

Semiclassical magnetotransport in Weyl semimetals with skyrmion-induced emergent magnetic fields

Azaz Ahmad ^{1,2,3,*} and Takami Tohyama ³

¹Department of Physics, *Indian Institute of Technology, Kanpur 208016, India*

²School of Physical Sciences, *Indian Institute of Technology Mandi, Mandi 175005, India*

³Department of Applied Physics, *Tokyo University of Science, Katsushika, Tokyo 125-8585, Japan*



(Received 18 March 2026; accepted 18 August 2026; published 26 August 2026)

We investigate magnetotransport in a time-reversal symmetry-broken, untilted Weyl semimetal in the simultaneous presence of momentum-space Berry curvature and skyrmion-induced emergent magnetic fields $\mathbf{B}_{\text{emer}}$. Using a semiclassical Boltzmann approach incorporating Berry-curvature corrections and intervalley scattering, we analyze the longitudinal magnetoconductivity and planar Hall conductivity in this mixed-topology regime. In the absence of $\mathbf{B}_{\text{emer}}$, increasing intervalley scattering drives a strong sign reversal of the longitudinal magnetoconductivity. A finite $\mathbf{B}_{\text{emer}}$ introduces an additional shift of the parabolic magnetic-field dependence, leading to a weak sign-reversal regime without altering the curvature. The coexistence of these effects naturally produces a strong-and-weak sign-reversal regime, demonstrating that intervalley scattering and real-space topology control distinct geometric features of the response. The emergent field further induces asymmetry in the angular dependence of both longitudinal and planar Hall conductivities. We show that a finite planar Hall response can arise solely from $\mathbf{B}_{\text{emer}}$ when its direction is varied, providing a transport signature of real-space topology. Our results establish that the skyrmion-induced emergent field acts as an independent topological tuning parameter, revealing measurable consequences of the interplay between real- and momentum-space Berry curvature in Weyl systems.

DOI: [10.1103/15sf-rdsh](https://doi.org/10.1103/15sf-rdsh)

I. INTRODUCTION

Topological semimetals realized in kagome-lattice materials provide a natural platform in which distinct forms of topology can coexist and mutually influence electronic transport [1–4]. In Weyl semimetals (WSMs), momentum $\mathbf{k}$ -space topology arises from isolated band-touching points (Weyl nodes) that act as monopoles of Berry curvature $\Omega_{\mathbf{k}}^{\chi}$, with chirality $\chi = \pm 1$ [5,6]. The associated Berry curvature profoundly modifies semiclassical transport, generating anomalous velocity contributions and phase-space corrections that underlie phenomena such as the anomalous Hall effect and the chiral anomaly [7–12]. These effects are now well established in Weyl systems [13–18].

Kagome-lattice materials, however, offer an additional ingredient: nontrivial real-space spin textures. Owing to geometrical frustration, strong spin-orbit coupling, and magnetic ordering tendencies, kagome systems can host complex magnetic configurations including noncoplanar

*Contact author: Aa178342@gmail.com

Published by the American Physical Society under the terms of the [Creative Commons Attribution 4.0 International](https://creativecommons.org/licenses/by/4.0/) license. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI.

spin states and skyrmion textures [19–21]. When the electronic spins adiabatically follow such a smoothly varying magnetization texture $\mathbf{n}(\mathbf{r})$, they acquire a real-space Berry phase that manifests as an emergent magnetic field $\mathbf{B}_{\text{emer}} = \frac{\hbar}{2e} \mathbf{n} \cdot (\partial_x \mathbf{n} \times \partial_y \mathbf{n}) \hat{e}$, where $\hat{e}$ is the unit vector pointing in the direction of the field. $\mathbf{B}_{\text{emer}}$ acts on the charge carriers analogously to an external magnetic field $\mathbf{B}$ [16]. This mechanism is responsible for the topological Hall effect in chiral magnets and has been widely discussed in systems with noncoplanar spin order [16,22–24].

In a kagome WSM hosting both Weyl nodes and a nontrivial magnetic texture, electrons therefore experience two distinct topological fields: the Berry curvature in momentum space and the emergent magnetic field in real space [1–4]. Within the semiclassical Boltzmann framework, the equations of motion depend only on the total magnetic field $\mathbf{B}_{\text{tot}} = \mathbf{B} + \mathbf{B}_{\text{emer}}$. As a result, the modified phase-space factor that controls the dynamics of the electrons becomes $\mathcal{D}^\chi = (1 + \frac{e}{\hbar} \mathbf{B}_{\text{tot}} \cdot \boldsymbol{\Omega}_{\mathbf{k}}^\chi)^{-1}$ [7,25]. This expression explicitly contains the cross term $\mathbf{B}_{\text{emer}} \cdot \boldsymbol{\Omega}_{\mathbf{k}}^\chi$, which encodes the interplay between real-space and momentum-space topology. Importantly, the intrinsic Berry curvature of the Weyl nodes remains unchanged; rather, it is the emergent field that modifies the effective magnetic environment in which the Weyl quasiparticles propagate. Consequently, magnetotransport in such systems carries signatures of a mixed real-momentum topological response. In particular, longitudinal magnetoconductivity (LMC) and planar Hall conductance (PHC) acquire corrections controlled by $\mathbf{B}_{\text{emer}}$, allowing magnetic texture effects to directly influence anomaly-driven transport. Kagome Weyl materials therefore provide an experimentally relevant setting in which this real-momentum topological mixing can be probed through standard magnetotransport measurements. A central conceptual advance of this work is the identification of a *mixed-topological magnetotransport mechanism* arising from the interplay between real-space and momentum-space topology. While previous studies have focused on strain-induced axial fields that are introduced as chirality-dependent gauge fields by construction [10,18,26–29], here the chirality dependence of the transport response emerges *dynamically* from the coupling between the real-space emergent field and the intrinsic Berry curvature of Weyl fermions.

Although the emergent magnetic field $\mathbf{B}_{\text{emer}}$ enters the semiclassical equations of motion in a form analogous to an external magnetic field, its physical origin is fundamentally different. In particular, $\mathbf{B}_{\text{emer}}$ is generated by the scalar spin chirality associated with the underlying real-space magnetic texture and is therefore directly tied to the skyrmion density and chirality. Consequently, transport responses arising from $\mathbf{B}_{\text{emer}}$ are expected to correlate with the creation, annihilation, or chirality reversal of skyrmion textures, unlike ordinary external magnetic-field-induced responses. Furthermore, in our framework, although $\mathbf{B}_{\text{emer}}$ is identical for both Weyl nodes, its projection onto the Berry curvature $\mathbf{B}_{\text{emer}} \cdot \boldsymbol{\Omega}_{\mathbf{k}}^\chi$ automatically acquires opposite signs for opposite chiralities due to $\boldsymbol{\Omega}_{\mathbf{k}}^\chi \propto \chi$ [25]. The resulting transport corrections, therefore, mimic axial-field-like magnetotransport without invoking any explicit chiral gauge field. This demonstrates that axial-type longitudinal and planar Hall responses can arise purely from the coexistence of momentum-space monopoles and real-space skyrmion topology. Our results thus establish a direct bridge between two distinct topological structures: the Weyl monopole in momentum space and the skyrmion winding number in real space. The magnetotransport response can be interpreted as a mixed-topological coupling proportional to the product of these two topological charges, providing a unified description of anomalylike transport phenomena driven by magnetic texture topology.

In the present work, the interplay between real-space and momentum-space topology is investigated within the semiclassical transport regime, where the dominant effect of the magnetic texture is captured through the emergent magnetic field $\mathbf{B}_{\text{emer}}$. We note that the appearance of additive or multiplicative transport contributions does not necessarily imply the absence of interplay between the two topological structures. Indeed, recent phase-space Berry-curvature formulations treating real-space and momentum-space coordinates on equal footing have shown that the leading transport responses can remain effectively separable, while genuine mixed phase-space curvature corrections appear only at higher order [17,23,30]. The present work, therefore, focuses on the leading semiclassical transport consequences of the coexistence of real-space magnetic topology and momentum-space Berry curvature in Weyl semimetals.

The semiclassical Boltzmann framework employed in the present work builds upon earlier transport formalisms developed for magnetic-field-driven transport in topological systems [25,26,31–42]. In this work, our focus is the investigation of Weyl-semimetal magnetotransport in the simultaneous presence of external and texture-induced emergent magnetic fields generated by topologically nontrivial magnetic textures. Specifically, we analyze how emergent electrodynamics associated with real-space magnetic topology modifies Berry-curvature- and chiral-anomaly-driven transport responses within the beyond-relaxation time approximation (RTA) framework.

