} \times 2^{2N}$ matrices

$$\begin{aligned} V_1(\{s_i\}_j, \{s_i\}_j) &= e^{K_1 \sum_{i=1}^{2N} (s_i)_j (s_{i+1})_j}, \\ V_1'(\{s_i\}_j, \{s_i\}_j) &= e^{K_3 \sum_{i=1}^{2N} (s_i)_j (s_{i+1})_j}, \\ V_2(\{s_i\}_j, \{s_i\}_{j+1}) &= e^{K_2 \sum_{i=1}^{2N} (s_i)_j (s_i)_{j+1}}, \end{aligned} \tag{5}$$

with $K_l = \beta J_l$ $(l = 1, 2, 3)$ and $(s_{2N+1})_j = (s_1)_j$. Diagonal matrices V_1 and V_1' take into account the Boltzmann factors associated with the interactions within each row, while matrix V_2 corresponds to those associated with the interactions between neighboring rows. According to Eq. (2), $Z(J_3)$ in our case can be expressed as

$$\begin{aligned} Z(J_3) = & \sum_{\{s_i\}_1 = \pm 1} \cdots \sum_{\{s_i\}_{M+1} = \pm 1} \sum_{\{s_i\}_0 = \pm 1} V_1(\{s_i\}_1, \{s_i\}_1) V_2(\{s_i\}_1, \{s_i\}_2) \\ & \times \cdots \times (V_1')^{-1}(\{s_i\}_{M+1}, \{s_i\}_{M+1}) V_2(\{s_i\}_{M+1}, \{s_i\}_0) V_1'(\{s_i\}_0, \{s_i\}_0) V_2(\{s_i\}_0, \{s_i\}_1) \\ = & \sum_{\{s_i\}_1 = \pm 1} \cdots \sum_{\{s_i\}_{M+1} = \pm 1} \sum_{\{s_i\}_0 = \pm 1} (V_1 V_2)(\{s_i\}_1, \{s_i\}_2) \\ & \times \cdots \times [(V_1')^{-1} V_2](\{s_i\}_{M+1}, \{s_i\}_0) \times (V_1' V_2)(\{s_i\}_0, \{s_i\}_1) \\ = & \operatorname{Tr}[(V_1 V_2)^M (V_1')^{-1} V_2 V_1' V_2]. \end{aligned} \tag{6}$$

The partition function is now written as the trace of a $2^{2N} \times 2^{2N}$ matrix. Equation (4) then becomes

$$Z_{\text{B-K}} = \frac{1}{4} \lim_{K_3 \to +\infty} \frac{1}{e^{4NK_3}} \operatorname{Tr}[(V_1 V_2)^M (V_1')^{-1} V_2 V_1' V_2]. \tag{7}$$

The explicit expressions of V_1, V_1', and V_2 are clearly given in the Pauli representation [7,22,24]. Use the notations

$$\begin{aligned} \sigma_i^z &= \mathbf{1} \otimes \cdots \otimes \mathbf{1} \otimes \underbrace{\sigma^z}_{i\text{th}} \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1}, \\ \sigma_i^x &= \mathbf{1} \otimes \cdots \otimes \mathbf{1} \otimes \underbrace{\sigma^x}_{i\text{th}} \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1}, \end{aligned} \tag{8}$$

where $\mathbf{1}$ is the identity matrix and σ^z and σ^x are two of the Pauli matrices:

$$\sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{9}$$

V_1, V_1', and V_2 can be given explicitly:

$$\begin{aligned} V_1 &= e^{K_1(\sigma_1^z\sigma_2^z+\cdots+\sigma_{2N-1}^z\sigma_{2N}^z+\sigma_{2N}^z\sigma_1^z)}, \\ V_1' &= e^{K_3(\sigma_1^z\sigma_2^z+\cdots+\sigma_{2N-1}^z\sigma_{2N}^z+\sigma_{2N}^z\sigma_1^z)}, \\ V_2 &= (2\sinh 2K_2)^N e^{K_2^*(\sigma_1^x+\cdots+\sigma_{2N}^x)}, \end{aligned} \tag{10}$$

with K_2^* defined as

$$\tanh K_2^* = e^{-2K_2} \tag{11a}$$

or equivalently

$$\sinh 2K_2 \sinh 2K_2^* = 1. \tag{11b}$$

Note that in this representation the matrices have the same index of rows and of columns (i.e., the same order of spin configurations appearing in rows and in columns). From Eq. (10) it is clear to verify that the transfer matrix of a two-dimensional classical Ising model is mapped into the density operator of a one-dimensional quantum Ising model.

As suggested by Ref. [24], we perform a canonical transformation $\sigma^x \to \sigma^z$, $\sigma^z \to -\sigma^x$ that keeps the commutation rules of Pauli matrices invariant. Under this transformation, V_1, V_1', and V_2 are written as

$$\begin{aligned} V_1 &= e^{K_1(\sigma_1^x\sigma_2^x+\cdots+\sigma_{2N-1}^x\sigma_{2N}^x+\sigma_{2N}^x\sigma_1^x)}, \\ V_1' &= e^{K_3(\sigma_1^x\sigma_2^x+\cdots+\sigma_{2N-1}^x\sigma_{2N}^x+\sigma_{2N}^x\sigma_1^x)}, \\ V_2 &= (2\sinh 2K_2)^N e^{K_2^*(\sigma_1^z+\cdots+\sigma_{2N}^z)}, \end{aligned} \tag{12}$$

and Eq. (7) still holds. Now the transfer matrices can be chosen to be symmetric:

$$V = V_2^{1/2} V_1 V_2^{1/2}, \quad V'' = V_2^{1/2}(V_1')^{-1}V_2^{1/2}, \quad V' = V_2^{1/2} V_1' V_2^{1/2}, \tag{13}$$

and the partition function in Eq. (7) is expressed using these matrices:

$$Z_{\text{B-K}} = \frac{1}{4}\lim_{K_3\to+\infty}\frac{1}{e^{4NK_3}}\text{Tr}[V^M V'' V']. \tag{14}$$

The fermionic representation will be introduced based on Eq. (12) in the next section.