The remainder of the paper is organized as follows. In Sec. II, we introduce the minimal model of a time-reversal symmetry-broken WSM, and Sec. III outlines the semiclassical Boltzmann formalism incorporating Berry-curvature corrections and the emergent magnetic field arising from real-space topology. In Sec. IV, we classify different regimes of sign reversal and present detailed results for the LMC and PHC in the presence of mixed real- and momentum-space topology. Finally, in Sec. V, we summarize our main findings and discuss their physical implications.

II. LOW-ENERGY MODEL FOR MIXED REAL- AND MOMENTUM-SPACE TOPOLOGY

To investigate transport arising from the interplay of momentum-space and real-space topology, we begin with the minimal low-energy Hamiltonian describing quasiparticles near a pair of Weyl nodes. In the absence of magnetic texture, the effective Hamiltonian reads [6,43–45]

$$H_{\text{WSM}}^\chi(\mathbf{k}) = \chi \hbar v_F \mathbf{k} \cdot \boldsymbol{\sigma}, \quad (1)$$

where $\chi = \pm 1$ denotes the chirality (topological charge) of the Weyl node, v_F is the Fermi velocity, and $\boldsymbol{\sigma}$ represents the Pauli matrices acting in pseudospin space. The wave vector $\mathbf{k}$ is measured from the Weyl node. For clarity and to isolate the essential mixed-topological contribution, we restrict our analysis to a single pair of Weyl cones, noting that additional pairs modify only the total response quantitatively [25,35]. Diagonalization of Eq. (1) yields the linear dispersion

$$\epsilon_{\mathbf{k}} = \pm \hbar v_F |\mathbf{k}|. \quad (2)$$

The orbital magnetic moment (OMM), arising from the self-rotation of the Bloch wave packet, plays an important role in the semiclassical dynamics. It is given by [15]

$$\begin{aligned} \mathbf{m}_{\mathbf{k}}^\chi &= -\frac{ie}{2\hbar} \text{Im} \langle \nabla_{\mathbf{k}} u^\chi(\mathbf{k}) | [\epsilon_{\mathbf{k}} - \hat{H}_{\text{WSM}}^\chi(\mathbf{k})] | \nabla_{\mathbf{k}} u^\chi(\mathbf{k}) \rangle \\ &\equiv -\chi e v_F \mathbf{k} / 2k^2, \end{aligned} \quad (3)$$

where $u^\chi(\mathbf{k})$ is the Bloch wave packet in the WSM. The coupling of the OMM to a total magnetic field $\mathbf{B}_{\text{tot}}$ modifies the band dispersion according to $\epsilon_{\mathbf{k}}^\chi = \epsilon_{\mathbf{k}} - \mathbf{m}_{\mathbf{k}}^\chi \cdot \mathbf{B}_{\text{tot}}$. Consequently, the constant-energy contour at the Fermi energy ϵ_F is obtained as [25,33,46]

$$k_F^\chi = \frac{\epsilon_F + \sqrt{\epsilon_F^2 - n \chi e v_F B_{\text{tot}} \beta_{\theta\phi}}}{n}, \quad (4)$$

where $n = 2\hbar v_F$. By using θ and ϕ as polar and azimuthal angles of the Bloch eigenvectors, respectively (see below), the geometric factor $\beta_{\theta\phi}$ is given by

$$\beta_{\theta\phi} = \sin \theta \cos \phi (\cos \gamma + \cos \gamma_{\text{emer}}) + \cos \theta (\sin \gamma + \sin \gamma_{\text{emer}}), \quad (5)$$

where the direction of the magnetic fields is assumed in the xz plane, measured with respect to the $\hat{x}$ axis, i.e., $\mathbf{B} = B(\cos \gamma, 0, \sin \gamma)$ and $\mathbf{B}_{\text{emer}} = B_{\text{emer}}(\cos \gamma_{\text{emer}}, 0, \sin \gamma_{\text{emer}})$. The nontrivial topology of the Weyl node is encoded in its Bloch eigenvectors,

$$|u^+\rangle^T = [e^{-i\phi} \cos(\theta/2), \sin(\theta/2)], \quad (6)$$

$$|u^-\rangle^T = [-e^{-i\phi} \sin(\theta/2), \cos(\theta/2)], \quad (7)$$

which generate a singular Berry curvature in momentum space,

$$\mathbf{\Omega}_{\mathbf{k}}^{\chi} = i\nabla_{\mathbf{k}} \times \langle u^{\chi}(\mathbf{k}) | \nabla_{\mathbf{k}} | u^{\chi}(\mathbf{k}) \rangle \equiv -\chi \frac{\mathbf{k}}{2k^3}, \quad (8)$$

acting as a monopole of Berry flux located at the Weyl node [7,15]. This momentum-space Berry curvature underlies anomalous transport responses such as the anomalous Hall effect and chiral-anomaly-induced magnetotransport [25,32–36,39,46–63].

When the WSM hosts a slowly varying magnetic texture $\mathbf{n}(\mathbf{r})$, such as a skyrmion lattice, electrons traversing the texture acquire an additional real-space Berry phase. In the adiabatic regime where the magnetic texture varies on a length scale much larger than the electronic wavelength, the effect of the texture can be captured through an emergent magnetic field [16,64], $\mathbf{B}_{\text{emer}} = \frac{\hbar}{2e} \mathbf{n} \cdot (\partial_x \mathbf{n} \times \partial_y \mathbf{n}) \hat{e}$, which is directly proportional to the skyrmion density. Unlike the external magnetic field, $\mathbf{B}_{\text{emer}}$ originates purely from real-space topology and exists even in the absence of applied fields. In systems where Weyl band topology coexists with skyrmion textures, as realized in magnetic kagome materials and related chiral magnets [1–4], charge carriers therefore experience two independent sources of Berry curvature: (1) momentum-space Berry curvature $\mathbf{\Omega}_{\mathbf{k}}^{\chi}$ associated with Weyl nodes and (2) Real-space Berry curvature encoded in $\mathbf{B}_{\text{emer}}$. Within the semiclassical description, both fields enter the equations of motion through the modified phase-space factor

$$\mathcal{D}^{\chi} = \left[1 + \frac{e}{\hbar} (\mathbf{B} + \mathbf{B}_{\text{emer}}) \cdot \mathbf{\Omega}_{\mathbf{k}}^{\chi} \right]^{-1}. \quad (9)$$

The crucial new ingredient is the mixed scalar product $\mathbf{B}_{\text{emer}} \cdot \mathbf{\Omega}_{\mathbf{k}}^{\chi}$, which vanishes unless both real-space and momentum-space topology are simultaneously present. This term modifies the density of states and transport coefficients, giving rise to additional contributions to LMC and PHC that are absent in ordinary metals and in systems hosting only one type of topology. Thus, the effective low-energy model of Eq. (1), supplemented by the emergent field $\mathbf{B}_{\text{emer}}$, provides the minimal theoretical framework required to describe mixed real-momentum space topological transport.

We assume that the length scale of the skyrmion textures is much smaller than the length of the system, so that the charge carrier experiences an average, uniform $\mathbf{B}_{\text{emer}}$. This leads to the momentum-dependent but position-independent phase-space factor $\mathcal{D}_{\mathbf{k}}^{\chi}$. The magnitude of the emergent magnetic field generated by a skyrmion texture is determined by the skyrmion density through $B_{\text{emer}} = n_{\text{sk}} \Phi_0$, where $\Phi_0 = h/e$ is the flux quantum. For typical skyrmion sizes of ~ 50 – 100 nm observed in chiral magnets, the corresponding emergent magnetic field lies in the range 10^{-2} – 1 T. Therefore, the parameter range $B_{\text{emer}} = 0$ – 0.25 T used in our calculations falls well within experimentally realistic values [65].

III. MAXWELL BOLTZMANN TRANSPORT FORMALISM FOR ELECTRON TRANSPORT IN SKYRMION

Using the quasiclassical Boltzmann transport theory, we study transport in WSMs in the limit of weak electric and magnetic fields. Since the quasiclassical Boltzmann theory is valid when the chemical potential μ is away from the nodal point such that $\mu^2 \gg \hbar v_F^2 e |\mathbf{B}_{\text{tot}}|$, we assume that μ lies in the conduction band without loss of generality. The phenomenological Boltzmann equation for the nonequilibrium distribution function $f_{\mathbf{k}}^{\chi}$ can be expressed as [31,51,66–69]

$$\left(\frac{\partial}{\partial t} + \dot{\mathbf{r}}^{\chi} \cdot \nabla_{\mathbf{r}} + \dot{\mathbf{k}}^{\chi} \cdot \nabla_{\mathbf{k}} \right) f_{\mathbf{k}}^{\chi} = \mathcal{I}_{\text{coll}}[f_{\mathbf{k}}^{\chi}], \quad (10)$$

where the collision term $\mathcal{I}_{\text{coll}}$ incorporates impurity scattering. In the presence of an external magnetic field $\mathbf{B}$, a real-space emergent magnetic field $\mathbf{B}_{\text{emer}}$, and an electric field $\mathbf{E}$, the semiclassical

equations of motion become [7,14,15,31,70]

$$\begin{aligned}\dot{\mathbf{r}}^\chi &= \mathcal{D}_\mathbf{k}^\chi \left[\mathbf{v}_\mathbf{k}^\chi + \frac{e}{\hbar} \mathbf{E} \times \boldsymbol{\Omega}_\mathbf{k}^\chi + \frac{e}{\hbar} (\mathbf{v}_\mathbf{k}^\chi \cdot \boldsymbol{\Omega}_\mathbf{k}^\chi) \mathbf{B}_\text{tot} \right], \\ \dot{\mathbf{k}}^\chi &= -\frac{e}{\hbar} \mathcal{D}_\mathbf{k}^\chi \left[\mathbf{E} + \mathbf{v}_\mathbf{k}^\chi \times \mathbf{B}_\text{tot} + \frac{e}{\hbar} (\mathbf{E} \cdot \mathbf{B}_\text{tot}) \boldsymbol{\Omega}_\mathbf{k}^\chi \right],\end{aligned}\quad (11)$$

where $\mathbf{v}_\mathbf{k}^\chi$ is the band velocity calculated using $\epsilon_\mathbf{k}^\chi$. The present study primarily focuses on the transport contributions arising from the chiral anomaly, while the conventional Lorentz-force contribution is neglected. More specifically, the Lorentz-force contribution enters the Boltzmann equation through the terms $\propto [(\mathbf{v}_\mathbf{k}^\chi \times \mathbf{B}_\text{tot}) \cdot \nabla_\mathbf{k} g_\mathbf{k}^\chi]$, which produces corrections proportional to the cyclotron parameter $\omega_c \tau$ [71–74]:

$$\omega_c \tau = \frac{eB}{m} \tau \equiv \frac{ev_F^2 B}{\epsilon_F} \tau \ll 1, \quad (12)$$

where τ denotes the relevant transport relaxation time. In the weak-field metallic regime $\omega_c \tau \ll 1$, these contributions generate higher-order angular distortions of the distribution function that remain subleading compared to the anomaly-induced contribution $\propto (\mathbf{E} \cdot \mathbf{B}_\text{tot}) \boldsymbol{\Omega}_\mathbf{k}^\chi$. Consequently, the dominant field-dependent contribution to the magnetotransport considered in the present work originates from the chiral anomaly terms. Similar approximations have been employed extensively in previous studies of anomaly-driven magnetotransport within the semiclassical regime [25,31,47,48,55,59,69,74–81].

The collision integral includes both intranode and internode scattering [31,35,71,72,82]:

$$\mathcal{I}_\text{coll}[f_\mathbf{k}^\chi] = \sum_{\chi'} \sum_{\mathbf{k}'} W_{\mathbf{k}\mathbf{k}'}^{\chi\chi'} (f_{\mathbf{k}'}^{\chi'} - f_\mathbf{k}^\chi), \quad (13)$$

with scattering rate calculated using Fermi's golden rule [71,72,82,83],

$$W_{\mathbf{k}\mathbf{k}'}^{\chi\chi'} = \frac{2\pi}{\hbar} \frac{n_i}{\mathcal{V}} |\langle u^{\chi'}(\mathbf{k}') | U_{\mathbf{k}\mathbf{k}'}^{\chi\chi'} | u^\chi(\mathbf{k}) \rangle|^2 \delta(\epsilon_{\mathbf{k}'}^{\chi'} - \epsilon_\mathbf{k}^\chi), \quad (14)$$

where n_i is the concentration of impurities, $\mathcal{V}$ is the volume of the system, and $U_{\mathbf{k}\mathbf{k}'}^{\chi\chi'}$ is the profile of the scattering potential. Assuming nonmagnetic pointlike impurities with momentum-independent and chirality-dependent amplitudes, i.e., $U_{\mathbf{k}\mathbf{k}'}^{\chi\chi'} = U^{\chi\chi'}$, we can express the square of the matrix elements as $\mathcal{G}^{\chi\chi'} \equiv |\langle u^{\chi'}(\mathbf{k}') | U_{\mathbf{k}\mathbf{k}'}^{\chi\chi'} | u^\chi(\mathbf{k}) \rangle|^2 = |U^{\chi\chi'}|^2 \mathcal{V}^2 [1 + \chi\chi'(\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos\phi\cos\phi' + \sin\theta\sin\theta'\sin\phi\sin\phi')]$. Here, it is worth highlighting that, in our formalism, the relative strength of the intra- and internode scattering channels can be tuned using the parameter defined as

$$\alpha = \frac{U^{\chi\chi'}}{U^{\chi\chi}} (\chi' \neq \chi). \quad (15)$$

For the weak-field regime considered here, we can expand the nonequilibrium distribution function as $f_\mathbf{k}^\chi = f_0 + g_\mathbf{k}^\chi$, with the Fermi distribution function f_0 and first-order field-induced deviation $g_\mathbf{k}^\chi = e(-\frac{\partial f_0}{\partial \epsilon_\mathbf{k}^\chi}) \mathbf{E} \cdot \boldsymbol{\Lambda}_\mathbf{k}^\chi$ [53,61,84–86]. Here, $\boldsymbol{\Lambda}_\mathbf{k}^\chi$ is an unknown function to be evaluated. Fixing $\mathbf{E} = E\hat{z}$, the component of $\boldsymbol{\Lambda}_\mathbf{k}^\chi$ along $\mathbf{E}$ would be relevant and one can write $g_\mathbf{k}^\chi = e(-\frac{\partial f_0}{\partial \epsilon_\mathbf{k}^\chi}) E \Lambda_{\mathbf{k},z}^\chi$, where $\Lambda_{\mathbf{k},z}^\chi$ represents the z component of unknown function (ansatz) [53,61,84,85]. For the sake of complexity, we will drop the irrelevant index, z , in $\Lambda_{\mathbf{k},z}^\chi$.

Keeping terms linear in $\mathbf{E}$, the Boltzmann equation becomes [31,33,71,72,85,87]

$$\mathcal{D}_\mathbf{k}^\chi \left[v_\mathbf{k}^{\chi z} + \frac{e(B \sin \gamma + B_\text{emer} \sin \gamma_\text{emer})}{\hbar} (\boldsymbol{\Omega}_\mathbf{k}^\chi \cdot \mathbf{v}_\mathbf{k}^\chi) \right] = \sum_{\chi'} \sum_{\mathbf{k}'} W_{\mathbf{k}\mathbf{k}'}^{\chi\chi'} (\Lambda_{\mathbf{k}'}^{\chi'} - \Lambda_\mathbf{k}^\chi). \quad (16)$$

Here, we define the band scattering rate as

$$\frac{1}{\tau_{\mathbf{k}}^{\chi}(\theta, \phi)} = \sum_{\chi'} \nu \int \frac{d^3 \mathbf{k}'}{(2\pi)^3} (\mathcal{D}_{\mathbf{k}'}^{\chi'})^{-1} W_{\mathbf{k}\mathbf{k}'}^{\chi\chi'}. \quad (17)$$

Using Eq. (17), we can write Eq. (16) as

$$h_{\mathbf{k}}^{\chi}(\theta, \phi) + \frac{\Lambda_{\mathbf{k}}^{\chi}(\theta, \phi)}{\tau_{\mathbf{k}}^{\chi}(\theta, \phi)} = \nu \sum_{\chi'} \int \frac{d^3 \mathbf{k}'}{(2\pi)^3} (\mathcal{D}_{\mathbf{k}'}^{\chi'})^{-1} W_{\mathbf{k}\mathbf{k}'}^{\chi\chi'} \Lambda_{\mathbf{k}'}^{\chi'}(\theta', \phi'). \quad (18)$$