B. Fermionic representation

1. Jordan-Wigner transformation

Define the spin raising and lowering operators

$$\sigma_i^+ = \tfrac{1}{2}\left(\sigma_i^x + i\sigma_i^y\right), \quad \sigma_i^- = \tfrac{1}{2}\left(\sigma_i^x - i\sigma_i^y\right). \tag{15}$$

It is straightforward to find

$$\sigma_i^x = \sigma_i^+ + \sigma_i^-, \quad \sigma_i^z = 2\sigma_i^+\sigma_i^- - \mathbf{1} \tag{16}$$

(here, $\mathbf{1}$ is a $2^{2N}\times 2^{2N}$ identity matrix). Equation (12) is then written as

$$\begin{aligned} V_1 &= e^{K_1[(\sigma_1^+ + \sigma_1^-)(\sigma_2^+ + \sigma_2^-)+\cdots+(\sigma_{2N}^+ + \sigma_{2N}^-)(\sigma_1^+ + \sigma_1^-)]}, \\ V_1' &= e^{K_3[(\sigma_1^+ + \sigma_1^-)(\sigma_2^+ + \sigma_2^-)+\cdots+(\sigma_{2N}^+ + \sigma_{2N}^-)(\sigma_1^+ + \sigma_1^-)]}, \\ V_2 &= (2\sinh 2K_2)^N e^{K_2^*[(2\sigma_1^+\sigma_1^- - \mathbf{1})+\cdots+(2\sigma_{2N}^+\sigma_{2N}^- - \mathbf{1})]}. \end{aligned} \tag{17}$$

Now we employ the SML method and use the Jordan-Wigner transformation [55] to introduce the fermion annihilation and creation operators

$$C_m = e^{i\pi \sum_{i=1}^{m-1} \sigma_i^+ \sigma_i^-} \sigma_m^-, \quad C_m^\dagger = e^{i\pi \sum_{i=1}^{m-1} \sigma_i^+ \sigma_i^-} \sigma_m^+, \quad m = 1, \ldots, 2N. \tag{18}$$

The anticommutation rules of fermion operators can be examined:

$$\{C_m, C_n\} = 0, \quad \{C_m^\dagger, C_n^\dagger\} = 0, \quad \{C_m, C_n^\dagger\} = \delta_{mn}. \tag{19}$$

It is straightforward to verify $e^{2i\pi \sum_{i=1}^{m-1} \sigma_i^+ \sigma_i^-} = \mathbf{1}$ and

$$\sigma_m^+ \sigma_m = C_m^\dagger C_m, \tag{20}$$

so that the inverse transformation is given by

$$\sigma_m^- = e^{i\pi \sum_{i=1}^{m-1} C_i^\dagger C_i} C_m, \quad \sigma_m^+ = e^{i\pi \sum_{i=1}^{m-1} C_i^\dagger C_i} C_m^\dagger, \quad m = 1, \ldots, 2N. \tag{21}$$

To translate the Pauli representation into the fermionic representation, we need to do some algebra. Using the fermion anticommutation relations [Eq. (19)], we can identify

$$\begin{aligned} C_m e^{i\pi C_m^\dagger C_m} &= -C_m, \quad e^{i\pi C_m^\dagger C_m} C_m = C_m, \\ C_m^\dagger e^{i\pi C_m^\dagger C_m} &= C_m^\dagger, \quad e^{i\pi C_m^\dagger C_m} C_m^\dagger = -C_m^\dagger. \end{aligned} \tag{22}$$

Then, we can derive the following identities from Eqs. (21) and (22):

$$\begin{aligned} \sigma_m^+ \sigma_{m+1}^+ &= C_m^\dagger C_{m+1}^\dagger, \quad \sigma_m^+ \sigma_{m+1}^- = C_m^\dagger C_{m+1}, \\ \sigma_m^- \sigma_{m+1}^+ &= -C_m C_{m+1}^\dagger, \quad \sigma_m^- \sigma_{m+1}^- = -C_m C_{m+1}. \end{aligned} \tag{23}$$

Now V_1, V_1', and V_2 are represented by the fermion operators by substituting Eqs. (20) and (23) into Eq. (17):

$$\begin{aligned} V_1 &= e^{K_1 [\sum_{j=1}^{2N-1} (C_j^\dagger - C_j)(C_{j+1}^\dagger + C_{j+1}) - U(C_{2N}^\dagger - C_{2N})(C_1^\dagger + C_1)]}, \\ V_1' &= e^{K_3 [\sum_{j=1}^{2N-1} (C_j^\dagger - C_j)(C_{j+1}^\dagger + C_{j+1}) - U(C_{2N}^\dagger - C_{2N})(C_1^\dagger + C_1)]}, \\ V_2 &= (2 \sinh 2K_2)^N e^{K_2^* \sum_{j=1}^{2N} (2C_j^\dagger C_j - \mathbf{1})}, \end{aligned} \tag{24}$$

with

$$U = e^{i\pi \sum_{j=1}^{2N} C_j^\dagger C_j}. \tag{25}$$

To deal with the difficulty caused by the operator U in Eq. (24), we adopt a technique used in Refs. [22,25]. Notice that all terms in V_1, V_1', and V_2 involve bilinear products of fermion operators; the evenness or oddness of the total number $\sum_{j=1}^{2N} C_j^\dagger C_j$ is conserved [24]. Therefore, U commutes with each of them:

$$[U, V_1] = [U, V_1'] = [U, V_2] = 0. \tag{26}$$

More exactly, U commutes with bilinear products of fermion operators. By using $U^2 = \mathbf{1}$, we find

$$\begin{aligned} (1+U) \times U(C_{2N}^\dagger - C_{2N})(C_1^\dagger + C_1) &= (1+U)(C_{2N}^\dagger - C_{2N})(C_1^\dagger + C_1), \\ (1-U) \times U(C_{2N}^\dagger - C_{2N})(C_1^\dagger + C_1) &= -(1-U)(C_{2N}^\dagger - C_{2N})(C_1^\dagger + C_1). \end{aligned} \tag{27}$$