Here, $h_{\mathbf{k}}^{\chi}(\theta, \phi) = \mathcal{D}_{\mathbf{k}}^{\chi} [v_{z,\mathbf{k}}^{\chi} + \frac{eB \sin \gamma + B_{\text{emer}} \sin \gamma_{\text{emer}}}{\hbar} (\boldsymbol{\Omega}_{\mathbf{k}}^{\chi} \cdot \mathbf{v}_{\mathbf{k}}^{\chi})]$ and is evaluated at the Fermi energy. In the low-temperature limit, for a constant Fermi energy surface, Eq. (17) and the right-hand side of Eq. (18) are reduced to integration over θ' and ϕ' ,

$$\begin{aligned} \frac{1}{\tau_{\mathbf{k}}^{\chi}(\theta, \phi)} &= \nu \sum_{\chi'} \Pi^{\chi\chi'} \iint \frac{(k_{\text{F}}^{\chi'})^3 \sin \theta'}{|\mathbf{v}_{\mathbf{k}_{\text{F}}}^{\chi'} \cdot \mathbf{k}_{\text{F}}^{\chi'}|} d\theta' d\phi' \mathcal{G}^{\chi\chi'} (\mathcal{D}_{\mathbf{k}'}^{\chi'})^{-1} \\ &= \nu \sum_{\chi'} \Pi^{\chi\chi'} \iint f^{\chi'}(\theta', \phi') \mathcal{G}^{\chi\chi'} \Lambda_{\mathbf{k}}^{\chi'} / \tau_{\mathbf{k}}^{\chi} d\theta' d\phi', \end{aligned} \quad (19)$$

where $\Pi^{\chi\chi'} = n_i / 4\pi^2 \hbar^2$ and $f^{\chi}(\theta, \phi) = \frac{(k_{\text{F}}^{\chi})^3}{|\mathbf{v}_{\mathbf{k}_{\text{F}}}^{\chi} \cdot \mathbf{k}_{\text{F}}^{\chi}|} \sin \theta (\mathcal{D}_{\mathbf{k}}^{\chi})^{-1} \tau_{\mathbf{k}}^{\chi}(\theta, \phi)$. Using ansatz $\Lambda_{\mathbf{k}}^{1,\chi} = (d^{\chi} - h_{\mathbf{k}}^{\chi} + a^{\chi} \cos \theta + b^{\chi} \sin \theta \cos \phi + c^{\chi} \sin \theta \sin \phi) \tau_{\mathbf{k}}^{\chi}(\theta, \phi)$, Eq. (19) can be written in the following form:

$$\begin{aligned} &d^{\chi} + a^{\chi} \cos \theta + b^{\chi} \sin \theta \cos \phi + c^{\chi} \sin \theta \sin \phi \\ &= \sum_{\chi'} \Pi^{\chi\chi'} \iint f^{\chi'}(\theta', \phi') \mathcal{G}^{\chi\chi'} d\theta' d\phi' \\ &\quad \times (d^{\chi'} - h_{\mathbf{k}}^{\chi'} + a^{\chi'} \cos \theta' + b^{\chi'} \sin \theta' \cos \phi' + c^{\chi'} \sin \theta' \sin \phi'). \end{aligned} \quad (20)$$

When the aforementioned equation is explicitly stated, it appears as seven simultaneous independent equations that must be solved for eight variables. The conservation of particle numbers provides an additional restriction:

$$\sum_{\chi} \sum_{\mathbf{k}} g_{\mathbf{k}}^{\chi} = 0. \quad (21)$$

For eight unknown variables ($d^{\pm 1}, a^{\pm 1}, b^{\pm 1}, c^{\pm 1}$), Eqs. (20) and (21) are simultaneously solved with Eq. (19). Due to the intricate structure of the equations, all two-dimensional integrals with respect to θ' and ϕ' are carried out numerically and the solution of simultaneous equations is obtained.

Once the distribution function $f_{\mathbf{k}}^{\chi}$ is evaluated, the current density can be evaluated as

$$\mathbf{J} = -e \sum_{\chi, \mathbf{k}} f_{\mathbf{k}}^{\chi} \mathbf{r}_{\mathbf{k}}^{\chi}. \quad (22)$$

The generalized velocity $\mathbf{r}_{\mathbf{k}}^{\chi}$ can be written as $\mathbf{r}_{\mathbf{k}}^{\chi} = \mathbf{v}_0^{\chi} + \mathbf{v}_1^{\chi}$, where $\mathbf{v}_0^{\chi}$ and $\mathbf{v}_1^{\chi}$ correspond to the velocities without and with the electric field, respectively. The explicit expressions are

$$\mathbf{v}_0^{\chi} = \left(\frac{e \mathcal{D}_{\mathbf{k}}^{\chi}}{\hbar} \right) \left(\mathbf{v}_{\mathbf{k}}^{\chi} \cdot \boldsymbol{\Omega}_{\mathbf{k}}^{\chi} \mathbf{B}_{\text{tot}} + \frac{\hbar}{e} \mathbf{v}_{\mathbf{k}}^{\chi} \right), \quad \mathbf{v}_1^{\chi} = \left(\frac{e \mathcal{D}_{\mathbf{k}}^{\chi}}{\hbar} \right) (\mathbf{E} \times \boldsymbol{\Omega}_{\mathbf{k}}^{\chi}). \quad (23)$$

Now, from Eq. (22), the generalized current response is given by

$$\mathbf{J} = -e \sum_{\chi, \mathbf{k}} (\mathbf{v}_0^{\chi} f_0^{\chi} + \mathbf{v}_1^{\chi} f_0^{\chi} + \mathbf{v}_0^{\chi} g_{\mathbf{k}}^{\chi}). \quad (24)$$

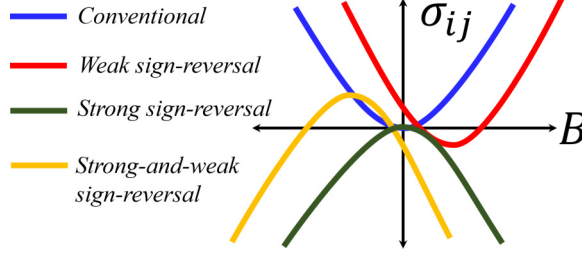


FIG. 1. A schematic illustration of the characteristic regimes of magnetoconductivity σ_{ij} in WSMs in the presence of both external magnetic field B and emergent field B_{emer} . The figure highlights weak sign reversal (shifted parabola), strong sign reversal (curvature inversion), and the coexisting strong-and-weak sign-reversal regime, in contrast to the conventional quadratic-in- B response. In the present mixed-topology setting, these regimes arise from the interplay between intervalley scattering and the real-space Berry-curvature contribution encoded in B_{emer} [26,46,88].

In the next sections, we evaluate LMC, σ_{zz} , and PHC, σ_{xz} , defined as $J_z = \sigma_{zz}E$ and $J_x = \sigma_{xz}E$, respectively. Unless otherwise stated, we use the following parameter values: Fermi velocity $v_F = 10^6$ m/s and Fermi energy $\epsilon_F = 0.05$ eV.

IV. RESULTS AND DISCUSSIONS

Before presenting the detailed numerical results, we first clarify the classification of different types of sign reversal of LMC in the present mixed-topology setting. The sign of LMC in WSMs has been extensively investigated in earlier works [31,33,75]. In an untitled WSM, the chiral anomaly typically produces a positive LMC under weak intervalley scattering; however, when intervalley scattering becomes sufficiently strong, the LMC undergoes a sign reversal. This mechanism originates from the competition between anomaly-driven charge pumping and relaxation processes. In the present work, an additional ingredient enters the problem to enrich the dynamics of the electrons: the emergent magnetic field $\mathbf{B}_{\text{emer}}$, which represents the coarse-grained real-space Berry curvature generated by a skyrmion texture. Although $\mathbf{B}_{\text{emer}}$ enters the semiclassical equations of motion in the same manner as an external magnetic field $\mathbf{B}$ through $\mathbf{B}_{\text{tot}} = \mathbf{B} + \mathbf{B}_{\text{emer}}$, its role is fundamentally distinct. $\mathbf{B}_{\text{emer}}$ encodes real-space topology and couples to the Weyl-node Berry curvature $\Omega_{\mathbf{k}}^x$ via the modified phase-space factor $\mathcal{D}^x$.

To characterize these behaviors systematically, we adopt the general quadratic fitting form of the magnetoconductivity tensor [26]

$$\sigma_{ij}(B) = \sigma_{ij}^{(0)} + (B - B_0)^2 \sigma_{ij}^{(2)}, \quad (25)$$

which provides a unified framework to describe (1) conventional quadratic-in- B magnetoconductivity, (2) shifted parabolic behavior, and (3) complete inversion of the curvature, as schematically illustrated in Fig. 1. Within this parametrization, we define the following three regimes. *Weak sign reversal* is characterized by (1) $B_0 \neq 0$, (2) $\sigma_{ij}^{(0)} \neq \sigma_{ij}(B = 0)$, and (3) $\text{sign}(\sigma_{ij}^{(2)}) > 0$. In this case, the vertex of the parabola in $\sigma_{ij}(B)$ shifts away from the origin, and $\sigma_{ij}(B)$ exhibits opposite signs for either small positive or negative magnetic fields. Importantly, the overall curvature remains positive. In the present context, such behavior can arise from the additional real-space topological contribution induced by a finite B_{emer} , which effectively shifts the balance between anomaly-driven enhancement and scattering-induced suppression. *Strong sign reversal*, on the other hand, corresponds to $\text{sign}(\sigma_{ij}^{(2)}) < 0$, with $B_0 = 0$, indicating a complete inversion of the parabolic curvature. This regime is typically driven by strong intervalley scattering, which overwhelms the anomaly contribution and reverses the overall magnetic-field response. Finally, when both a curvature inversion [$\text{sign}(\sigma_{ij}^{(2)}) < 0$] and a finite shift ($B_0 \neq 0$) occur simultaneously, the system exhibits a

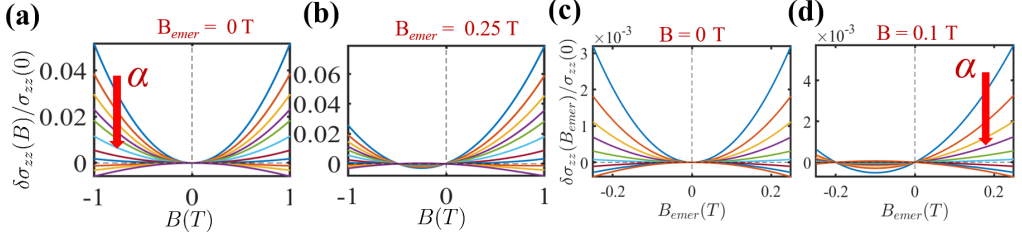


FIG. 2. Normalized LMC for a minimal model of a time-reversal symmetry-broken, untitled WSM in the presence of an emergent magnetic field $\mathbf{B}_{\text{emer}}$ arising from real-space topology. The angle of $\mathbf{B}_{\text{emer}}$ and an external magnetic field $\mathbf{B}$ is set to be $\gamma_{\text{emer}} = \gamma = \pi/2$. In all the panels, α varies from 0.20 to 1.50 along the red arrow. (a) $B_{\text{emer}} = 0$ T and changing B . At $\alpha_c \simeq 0.80$, the anomaly-driven sign change of LMC occurs [34]. (b) $B_{\text{emer}} = 0.25$ T and changing B . For $\alpha < \alpha_c$ ($\alpha > \alpha_c$), weak sign-reversal (strong-and-weak sign-reversal) behaviors emerge. (c) $B = 0$ T and changing B_{emer} . The sign change occurs at α_c . (d) $B = 0.1$ T and changing B_{emer} .

strong-and-weak sign-reversal regime. In the mixed-topology scenario considered here, this regime naturally emerges from the cooperative interplay between momentum-space Berry curvature (Weyl monopoles) and real-space topology (skyrmion-induced emergent field). We are now in the position to discuss the magnetoconductivity in the presence of real- and momentum-space topology.

A. LMC in the presence of real- and momentum-space topology

We begin by recovering the well-established behavior of the LMC in the absence of real-space topology ($B_{\text{emer}} = 0$). Figure 2(a) shows the normalized LMC defined as

$$\frac{\delta\sigma_{zz}(B)}{\sigma_{zz}(0)} = \frac{\sigma_{zz}(B) - \sigma_{zz}(0)}{\sigma_{zz}(0)} \quad (26)$$

for the angle $\gamma = \pi/2$. Increasing the intervalley scattering strength α drives a sign reversal of the LMC across the critical value α_c . For $\alpha < \alpha_c$, the anomaly-induced positive LMC dominates, whereas for $\alpha > \alpha_c$, strong intervalley relaxation overwhelms the chiral pumping mechanism and inverts the curvature of the magnetoconductivity [25,26,31,33–36,46,88,89].

We now incorporate real-space topology through a finite emergent magnetic field $\mathbf{B}_{\text{emer}}$ with $\gamma_{\text{emer}} = \pi/2$, representing the coarse-grained Berry curvature generated by a skyrmion texture. The corresponding LMC is shown in Fig. 2(b). Even an infinitesimal B_{emer} qualitatively enriches the magnetic-field dependence of the LMC: The vertex of the parabola shifts away from the origin while its overall curvature remains unchanged for fixed α . According to the classification discussed earlier, this behavior corresponds to weak sign reversal. The shift originates from terms linear in B_{emer} that arise through the modified phase-space factor $\mathcal{D}_{\mathbf{k}}^X$. Physically, the emergent field modifies the effective magnetic environment experienced by Weyl quasiparticles without altering the intrinsic anomaly mechanism.

Importantly, for a fixed value of B_{emer} , increasing α still leads to inversion of the parabola, i.e., a transition to strong sign reversal, as shown in Fig. 2(b). Thus, intervalley scattering controls the curvature of the LMC, while the emergent field controls the shift of its extremum. The coexistence of these two effects naturally gives rise to the strong-and-weak sign-reversal regime. To gain further insight, we plot the normalized LMC for two fixed values of α (one below and one above α_c) at different values of B_{emer} in Fig. 3. For $\alpha < \alpha_c$ [Fig. 3(a)], increasing B_{emer} progressively shifts the parabola, producing weak sign reversal without changing its positive curvature. In contrast, for $\alpha > \alpha_c$ [Fig. 3(b)], the curvature is already inverted; the addition of B_{emer} introduces an additional shift, leading to the combined strong-and-weak sign-reversal regime. These results demonstrate that real-space topology and intervalley scattering affect distinct geometrical features of the LMC curve.

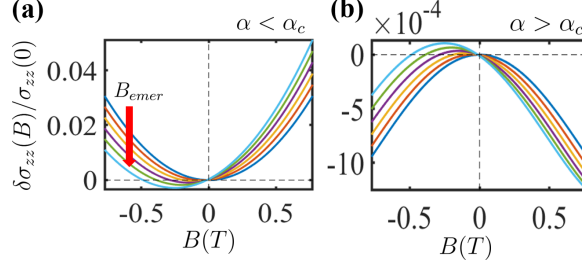


FIG. 3. Normalized LMC as a function of the external magnetic field $\mathbf{B}$, with $\gamma = \pi/2$ for a minimal model of an untilted time-reversal symmetry-broken WSM. (a) Weak sign-reversal regime ($\alpha = 0.2 < \alpha_c \simeq 0.80$) and (b) strong-and-weak sign-reversal regime ($\alpha = 0.90 > \alpha_c$). The progression from the black to the green curves (indicated by the red arrow) corresponds to increasing the emergent magnetic field B_{emer} from 0 to 0.25 T, with $\gamma_{\text{emer}} = \pi/2$.

We next analyze the LMC as a function of B_{emer} itself, both in the absence and presence of an external magnetic field, as shown in Figs. 2(c) and 2(d), respectively. In the absence of an external field, the LMC as a function of B_{emer} exhibits a strong sign reversal, reflecting the dominant role of the emergent field in controlling the curvature. When an external magnetic field is present, the interplay between $\mathbf{B}$ and $\mathbf{B}_{\text{emer}}$ produces weak or strong-and-weak sign reversal depending on the value of α . This confirms that B_{emer} acts as an independent topological tuning parameter rather than merely renormalizing the external magnetic field.

It is important to emphasize that although B_{emer} enters the semiclassical equations of motion through the linear combination $\mathbf{B}_{\text{tot}} = \mathbf{B} + \mathbf{B}_{\text{emer}}$ and has the same sign at both Weyl nodes, its transport effect is not a trivial field renormalization. The key point is that B_{emer} couples to the momentum-space Berry curvature $\Omega_{\mathbf{k}}^x$, which is opposite for opposite chiralities. Consequently, the mixed term $\mathbf{B}_{\text{emer}} \cdot \Omega_{\mathbf{k}}^x$ generates chirality-dependent corrections to the phase-space factor and generalized velocity, thereby modifying the anomaly-driven LMC in a genuinely topological manner. This mixed real- and momentum-space coupling constitutes a central result of the present work.