This leads to

$$(1+U)V_1 = (1+U)V_1^+, \quad (1-U)V_1 = (1-U)V_1^-, \tag{28}$$

where

$$V_1^{\pm} = e^{K_1\left[\sum_{j=1}^{2N}(C_j^\dagger - C_j)(C_{j+1}^\dagger + C_{j+1})\right]}, \tag{29}$$

with anticyclic definition of $C_{2N+1}^\dagger$ and C_{2N+1}:

$$C_{2N+1}^\dagger = -C_1^\dagger, \quad C_{2N+1} = -C_1 \tag{30}$$

for V_1^+ and cyclic definition

$$C_{2N+1}^\dagger = C_1^\dagger, \quad C_{2N+1} = C_1 \tag{31}$$

for V_1^-. $V_1'^{\pm}$ are determined similarly. Now we can express the transfer matrix V in Eq. (13) by using the projection operators $\frac{1}{2}(1 \pm U)$:

$$\begin{aligned} V &= \tfrac{1}{2}(1+U)V + \tfrac{1}{2}(1-U)V \\ &= \tfrac{1}{2}(1+U)V_2^{1/2}V_1^+V_2^{1/2} + \tfrac{1}{2}(1-U)V_2^{1/2}V_1^-V_2^{1/2} \\ &\equiv \tfrac{1}{2}(1+U)V^+ + \tfrac{1}{2}(1-U)V^-. \end{aligned} \tag{32}$$

$V''^{\pm}$ and $V'^{\pm}$ [see Eq. (13)] are defined similarly. It is seen that the eigenvectors of $\frac{1}{2}(1+U)V^+$ (also those of $\frac{1}{2}(1+U)V''^+$ and $\frac{1}{2}(1+U)V'^+$) involve even numbers of fermions, and those of $\frac{1}{2}(1-U)V^-$ (also those of $\frac{1}{2}(1-U)V''^-$ and $\frac{1}{2}(1-U)V'^-$) involve odd numbers of fermions.

The next step is to consider $V^M V'' V'$ in Eq. (14). Making use of

$$\left[\tfrac{1}{2}(1+U)\right]^k = \tfrac{1}{2}(1+U), \quad \left[\tfrac{1}{2}(1-U)\right]^k = \tfrac{1}{2}(1-U), \quad k \in N_+ \tag{33a}$$

and

$$(1+U)(1-U) = 0, \tag{33b}$$

it is clear to show that

$$V^M V'' V' = \tfrac{1}{2}(1+U)(V^+)^M V''^+ V'^+ + \tfrac{1}{2}(1-U)(V^-)^M V''^- V'^-. \tag{34}$$

Like we have analyzed for V, $V^M V'' V'$ is divided into two parts, which are referred to as the even and odd parts henceforth. Now the operator U in the terms of V_1 and V_1' is eliminated, and $V_1^{\pm}$ and $V_1'^{\pm}$ are introduced. Further transformation in the fermionic representation is based on Eqs. (29)–(31). The trace of Eq. (34) is the sum of four terms. Note that when the horizontal interactions are uniform (the usual toroidal case), the trace is actually Kaufman's solution [22], and the four terms in the trace can be seen as four Pfaffians [21].

2. Direct product decomposition

Via the Jordan-Wigner transformation, we obtain the expressions of $V_1^{\pm}$, $V_1'^{\pm}$, and V_2 in terms of fermion operators. It is expected to perform a direct product decomposition to these matrices so that the diagonalization will be simplified. Following the SML method, we take the linear canonical transformation to a new set of fermion operators

$$\eta_q = (2N)^{-1/2}e^{i\pi/4}\sum_{j=1}^{2N} e^{-iqj}C_j, \quad \eta_q^\dagger = (2N)^{-1/2}e^{-i\pi/4}\sum_{j=1}^{2N} e^{iqj}C_j^\dagger. \tag{35}$$

It can be examined that η_q's and $\eta_q^\dagger$'s obey the fermion anticommutation rules [as in Eq. (19)]. The inverse transformation is straightforward:

$$C_j = (2N)^{-1/2} e^{-i\pi/4} \sum_q e^{iqj} \eta_q, \quad C_j^\dagger = (2N)^{-1/2} e^{i\pi/4} \sum_q e^{-iqj} \eta_q^\dagger. \tag{36}$$

The index q is set as

$$q = \pm \frac{(2l-1)\pi}{2N}, \quad l = 1, \ldots, N \tag{37}$$

for the anticyclic condition [Eq. (30)] for V_1^+ in the even part, and as

$$q = 0, \pi, \pm \frac{2l\pi}{2N}, \quad l = 1, \ldots, N-1 \tag{38}$$

for the cyclic condition [Eq. (31)] for V_1^- in the odd part. With this notations, we can also refer to η_q as $\eta_{\pm(2l-1)}$ or $\eta_{\pm(2l)}$ without loss of clarity.

Now we can express $V_1^\pm$, $V_1'^\pm$, and V_2 in terms of η_q's and $\eta_q^\dagger$'s. To do this, some algebra is needed. First, it is easy to see

$$\sum_{j=1}^{2N} C_j^\dagger C_j = \sum_q \eta_q^\dagger \eta_q. \tag{39}$$

This yields V_2 in Eq. (24) in the form

$$V_2^+ = (2\sinh 2K_2)^N e^{2K_2^* \sum_{l=1}^{N} (\eta_{2l-1}^\dagger \eta_{2l-1} + \eta_{-(2l-1)}^\dagger \eta_{-(2l-1)} - 1)} \tag{40}$$

for the even part and

$$V_2^- = (2\sinh 2K_2)^N e^{2K_2^* \left[\sum_{l=1}^{N-1} (\eta_{2l}^\dagger \eta_{2l} + \eta_{-2l}^\dagger \eta_{-2l} - 1) + \left(\eta_0^\dagger \eta_0 - \frac{1}{2}\right) + \left(\eta_\pi^\dagger \eta_\pi - \frac{1}{2}\right) \right]} \tag{41}$$

for the odd part. Next, we calculate

$$\begin{aligned}
\sum_{j=1}^{2N} (C_j^\dagger - C_j)(C_{j+1}^\dagger + C_{j+1}) &= \sum_{j=1}^{2N} \left(C_j^\dagger C_{j+1}^\dagger + C_{j+1} C_j + C_j^\dagger C_{j+1} + C_{j+1}^\dagger C_j \right) \\
&= \frac{1}{2N} \sum_{j=1}^{2N} \sum_q \sum_{q'} \left[e^{i\frac{\pi}{2}} e^{-i(q+q')j} e^{-iq'} \eta_q^\dagger \eta_{q'}^\dagger + e^{-i\frac{\pi}{2}} e^{i(q+q')j} e^{iq} \eta_q \eta_{q'} \right. \\
&\qquad \left. + e^{-i(q-q')j} e^{iq'} \eta_q^\dagger \eta_{q'} + e^{-i(q-q')j} e^{-iq} \eta_q^\dagger \eta_{q'} \right] \\
&= \sum_q \left[e^{i\frac{\pi}{2}} e^{iq} \eta_q^\dagger \eta_{-q}^\dagger + e^{-i\frac{\pi}{2}} e^{iq} \eta_q \eta_{-q} + (e^{iq} + e^{-iq}) \eta_q^\dagger \eta_q \right] \\
&= \sum_q \left[2\cos q \times \eta_q^\dagger \eta_q + i e^{iq} (\eta_q^\dagger \eta_{-q}^\dagger - \eta_q \eta_{-q}) \right].
\end{aligned} \tag{42}$$

For the even part, the above result can be written as

$$\begin{aligned}
\sum_{j=1}^{2N} (C_j^\dagger - C_j)(C_{j+1}^\dagger + C_{j+1}) = 2 \sum_{l=1}^{N} \left[\cos \frac{(2l-1)\pi}{2N} \left(\eta_{2l-1}^\dagger \eta_{2l-1} + \eta_{-(2l-1)}^\dagger \eta_{-(2l-1)} \right) \right. \\
\left. + \sin \frac{(2l-1)\pi}{2N} \left(\eta_{2l-1} \eta_{-(2l-1)} - \eta_{2l-1}^\dagger \eta_{-(2l-1)}^\dagger \right) \right],
\end{aligned} \tag{43}$$

while for the odd part it is written as

$$\sum_{j=1}^{2N}(C_j^\dagger - C_j)(C_{j+1}^\dagger + C_{j+1}) = 2\sum_{l=1}^{N-1}\Bigg[\cos\frac{2l\pi}{2N}(\eta_{2l}^\dagger\eta_{2l} + \eta_{-2l}^\dagger\eta_{-2l}) + \sin\frac{2l\pi}{2N}(\eta_{2l}\eta_{-2l} - \eta_{2l}^\dagger\eta_{-2l}^\dagger)\Bigg] + 2\eta_0^\dagger\eta_0 - 2\eta_\pi^\dagger\eta_\pi. \quad (44)$$

This yields $V_1^\pm$ in Eq. (29) in the form

$$V_1^+ = e^{2K_1\sum_{l=1}^{N}\left[\cos\frac{(2l-1)\pi}{2N}(\eta_{2l-1}^\dagger\eta_{2l-1}+\eta_{-(2l-1)}^\dagger\eta_{-(2l-1)})+\sin\frac{(2l-1)\pi}{2N}(\eta_{2l-1}\eta_{-(2l-1)}-\eta_{2l-1}^\dagger\eta_{-(2l-1)}^\dagger)\right]} \quad (45)$$

and

$$V_1^- = e^{2K_1\left\{\sum_{l=1}^{N-1}\left[\cos\frac{2l\pi}{2N}\left(\eta_{2l}^\dagger\eta_{2l}+\eta_{-2l}^\dagger\eta_{-2l}\right)+\sin\frac{2l\pi}{2N}\left(\eta_{2l}\eta_{-2l}-\eta_{2l}^\dagger\eta_{-2l}^\dagger\right)\right]+\eta_0^\dagger\eta_0-\eta_\pi^\dagger\eta_\pi\right\}}. \quad (46)$$

$V_1'^\pm$ are given similarly.

Now we have succeeded in constructing a direct product decomposition for the transfer matrices from Eqs. (40), (41), (45), and (46):

$$V_1^+ = \prod_{l=1}^{N}{}_\otimes\, V_{1,l}^+, \quad V_1^- = \left(\prod_{l=1}^{N-1}{}_\otimes\, V_{1,l}^-\right)\otimes V_{1,0}^-\otimes V_{1,\pi}^-, \quad (47)$$

with

$$\begin{aligned} V_{1,l}^+ &= e^{2K_1\left[\cos\frac{(2l-1)\pi}{2N}\left(\eta_{2l-1}^\dagger\eta_{2l-1}+\eta_{-(2l-1)}^\dagger\eta_{-(2l-1)}\right)+\sin\frac{(2l-1)\pi}{2N}\left(\eta_{2l-1}\eta_{-(2l-1)}-\eta_{2l-1}^\dagger\eta_{-(2l-1)}^\dagger\right)\right]},\\ V_{1,l}^- &= e^{2K_1\left[\cos\frac{2l\pi}{2N}\left(\eta_{2l}^\dagger\eta_{2l}+\eta_{-2l}^\dagger\eta_{-2l}\right)+\sin\frac{2l\pi}{2N}\left(\eta_{2l}\eta_{-2l}-\eta_{2l}^\dagger\eta_{-2l}^\dagger\right)\right]},\\ V_{1,0}^- &= e^{2K_1\eta_0^\dagger\eta_0}, \quad V_{1,\pi}^- = e^{-2K_1\eta_\pi^\dagger\eta_\pi}, \end{aligned} \quad (48)$$

and

$$\begin{aligned} V_2^+ &= (2\sinh 2K_2)^N\prod_{l=1}^{N}{}_\otimes\, V_{2,l}^+,\\ V_2^- &= (2\sinh 2K_2)^N\left(\prod_{l=1}^{N-1}{}_\otimes\, V_{2,l}^-\right)\otimes V_{2,0}^-\otimes V_{2,\pi}^-, \end{aligned} \quad (49)$$

with

$$\begin{aligned} V_{2,l}^+ &= e^{2K_2^*\left(\eta_{2l-1}^\dagger\eta_{2l-1}+\eta_{-(2l-1)}^\dagger\eta_{-(2l-1)}-1\right)},\\ V_{2,l}^- &= e^{2K_2^*\left(\eta_{2l}^\dagger\eta_{2l}+\eta_{-2l}^\dagger\eta_{-2l}-1\right)},\\ V_{2,0}^- &= e^{2K_2^*\left(\eta_0^\dagger\eta_0-\frac{1}{2}\right)}, \quad V_{2,\pi}^- = e^{2K_2^*\left(\eta_\pi^\dagger\eta_\pi-\frac{1}{2}\right)}. \end{aligned} \quad (50)$$