Finally, we investigate the angular γ dependence of the normalized LMC in the presence of B_{emer} at $B = 0.5$ T, shown in Fig. 4(a). We observe that the emergent field introduces a pronounced asymmetry in the angular response around $\gamma = \pi$. This asymmetry originates from the fact that B_{emer} has the same sign at both Weyl nodes and therefore breaks the effective inversion symmetry of the magnetotransport response. The resulting angular dependence provides an experimentally accessible signature of the mixed real-space and momentum-space topology. To deepen our understanding of B_{emer} in LMC, we have plotted it as a function of γ_{emer} at different values of B_{emer} in Fig. 4(c) at $B = 0$. The angular anisotropy is clearly visible in this plot. Although the angular dependence remains intact for all values of B_{emer} , the magnitude increases with increasing B_{emer} and the positive portion increases more rapidly than negative one. As discussed earlier, due to the same sign of B_{emer} for both nodes, the Fermi surface distorted along the direction of B_{emer} , and this leads to an anisotropic response and shows a maximum LMC for $\mathbf{B}_{\text{emer}} \parallel \hat{z}$.

B. PHC in the presence of real- and momentum-space topology

Having discussed the LMC, we now turn to PHC. The PHC exhibits a characteristic angular dependence proportional to $\sin(2\gamma)$, where γ is the angle of the applied magnetic field measured from the x axis [47]. In Fig. 4(b), we plot this angular dependence for different values of B_{emer} . In the presence of B_{emer} , the angular profile becomes asymmetric about $\gamma = \pi$. Although the functional dependence remains similar for $\gamma \in (\pi, 2\pi)$, the magnitude is reduced. This asymmetry arises from the fixed orientation of B_{emer} along the z direction.

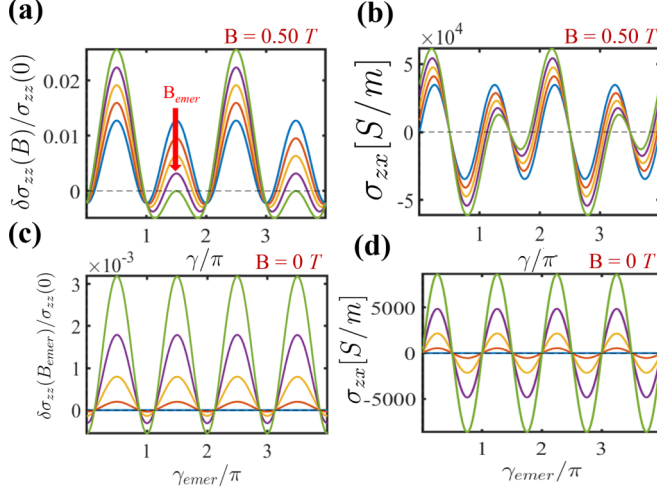


FIG. 4. Normalized LMC and PHC for a minimal model of untitled WSM as a function of γ and γ_{emer} . (a) Normalized LMC at $B = 0.5$ T. (b) PHC at $B = 0.5$ T. (c) Normalized LMC for given B_{emer} at $B = 0$ T. (d) PHC for given B_{emer} at $B = 0$ T. In panels (a) and (b), from black to green lines, B_{emer} is varied from 0 to 0.25 T, with $\gamma_{\text{emer}} = \pi/2$. In panels (c) and (d), $\gamma = \pi/2$.

To isolate the role of real-space topology, we first evaluate the PHC in the absence of B_{emer} . Figure 5(a) shows the planar Hall conductivity σ_{zx} for several values of the intervalley scattering strength α . In this case, the response is governed purely by momentum-space topology. Increasing α suppresses the magnitude of the PHC; however, unlike the LMC, no sign reversal is observed. When B_{emer} is switched on [Fig. 5(b)], the minima of the PHC parabola shift, but show weak sign reversal. Fixing α and varying B_{emer} leads to a qualitatively different behavior, shown in Fig. 6(a). In this case, the presence of real-space topology drives the system into a weak sign-reversal regime. In Fig. 6(b), we plot the PHC as a function of B_{emer} for different values of the external magnetic field. For $B = 0$, the PHC seems to vanish, indicating that an emergent field directed along z alone does not generate a planar Hall response of the same magnitude as the external field does. Increasing the external magnetic field enhances the magnitude of the PHC within the studied parameter range.

To gain further insight, we consider the PHC in the presence of B_{emer} alone while varying its direction within the xz plane, as shown in Fig. 4(d), which shows that the PHC has very less magnitude, but is still finite. It also confirms that B_{emer} can alone give a finite PHC. It is clear that $\sigma_{zx} \propto \sin 2\gamma_{\text{emer}}$, which is intact for all finite values of B_{emer} taken. This has never been explored before!

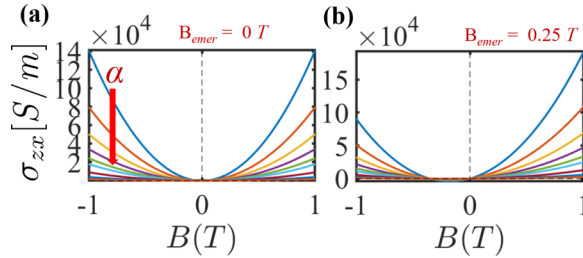


FIG. 5. PHC as a function of external magnetic field B for a minimal model of an untitled time-reversal symmetry-broken WSM. (a) The emergent magnetic field $B_{\text{emer}} = 0$ T. (b) $B_{\text{emer}} = 0.25$ T. The angle $\gamma = \pi/4$ and $\gamma_{\text{emer}} = \pi/2$. In panels (a) and (b), α is changed from 0.20 to 1.5 from the top to bottom lines.

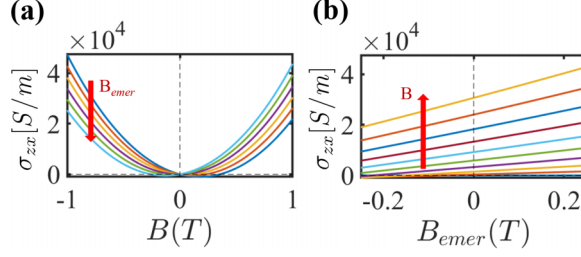


FIG. 6. PHC for a minimal model of an untilted time-reversal symmetry-broken WSM. (a) PHC as a function of the external magnetic field B . Along the red arrow, the emergent magnetic field B_{emer} changes from -0.25 to 0.25 T. (b) PHC as a function of B_{emer} . Along the red arrow, B changes from 0 to 1 T. In both, $\gamma = \pi/4$, $\gamma_{\text{emer}} = \pi/2$, and $\alpha < \alpha_c$.

V. CONCLUSIONS

The present work extends beyond RTA WSM transport theory to the regime where emergent electrodynamics generated by real-space magnetic textures coexists with momentum-space topological transport. We have investigated magnetotransport in a time-reversal symmetry-broken, untilted WSM in the simultaneous presence of momentum-space Berry curvature and real-space topology encoded through an emergent magnetic field $\mathbf{B}_{\text{emer}}$. Within a semiclassical Boltzmann framework, we systematically analyzed the LMC and PHC in this mixed-topology setting. In the absence of real-space topology, we recover the known behavior: The LMC undergoes a strong sign reversal as intervalley scattering increases beyond a critical value, reflecting the competition between anomaly-driven charge pumping and relaxation. Upon introducing a finite B_{emer} , the magnetoconductivity acquires an additional shift of its parabolic profile, corresponding to a weak sign-reversal regime. While intervalley scattering controls the curvature of the response, the emergent field controls the shift of its extremum. Their combined action naturally generates a strong-and-weak sign-reversal regime, demonstrating that real- and momentum-space topology affect distinct geometrical features of the magnetoconductivity.

We further showed that the emergent field induces a pronounced asymmetry in the angular dependence of both longitudinal and planar Hall conductivities. For the planar Hall response, $\mathbf{B}_{\text{emer}}$ modifies the magnitude and can drive weak sign-reversal behavior, while preserving the characteristic $\sin(2\gamma)$ dependence. Notably, a finite PHC can arise solely from $\mathbf{B}_{\text{emer}}$ when its direction is varied, confirming that real-space topology alone can generate a measurable transverse response. Our results establish that $\mathbf{B}_{\text{emer}}$ does not merely renormalize the external magnetic field through $\mathbf{B}_{\text{tot}} = \mathbf{B} + \mathbf{B}_{\text{emer}}$. Instead, through its coupling to momentum-space Berry curvature, it acts as an independent topological tuning parameter. The interplay between Weyl monopoles and skyrmion-induced emergent fields provides a direct transport signature of mixed real- and momentum-space topology. Our results describe the leading semiclassical transport consequences arising from the coexistence of real-space magnetic topology and momentum-space Berry curvature, while higher-order mixed phase-space curvature effects remain an interesting direction for future work [17,23,30]. Our results further suggest that transport signatures correlated with skyrmion chirality and texture evolution may provide a route to experimentally distinguish emergent-field-driven responses from conventional magnetic-field effects.