$V_{1,l}''^\pm$ and $V_{1,l}'^\pm$ are defined similarly. Obviously, the operator U has the form

$$\begin{aligned} U^+ &= \prod_{l=1}^{N}{}_\otimes\, e^{i\pi\left(\eta_{2l-1}^\dagger\eta_{2l-1}+\eta_{-(2l-1)}^\dagger\eta_{-(2l-1)}\right)}, && \text{for the even part;}\\ U^- &= \left[\prod_{l=1}^{N-1}{}_\otimes\, e^{i\pi\left(\eta_{2l}^\dagger\eta_{2l}+\eta_{-2l}^\dagger\eta_{-2l}\right)}\right]\otimes e^{i\pi\eta_0^\dagger\eta_0}\otimes e^{i\pi\eta_\pi^\dagger\eta_\pi}, && \text{for the odd part.} \end{aligned} \quad (51)$$

Then, we can return to Eq. (34). The even and odd parts of $V^M V'' V'$ can be given in terms of direct products, respectively:

$$\frac{1}{2}(1+U^+)(V^+)^M V''^+ V'^+ = \frac{1}{2}(2\sinh 2K_2)^{(M+2)N}\left\{\prod_{l=1}^{N}{}_{\otimes}\,\bar{V}_l^+ + \prod_{l=1}^{N}{}_{\otimes}\, e^{i\pi\left(\eta^\dagger_{2l-1}\eta_{2l-1}+\eta^\dagger_{-(2l-1)}\eta_{-(2l-1)}\right)}\bar{V}_l^+\right\}, \tag{52}$$

with

$$\bar{V}_l^+ = \left[(V_{2,l}^+)^{1/2}V_{1,l}^+(V_{2,l}^+)^{1/2}\right]^M\left[(V_{2,l}^+)^{1/2}V_{1,l}''^+(V_{2,l}^+)^{1/2}\right]\left[(V_{2,l}^+)^{1/2}V_{1,l}'^+(V_{2,l}^+)^{1/2}\right] \tag{53}$$

and

$$\begin{aligned}\frac{1}{2}(1-U^-)(V^-)^M V''^- V'^- = \frac{1}{2}(2\sinh 2K_2)^{(M+2)N}\Bigg\{&\left(\prod_{l=1}^{N-1}{}_{\otimes}\,\bar{V}_l^-\right)\otimes\bar{V}_0^-\otimes\bar{V}_\pi^- \\ &-\left[\prod_{l=1}^{N-1}{}_{\otimes}\, e^{i\pi\left(\eta^\dagger_{2l}\eta_{2l}+\eta^\dagger_{-2l}\eta_{-2l}\right)}\bar{V}_l^-\right]\otimes e^{i\pi\eta_0^\dagger\eta_0}\bar{V}_0^-\otimes e^{i\pi\eta_\pi^\dagger\eta_\pi}\bar{V}_\pi^-\Bigg\},\end{aligned} \tag{54}$$

with

$$\begin{aligned}\bar{V}_l^- &= \left[(V_{2,l}^-)^{1/2}V_{1,l}^-(V_{2,l}^-)^{1/2}\right]^M\left[(V_{2,l}^-)^{1/2}V_{1,l}''^-(V_{2,l}^-)^{1/2}\right]\left[(V_{2,l}^-)^{1/2}V_{1,l}'^-(V_{2,l}^-)^{1/2}\right],\\ \bar{V}_0^- &= \left[(V_{2,0}^-)^{1/2}V_{1,0}^-(V_{2,0}^-)^{1/2}\right]^M\left[(V_{2,0}^-)^{1/2}V_{1,0}''^-(V_{2,0}^-)^{1/2}\right]\left[(V_{2,0}^-)^{1/2}V_{1,0}'^-(V_{2,0}^-)^{1/2}\right],\\ \bar{V}_\pi^- &= \left[(V_{2,\pi}^-)^{1/2}V_{1,\pi}^-(V_{2,\pi}^-)^{1/2}\right]^M\left[(V_{2,\pi}^-)^{1/2}V_{1,\pi}''^-(V_{2,\pi}^-)^{1/2}\right]\left[(V_{2,\pi}^-)^{1/2}V_{1,\pi}'^-(V_{2,\pi}^-)^{1/2}\right].\end{aligned} \tag{55}$$

In the next section, the partition function will be derived based on the direct product decomposition [Eqs. (52)–(55)].

C. Partition function

The partition function $Z_{\mathrm{B-K}}$ is expressed in Eq. (14) as the trace of $V^M V' V''$ in the limit $\lim_{K_3\to+\infty}\frac{1}{e^{4NK_3}}$. We have already obtained the $\{\eta\}$ fermion representation and the form of direct product decomposition. Substituting Eqs. (52) and (54) into Eq. (14) yields

$$Z_{\mathrm{B\text{-}K}} = \frac{1}{8}(2\sinh 2K_2)^{(M+2)N}\left\{\prod_{l=1}^{N}①_l + \prod_{l=1}^{N}②_l + \left(\prod_{l=1}^{N-1}③_l\right)\times④\times⑤ - \left(\prod_{l=1}^{N-1}⑥_l\right)\times⑦\times⑧\right\}, \tag{56}$$

with

$$\begin{aligned}①_l &= \lim_{K_3\to+\infty}\frac{1}{e^{4K_3}}\mathrm{Tr}(\bar{V}_l^+),\\ ②_l &= \lim_{K_3\to+\infty}\frac{1}{e^{4K_3}}\mathrm{Tr}\left[e^{i\pi\left(\eta^\dagger_{2l-1}\eta_{2l-1}+\eta^\dagger_{-(2l-1)}\eta_{-(2l-1)}\right)}\bar{V}_l^+\right],\\ ③_l &= \lim_{K_3\to+\infty}\frac{1}{e^{4K_3}}\mathrm{Tr}(\bar{V}_l^-),\\ ④ &= \lim_{K_3\to+\infty}\frac{1}{e^{2K_3}}\mathrm{Tr}(\bar{V}_0^-),\\ ⑤ &= \lim_{K_3\to+\infty}\frac{1}{e^{2K_3}}\mathrm{Tr}(\bar{V}_\pi^-),\end{aligned}$$

$$
\begin{aligned}
⑥_l &= \lim_{K_3\to+\infty}\frac{1}{e^{4K_3}}\mathrm{Tr}\big[e^{i\pi(\eta_{2l}^\dagger\eta_{2l}+\eta_{-2l}^\dagger\eta_{-2l})}\bar{V}_l^-\big],\\
⑦ &= \lim_{K_3\to+\infty}\frac{1}{e^{2K_3}}\mathrm{Tr}\big(e^{i\pi\eta_0^\dagger\eta_0}\bar{V}_0^-\big),\\
⑧ &= \lim_{K_3\to+\infty}\frac{1}{e^{2K_3}}\mathrm{Tr}\big(e^{i\pi\eta_\pi^\dagger\eta_\pi}\bar{V}_\pi^-\big).
\end{aligned}
\tag{57}
$$

Therefore, the problem reduces to the calculation of the traces of the matrices associated with $\bar{V}_l^+$ and $\bar{V}_l^-$, which are defined in Eqs. (53) and (55). In expressing $Z_{\mathrm{B-K}}$ as Eqs. (56) and (57), we have partitioned the factor $\frac{1}{e^{4NK_3}}$. Below we will illustrate that this partition is reasonable.

The explicit expressions of all the relevant matrices can be given in the basis $|0_{-(2l-1)}0_{2l-1}\rangle$, $|1_{-(2l-1)}1_{2l-1}\rangle$, $|0_{-(2l-1)}1_{2l-1}\rangle$, $|1_{-(2l-1)}0_{2l-1}\rangle$ or $|0_{-2l}0_{2l}\rangle$, $|1_{-2l}1_{2l}\rangle$, $|0_{-2l}1_{2l}\rangle$, $|1_{-2l}0_{2l}\rangle$, which are denoted as $|00\rangle$, $|11\rangle$, $|01\rangle$, $|10\rangle$ for convenience. We first consider $①_l$ in the even part. The matrix $\bar{V}_l^+$ is given in Eq. (53), with $V_{1,l}^+$, $V_{1,l}^{\prime\prime+}$, $V_{1,l}^{\prime+}$, and $V_{2,l}^+$ defined in Eqs. (48) and (50). We observe that $|01\rangle$ and $|10\rangle$ are eigenvectors of $\bar{V}_l^+$ with the same eigenvalue, as

$$
\begin{aligned}
V_{1,l}^+|01\rangle(\mathrm{or}|10\rangle) &= e^{2K_1\cos\frac{(2l-1)\pi}{2N}}|01\rangle(\mathrm{or}|10\rangle),\\
V_{1,l}^{\prime\prime+}|01\rangle(\mathrm{or}|10\rangle) &= e^{-2K_3\cos\frac{(2l-1)\pi}{2N}}|01\rangle(\mathrm{or}|10\rangle),\\
V_{1,l}^{\prime+}|01\rangle(\mathrm{or}|10\rangle) &= e^{2K_3\cos\frac{(2l-1)\pi}{2N}}|01\rangle(\mathrm{or}|10\rangle),\\
V_{2,l}^+|01\rangle(\mathrm{or}|10\rangle) &= |01\rangle(\mathrm{or}|10\rangle).
\end{aligned}
\tag{58}
$$

We see that in the space of $|01\rangle$ and $|10\rangle$, the 2×2 matrix is

$$
\bar{V}_l^+ = e^{2MK_1\cos\frac{(2l-1)\pi}{2N}}\mathbf{1}. \tag{59}
$$

This matrix has zero contribution to $①_l$ in the limit $\lim_{K_3\to+\infty}\frac{1}{e^{4K_3}}$ [see Eq. (57)]. Thus, $①_l$ is determined in the space of $|00\rangle$ and $|11\rangle$. The same analysis also applies to $②_l$, $③_l$, and $⑥_l$, so that we know

$$
①_l = ②_l, \quad ③_l = ⑥_l. \tag{60}
$$

In the space of $|00\rangle$ and $|11\rangle$, the explicit forms of $V_{1,l}^+$, $V_{1,l}^{\prime\prime+}$, $V_{1,l}^{\prime+}$, and $V_{2,l}^+$ are given as

$$
\begin{aligned}
V_{1,l}^+ &= e^{2K_1\cos\frac{(2l-1)\pi}{2N}}\begin{pmatrix}a_l & b_l\\ b_l & c_l\end{pmatrix},\\
V_{1,l}^{\prime\prime+} &= e^{-2K_3\cos\frac{(2l-1)\pi}{2N}}\begin{pmatrix}d_l & -e_l\\ -e_l & f_l\end{pmatrix},\\
V_{1,l}^{\prime+} &= e^{2K_3\cos\frac{(2l-1)\pi}{2N}}\begin{pmatrix}f_l & e_l\\ e_l & d_l\end{pmatrix},\\
V_{2,l}^+ &= \begin{pmatrix}e^{-2K_2^*} & \\ & e^{2K_2^*}\end{pmatrix},
\end{aligned}
\tag{61}
$$

with

$$
\begin{aligned}
a_l &= \cosh 2K_1 + \sinh 2K_1\cos\frac{(2l-1)\pi}{2N},\\
b_l &= \sinh 2K_1\sin\frac{(2l-1)\pi}{2N},\\
c_l &= \cosh 2K_1 - \sinh 2K_1\cos\frac{(2l-1)\pi}{2N},
\end{aligned}
$$

$$d_l = \cosh 2K_3 - \sinh 2K_3 \cos\frac{(2l-1)\pi}{2N},$$
$$e_l = \sinh 2K_3 \sin\frac{(2l-1)\pi}{2N},$$
$$f_l = \cosh 2K_3 + \sinh 2K_3 \cos\frac{(2l-1)\pi}{2N}. \tag{62}$$

$V_{2,l}^{+}$ is obviously diagonal, and the details of determining the elements of $V_{1,l}^{+}$, $V_{1,l}^{\prime\prime +}$, and $V_{1,l}^{\prime +}$ follow closely those proposed in Ref. [24]. We take the limit of $[(V_{2,l}^{+})^{1/2}V_{1,l}^{\prime\prime +}(V_{2,l}^{+})^{1/2}]$ $[(V_{2,l}^{+})^{1/2}V_{1,l}^{\prime +}(V_{2,l}^{+})^{1/2}]$:

$$\lim_{K_3\to+\infty}\frac{1}{e^{4K_3}}\left[(V_{2,l}^{+})^{1/2}V_{1,l}^{\prime\prime +}(V_{2,l}^{+})^{1/2}\right]\left[(V_{2,l}^{+})^{1/2}V_{1,l}^{\prime +}(V_{2,l}^{+})^{1/2}\right] = \frac{1}{4}\sin\frac{(2l-1)\pi}{2N}\begin{pmatrix} g_l & -h_l \\ m_l & n_l \end{pmatrix}, \tag{63}$$

with

$$g_l = (e^{-4K_2^*} - 1)\sin\frac{(2l-1)\pi}{2N},$$
$$h_l = 2\sinh 2K_2^*\left(1 - \cos\frac{(2l-1)\pi}{2N}\right),$$
$$m_l = 2\sinh 2K_2^*\left(1 + \cos\frac{(2l-1)\pi}{2N}\right),$$
$$n_l = (e^{4K_2^*} - 1)\sin\frac{(2l-1)\pi}{2N}. \tag{64}$$