Our results suggest clear experimental signatures that can be probed using angle-dependent magnetotransport measurements in magnetic Weyl semimetals hosting nontrivial spin textures. By rotating the external magnetic field within the transport plane, one can measure the LMC and PHC as functions of both the field magnitude and orientation. The interplay between Berry curvature in momentum space and the emergent magnetic field generated by real-space topology is predicted to produce two distinct features: (1) a finite shift of the magnetoconductivity parabola away from $B = 0$ and (2) an angular asymmetry of the magnetotransport response about $\gamma = \pi$.

Since the emergent field is directly related to the density of magnetic textures, these transport signatures should evolve with the skyrmion density, which can be independently characterized using magnetic imaging techniques such as Lorentz transmission electron microscopy or magnetic force microscopy. Observation of these features would provide direct experimental evidence of mixed real- and momentum-space topology in Weyl systems.

ACKNOWLEDGMENTS

The authors are grateful to Amit Agarwal, Gautham Varma K, Tanay Nag, and Gargee Sharma for their insightful discussions and valuable scientific input. A.A. acknowledges funding from the Core Research Grant by Anusandhan National Research Foundation (ANRF) (Sanction No. CRG/2023/007003), Department of Science and Technology, India. T.T. acknowledges financial support by the Japan Society for the Promotion of Science, KAKENHI (Grant No. 25H01248) from the Ministry of Education, Culture, Sports, Science, and Technology, Japan. The authors are very grateful to IIT Mandi for providing computing facilities.

DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX A: REAL- AND MOMENTUM-SPACE MIXED TERM IN THE CURRENT

We evaluate the current solely from the nonequilibrium correction, which is first order in E :

$$\mathbf{J} = -e \sum_{\chi, \mathbf{k}} \mathbf{v}_0^\chi g_{\mathbf{k}}^{1, \chi}. \quad (\text{A1})$$

Using $\mathbf{v}_0^\chi = \mathcal{D}^\chi[\mathbf{v}_\mathbf{k}^\chi + \frac{e}{\hbar}(\mathbf{v}_\mathbf{k}^\chi \cdot \boldsymbol{\Omega}^\chi)\mathbf{B}_{\text{tot}}]$, the weak-field expansion gives $\mathbf{v}_0^\chi \simeq \mathbf{v}_\mathbf{k}^\chi + \frac{e}{\hbar}(\mathbf{v}_\mathbf{k}^\chi \cdot \boldsymbol{\Omega}^\chi)\mathbf{B}_{\text{tot}} - \frac{e}{\hbar}(\mathbf{B}_{\text{tot}} \cdot \boldsymbol{\Omega}^\chi)\mathbf{v}_\mathbf{k}^\chi$. In constant relaxation-time approximation, $g_{\mathbf{k}}^{1, \chi} = -e\tau\mathbf{E} \cdot \mathbf{v}_\mathbf{k}^\chi (-\frac{\partial f_0}{\partial \epsilon})$. Keeping only terms linear in $\mathbf{B}_{\text{emer}}$, we obtain

$$\mathbf{J}_{\text{cross}} = \frac{e^3 \tau}{\hbar} \sum_{\chi, \mathbf{k}} \left(-\frac{\partial f_0}{\partial \epsilon} \right) (\mathbf{E} \cdot \mathbf{v}_\mathbf{k}^\chi) [(\mathbf{v}_\mathbf{k}^\chi \cdot \boldsymbol{\Omega}^\chi)\mathbf{B}_{\text{emer}} - (\mathbf{B}_{\text{emer}} \cdot \boldsymbol{\Omega}^\chi)\mathbf{v}_\mathbf{k}^\chi]. \quad (\text{A2})$$

For longitudinal transport, $\mathbf{E} = E_z \hat{z}$ and $\mathbf{B}_{\text{emer}} = B_{\text{emer}} \hat{z}$. Then,

$$J_z^{\text{cross}} = \frac{e^3 \tau E_z B_{\text{emer}}}{\hbar} \sum_{\chi, \mathbf{k}} \left(-\frac{\partial f_0}{\partial \epsilon} \right) v_{k,z}^\chi [(\mathbf{v}_\mathbf{k}^\chi \cdot \boldsymbol{\Omega}^\chi) - \Omega_z^\chi v_{k,z}^\chi]. \quad (\text{A3})$$

Thus, the longitudinal conductivity acquires a term

$$\sigma_{zz}^{\text{cross}} \propto B B_{\text{emer}}, \quad (\text{A4})$$

which is linear in B_{emer} and linear in B (through the standard anomaly correction inside g^1 when solved self-consistently). Including the usual anomaly contribution $\propto B^2$, the conductivity takes the general form

$$\sigma_{zz}(B) = \sigma_{zz}^{(0)} + a_1 B^2 + b_1 B B_{\text{emer}}, \quad (\text{A5})$$

where a_1 originates from the usual chiral anomaly and b_1 arises from Eq. (A2). Completing the square,

$$\sigma_{zz}(B) = \sigma_{zz}^{(0)} + a_1 \left(B + \frac{b_1}{2a_1} B_{\text{emer}} \right)^2 - \frac{b_1^2}{4a_1} B_{\text{emer}}^2. \quad (\text{A6})$$

Hence, the extremum shifts to $B_0 = -\frac{b_1}{2a_1}B_{\text{emer}}$. For $\mathbf{B} = B(\cos \gamma \hat{x} + \sin \gamma \hat{z})$ and $\mathbf{B}_{\text{emer}} = B_{\text{emer}}\hat{z}$, the mixed term in Eq. (A2) produces contributions proportional to $BB_{\text{emer}} \sin \gamma$. Although this term is formally invariant under $\gamma \rightarrow \gamma + \pi$ when the reversal of the magnetic-field direction is taken into account, the presence of a fixed $\mathbf{B}_{\text{emer}}$ breaks the effective $\gamma \rightarrow \gamma + \pi$ symmetry of the transport response. As a result, $\sigma(\gamma) \neq \sigma(\gamma + \pi)$, leading to an angular asymmetry originating from the real- and momentum-space cross-coupling. Physically, since B_{emer} has the same sign at both Weyl nodes while Ω^χ changes sign with chirality, the mixed contribution remains chirality dependent and breaks the effective inversion symmetry of the angular magnetotransport response.

APPENDIX B: PHASE-SPACE BERRY CURVATURE ANALYSIS OF MIXED REAL- AND MOMENTUM-SPACE TOPOLOGY

In this Appendix, we study the topologies by formulating the problem within the full phase-space Berry curvature within the full phase-space Berry-curvature framework. We show that the leading-order magnetotransport response is additive in the external magnetic field $\mathbf{B}$ and the skyrmion-induced emergent magnetic field $\mathbf{B}_{\text{emer}}$, while the genuine mixed phase-space Berry-curvature contribution is parametrically suppressed in the semiclassical regime.

For a Bloch electron moving in a slowly varying magnetic texture, the natural semiclassical coordinates are $\xi^\alpha = (r_x, r_y, r_z, k_x, k_y, k_z)$, and the dynamics is governed by the full phase-space Berry-curvature tensor

$$\Omega_{\alpha\beta} = \begin{pmatrix} \Omega_{rr} & \Omega_{rk} \\ \Omega_{kr} & \Omega_{kk} \end{pmatrix}. \quad (\text{B1})$$

Here, Ω_{kk} denotes the momentum-space Berry curvature associated with the Weyl node, $\Omega_{kk}^{ij} = \epsilon^{ijl} \Omega_l^\chi(\mathbf{k})$, with $\Omega^\chi(\mathbf{k}) = -\chi \frac{\mathbf{k}}{2k^3}$, where $\chi = \pm 1$ labels the chirality. The real-space Berry curvature generated by a slowly varying skyrmion texture is

$$\Omega_{rr}^{ij} = \frac{1}{2} \hat{\mathbf{n}} \cdot (\partial_{r_i} \hat{\mathbf{n}} \times \partial_{r_j} \hat{\mathbf{n}}), \quad (\text{B2})$$

which may be written in terms of the emergent magnetic field: $B_{\text{emer},l} = \frac{\hbar}{e} \frac{1}{2} \epsilon_{ijl} \Omega_{rr}^{ij}$. Finally, the mixed phase-space Berry curvature is

$$\Omega_{kr}^{ij} = \frac{1}{2} \hat{\mathbf{d}} \cdot (\partial_{k_i} \hat{\mathbf{d}} \times \partial_{r_j} \hat{\mathbf{d}}), \quad (\text{B3})$$

where $\mathbf{d}(\mathbf{r}, \mathbf{k})$ denotes the vector entering the local two-band Hamiltonian. For a slowly varying magnetic texture, momentum-space and real-space derivatives scale as $\partial_k \sim k_F^{-1}$ and $\partial_r \sim L_s^{-1}$, where k_F is the Fermi wave vector and L_s is the characteristic skyrmion length scale. Equation (B3) therefore implies

$$\Omega_{kr} \sim \frac{1}{k_F L_s}. \quad (\text{B4})$$

The semiclassical regime requires $k_F^{-1} \gg L_s$, and hence

$$\Omega_{kr} \ll 1. \quad (\text{B5})$$

Equation (B5) establishes that the mixed phase-space Berry curvature is parametrically small compared with the leading real-space and momentum-space Berry curvature contributions. A detailed analysis of the different regimes can be found in the literature [17,23,30].