The result indicates that the partition of $\frac{1}{e^{4NK_3}}$ is reasonable. Then, we diagonalize $(V_{2,l}^{+})^{1/2}V_{1,l}^{+}(V_{2,l}^{+})^{1/2}$:

$$(V_{2,l}^{+})^{1/2}V_{1,l}^{+}(V_{2,l}^{+})^{1/2} = e^{2K_1\cos\frac{(2l-1)\pi}{2N}}T\begin{pmatrix} e^{\epsilon_l} & \\ & e^{-\epsilon_l} \end{pmatrix}T^{-1}, \tag{65}$$

where ϵ_l is the positive root of

$$\cosh\epsilon_l = \cosh 2K_1\cosh 2K_2^* - \sinh 2K_1\sinh 2K_2^*\cos\frac{(2l-1)\pi}{2N} \tag{66}$$

and

$$T = \begin{pmatrix} p_l & p_l \\ r_l & s_l \end{pmatrix}, \quad T^{-1} = \frac{1}{\sinh 2K_1\sin\frac{(2l-1)\pi}{2N}(e^{-\epsilon_l} - e^{\epsilon_l})}\begin{pmatrix} s_l & -p_l \\ -r_l & p_l \end{pmatrix}, \tag{67}$$

with

$$p_l = \sinh 2K_1\sin\frac{(2l-1)\pi}{2N},$$
$$r_l = e^{\epsilon_l} - e^{-2K_2^*}\left(\cosh 2K_1 + \sinh 2K_1\cos\frac{(2l-1)\pi}{2N}\right), \tag{68}$$
$$s_l = e^{-\epsilon_l} - e^{-2K_2^*}\left(\cosh 2K_1 + \sinh 2K_1\cos\frac{(2l-1)\pi}{2N}\right).$$

Now substituting Eqs. (63)–(68) into Eq. (53) and taking the trace in the limit $\lim_{K_3\to+\infty}\frac{1}{e^{4K_3}}$, we obtain the result of $①_l$ by elementary algebra:

$$\begin{aligned}①_l &= \lim_{K_3\to+\infty}\frac{1}{e^{4K_3}}\mathrm{Tr}\big(\bar{V}_l^+\big)\\ &= e^{2MK_1\cos\frac{(2l-1)\pi}{2N}}\times\frac{\sin^2\frac{(2l-1)\pi}{2N}(2\sinh 2K_2^*)^2}{4}\frac{e^{(M+1)\epsilon_l}-e^{-(M+1)\epsilon_l}}{e^{\epsilon_l}-e^{-\epsilon_l}}.\end{aligned}\tag{69}$$

Making use of an identity

$$\gamma^{M+1}-\gamma^{-(M+1)}=(\gamma-\gamma^{-1})\prod_{j=1}^{M}\left(\gamma+\gamma^{-1}-2\cos\frac{j\pi}{M+1}\right),\tag{70}$$

we have the form of $①_l$:

$$\begin{aligned}①_l = e^{2MK_1\cos\frac{(2l-1)\pi}{2N}}\times\frac{\sin^2\frac{(2l-1)\pi}{2N}(2\sinh 2K_2^*)^2}{4}\prod_{j=1}^{M}2\Big(&\cosh 2K_1\cosh 2K_2^*-\sinh 2K_1\sinh 2K_2^*\\ &\times\cos\frac{(2l-1)\pi}{2N}-\cos\frac{j\pi}{M+1}\Big).\end{aligned}\tag{71}$$

We then turn to the odd part. Notice that $V_{1,0}^-$, $V_{2,0}^-$, $V_{1,0}^{\prime\prime-}$, $V_{1,0}^{\prime-}$ and $V_{1,\pi}^-$, $V_{2,\pi}^-$, $V_{1,\pi}^{\prime\prime-}$, $V_{1,\pi}^{\prime-}$ are all diagonal, and

$$V_{1,0}^{\prime\prime-}=V_{1,\pi}^{\prime-}=\begin{pmatrix}1 & \\ & e^{-2K_3}\end{pmatrix},\quad V_{1,0}^{\prime-}=V_{1,\pi}^{\prime\prime-}=\begin{pmatrix}1 & \\ & e^{2K_3}\end{pmatrix}.\tag{72}$$

It is clear that V_0^- and V_π^- are independent of K_3. We immediately verify that

$$④=⑤=⑦=⑧=0\tag{73}$$

from Eq. (57). Thus, the odd part has no contribution to the partition function.

Now we can go to the final step. Only the even part of the transfer matrix contributes to the partition function. Substituting Eqs. (60), (71), and (73) into Eq. (56), and making use of $\sum_{l=1}^{N}\cos\frac{(2l-1)\pi}{2N}=0$, Eq. (11b), and the identity

$$\prod_{l=1}^{N}\sin\frac{(2l-1)\pi}{2N}=\frac{1}{2^{N-1}},\tag{74}$$

the final result turns out to be

$$\begin{aligned}Z_{\text{B-K}} &= \frac{1}{8}(2\sinh 2K_2)^{(M+2)N}\times 2\prod_{l=1}^{N}①_l\\ &= (2\sinh 2K_2)^{MN}\times\prod_{l=1}^{N}\prod_{j=1}^{M}2\Big(\cosh 2K_1\cosh 2K_2^*-\sinh 2K_1\sinh 2K_2^*\cos\frac{(2l-1)\pi}{2N}\\ &\qquad -\cos\frac{j\pi}{M+1}\Big)\\ &= 2^{2MN}\prod_{l=1}^{N}\prod_{j=1}^{M}\left(\cosh 2K_1\cosh 2K_2-\sinh 2K_1\cos\frac{(2l-1)\pi}{2N}-\sinh 2K_2\cos\frac{j\pi}{M+1}\right).\end{aligned}\tag{75}$$

Now we accomplish our derivation. Our solution is completely consistent with the result previously published [43,47–50].

IV. DISCUSSION AND SUMMARY

We have succeeded in deriving the solution of the square lattice Ising model under the B-K BCs by the SML method within the transfer matrix formalism. As shown in Eq. (75), the partition function has a double product form, which permits an analytic calculation of the Fisher zeros. In the variable $z = \sinh 2K_1$, the Fisher zeros are explicitly solved:

$$z_{lj,1,2} = \frac{\sinh 2K_2 \cos\frac{(2l-1)\pi}{2N}\cos\frac{j\pi}{M+1} \pm i\cosh 2K_2\sqrt{\sinh^2 2K_2 \sin^2\frac{j\pi}{M+1} + \sin^2\frac{(2l-1)\pi}{2N}}}{\cosh^2 2K_2 - \cos^2\frac{(2l-1)\pi}{2N}},$$
$$l = 1,\ldots,N; \quad j = 1,\ldots,M. \tag{76}$$

For finite M and N, the Fisher zeros in the complex z plane do not lie on the real axis. When the system approaches the thermodynamic limit $M, N \to \infty$, the Fisher loci cut the real axis at $z = \frac{1}{\sinh 2K_2}$ and $z = -\frac{1}{\sinh 2K_2}$. Hence, the system exhibits a phase transition at the critical temperature determined by $\sinh 2K_1 \sinh 2K_2 = 1$ (J_1 and J_2 have the same sign) or $\sinh 2K_1 \sinh 2K_2 = -1$ (J_1 and J_2 have opposite signs). The free energy per spin in the thermodynamic limit is directly obtained from Eq. (75):

$$\lim_{M,N\to\infty}\frac{1}{2MN}(-\beta F) = \lim_{M,N\to\infty}\frac{1}{2MN}\ln Z_{\text{B-K}} = \ln 2 + \frac{1}{8\pi^2}\int_0^{2\pi} d\theta \int_0^{2\pi} d\phi \ln(\cosh 2K_1 \cosh 2K_2$$
$$- \sinh 2K_1 \cos\theta - \sinh 2K_2 \cos\phi), \tag{77}$$

which is Onsager's well-known solution [7]. In the second step of Eq. (77), we have used the observation that $\ln Z_{\text{B-K}}$ can be expressed as a sum and the sum converges to the integral when the number of terms goes to infinity. The physical critical point can also be verified from taking the derivative of the solution Eq. (77) with respect to T.

Two special cases of the system are of interest. The first is the case that $K_2 = 0$, i.e., $J_2 = 0$ and the system reduces to the one-dimensional model. The Fisher zeros in Eq. (76) become

$$z_{l,1,2} = \pm i\frac{1}{\sin\frac{(2l-1)\pi}{2N}}, \quad l = 1,\ldots,N. \tag{78}$$

This result agrees with the previous studies of one-dimensional Ising model [56]. The second case is that the interactions are isotropic, i.e., $J_1 = J_2 \equiv J$ and $K_1 = K_2 \equiv K$. In terms of $\bar{z} = \sinh 2K$, the partition function in Eq. (75) is expressed as

$$Z_{\text{B-K}} = 2^{2MN}\bar{z}^{MN}\prod_{l=1}^{N}\prod_{j=1}^{M}\left(\bar{z} + \bar{z}^{-1} - \cos\frac{(2l-1)\pi}{2N} - \cos\frac{j\pi}{M+1}\right). \tag{79}$$

The Fisher zeros are

$$\bar{z}_{lj,1,2} = e^{\pm i\theta_{lj}}, \tag{80}$$

with

$$\theta_{lj} = \arccos\left[\frac{1}{2}\left(\cos\frac{(2l-1)\pi}{2N} + \cos\frac{j\pi}{M+1}\right)\right]. \tag{81}$$

For any finite lattice, the Fisher zeros lie on the unit circle in the complex $\bar{z}$ plane. In the thermodynamic limit, the Fisher loci form a continuous unit circle; thus, we find the physical critical point at $\bar{z} = 1$ ($J > 0$) or $\bar{z} = -1$ ($J < 0$). When we consider the variable $u = e^{2K}$, the Fisher loci form two circles $|u \pm 1| = \sqrt{2}$, which were first proposed by Fisher [44]. The physical critical point in variable u is $u = \sqrt{2} + 1$ ($J > 0$) or $u = \sqrt{2} - 1$ ($J < 0$).

Finally, we briefly discuss the difference between the transfer matrix approaches to the B-K BCs and to the toroidal BCs. In Sec. III, we transform the system under the B-K BCs into another

system under the toroidal BCs, by setting special interactions on the upper and lower boundaries and taking the limit $\lim_{J_3\to+\infty}\frac{1}{e^{4N\beta J_3}}$. By using the SML method, we express the transfer matrices in the fermionic representation, thereby translating the problem of a two-dimensional classical Ising model into the one of solving the eigenvalues of a one-dimensional fermion system. In Eq. (34), the product of transfer matrices is divided into the even and odd parts, which correspond to the mapping systems involving even and odd number of fermions, respectively. Both the even and odd parts have two subparts. The effect of the limit $\lim_{J_3\to+\infty}\frac{1}{e^{4N\beta J_3}}$ is that the contribution of the odd part is 0 and the contribution from two subparts in the even part is equal, so that we achieve the final double product form of the partition function. While under the usual toroidal BCs, i.e., the horizontal interactions are uniform in our model, $V^M V'' V'$ in Eq. (34) simply becomes V^{M+2}. In this case, each subpart in the even and odd parts has nonzero contribution; thus, the solution is the sum of four double products [22]. The details in the transfer matrix lead to the significant difference between the forms of two solutions. Therefore, it is natural to apply the technique of taking certain limit of certain interactions to other lattices or other BCs, such as the Kagomé lattice model [12] and the square lattice model under the open BCs [53], in the transfer matrix formalism. The transfer matrix approach in the lattice models under various BCs, in particular, the SML method using the fermionic representation, merits further investigation. Also, we hope that the fermionic representation can be further extended as a new approach to more problems of interest such as the spontaneous magnetization.

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

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