Phase-space measure. The invariant phase-space volume element is $dV = D(\xi) d^3r d^3k$, with $D(\xi) = \sqrt{\det \Omega}$. Expanding to the leading order in Berry curvatures gives

$$D(\xi) = 1 + \frac{e}{\hbar} \mathbf{B} \cdot \Omega^\chi + \frac{e}{\hbar} \mathbf{B}_{\text{emer}} \cdot \Omega^\chi + \text{Tr}[\Omega_{kr}] + \mathcal{O}(\Omega^2). \quad (\text{B6})$$

The first correction arises from the ordinary magnetic field, the second from the real-space Berry curvature through the emergent field, and the third represents the genuine mixed phase-space contribution. Using Eq. (B4), the latter scales as $\text{Tr}[\Omega_{kr}] \sim \frac{1}{k_F L_s}$, and is therefore subleading in the semiclassical limit.

Conductivity. The longitudinal conductivity can be written schematically as

$$\sigma_{ij} \sim \int [d\mathbf{k}] D(\xi) v_i \Lambda_j, \quad (\text{B7})$$

where Λ_j denotes the solution of the Boltzmann equation. Substituting Eq. (B6) into Eq. (B7) yields

$$\sigma_{ij} = \sigma_{ij}^{(0)} + \sigma_{ij}^{(B)} + \sigma_{ij}^{(\text{emer})} + \delta\sigma_{ij}^{(kr)} + \dots, \quad (\text{B8})$$

with $\sigma_{ij}^{(B)} \propto \mathbf{B} \cdot \boldsymbol{\Omega}^\chi$, $\sigma_{ij}^{(\text{emer})} \propto \mathbf{B}_{\text{emer}} \cdot \boldsymbol{\Omega}^\chi$, and $\delta\sigma_{ij}^{(kr)} \propto \Omega_{kr}$. Using Eq. (B4), one finds $\delta\sigma_{ij}^{(kr)} = \mathcal{O}(\frac{1}{k_F L_s})$, which is parametrically smaller than the leading contributions. Therefore, to leading order,

$$\sigma_{ij} = \sigma_{ij}^{(0)} + \sigma_{ij}^{(B)} + \sigma_{ij}^{(\text{emer})} \quad (\text{B9})$$

and the conductivity is additive in the external and emergent magnetic fields.

Application to anomaly-induced longitudinal magnetoconductivity. For a Weyl semimetal, the anomaly-induced conductivity depends on the total effective magnetic field $\mathbf{B}_{\text{tot}} = \mathbf{B} + \mathbf{B}_{\text{emer}}$. The leading longitudinal conductivity therefore, takes the form

$$\sigma_{zz} = \sigma_0 + C(\mathbf{B} + \mathbf{B}_{\text{emer}})^2, \quad (\text{B10})$$

where C is determined by the band structure and intervalley scattering. Expanding Eq. (B10) gives

$$\sigma_{zz} = \sigma_0 + CB^2 + 2C\mathbf{B} \cdot \mathbf{B}_{\text{emer}} + CB_{\text{emer}}^2. \quad (\text{B11})$$

The cross term proportional to $\mathbf{B} \cdot \mathbf{B}_{\text{emer}}$ arises solely from the square of the total effective magnetic field and survives even when $\Omega_{kr} = 0$. Consequently, the term proportional to $\mathbf{B} \cdot \mathbf{B}_{\text{emer}}$ does not represent a genuine mixed phase-space Berry-curvature effect. The true mixed contribution is instead contained in $\delta\sigma_{ij}^{(kr)}$, which is suppressed by the small parameter $(k_F L_s)^{-1}$. We therefore conclude that within the semiclassical regime $k_F^{-1} \ll L_s$, the leading magnetotransport response is governed by the additive contributions of momentum-space topology and real-space topology, while genuine mixed phase-space Berry-curvature effects enter only as higher-order corrections.

-
- [1] D. K. Le, E. Thareja, B. Konushbaev, G. Pantano, T. Saunderson, M.-H. Phan, Y. Mokrousov, and J. Gayles, Ultra-stable Weyl topology driven by magnetic textures in the shandite compound $\text{Co}_3\text{Sn}_2\text{S}_{(2-x)}\text{Se}_x$, [arXiv:2601.09922](#).
 - [2] L. Dominici, A. Rahmani, D. Colas, D. Ballarini, M. De Giorgi, G. Gigli, D. Sanvitto, F. P. Laussy, and N. Voronova, Coupled quantum vortex kinematics and Berry curvature in real space, [Commun. Phys. 6, 197 \(2023\)](#).
 - [3] B. He, M. Yao, Y. Pan, K. E. Arpino, D. Chen, F. M. Serrano-Sanchez, S. Ju, M. Shi, Y. Sun, and C. Felser, Enhanced Weyl semimetal signature in $\text{Co}_3\text{Sn}_2\text{S}_2$ kagome ferromagnet by chlorine doping, [Commun. Mater. 5, 275 \(2024\)](#).
 - [4] M. Raju, R. Romero III, D. Nishio-Hamane, R. Uesugi, M. Asakura, Z. Tagay, T. Higo, N. P. Armitage, C. Broholm, and S. Nakatsuji, Anisotropic anomalous transport in the kagome-based topological antiferromagnetic Mn_3Ga epitaxial thin films, [Phys. Rev. Mater. 8, 014204 \(2024\)](#).
 - [5] X. Wan, A. M. Turner, A. Vishwanath, and S. Y. Savrasov, Topological semimetal and Fermi-arc surface states in the electronic structure of pyrochlore iridates, [Phys. Rev. B 83, 205101 \(2011\)](#).
 - [6] N. P. Armitage, E. J. Mele, and A. Vishwanath, Weyl and Dirac semimetals in three-dimensional solids, [Rev. Mod. Phys. 90, 015001 \(2018\)](#).

- [7] D. T. Son and N. Yamamoto, Berry curvature, triangle anomalies, and the chiral magnetic effect in Fermi liquids, *Phys. Rev. Lett.* **109**, 181602 (2012).
- [8] A. A. Burkov, Anomalous Hall effect in Weyl metals, *Phys. Rev. Lett.* **113**, 187202 (2014).
- [9] T. Bevan, A. Manninen, J. Cook, J. Hook, H. Hall, T. Vachaspati, and G. Volovik, Momentum creation by vortices in superfluid ^3He as a model of primordial baryogenesis, *Nature (London)* **386**, 689 (1997).
- [10] G. Volovik, On induced *CPT*-odd Chern-Simons terms in the $3 + 1$ effective action, *J. Exp. Theor. Phys. Lett.* **70**, 1 (1999).
- [11] G. E. Volovik, *The Universe in a Helium Droplet* (Oxford University Press, Oxford, 2003), Vol. 117.
- [12] K. S. Novoselov, E. McCann, S. Morozov, V. I. Falko, M. Katsnelson, U. Zeitler, D. Jiang, F. Schedin, and A. Geim, Unconventional quantum Hall effect and Berry's phase of 2π in bilayer graphene, *Nat. Phys.* **2**, 177 (2006).
- [13] M. V. Berry, Quantal phase factors accompanying adiabatic changes, *Proc. R. Soc. London A* **392**, 45 (1984).
- [14] G. Sundaram and Q. Niu, Wave-packet dynamics in slowly perturbed crystals: Gradient corrections and Berry-phase effects, *Phys. Rev. B* **59**, 14915 (1999).
- [15] D. Xiao, M.-C. Chang, and Q. Niu, Berry phase effects on electronic properties, *Rev. Mod. Phys.* **82**, 1959 (2010).
- [16] N. Nagaosa and Y. Tokura, Topological properties and dynamics of magnetic skyrmions, *Nat. Nanotechnol.* **8**, 899 (2013).
- [17] N. Verma, Z. Addison, and M. Randeria, Unified theory of the anomalous and topological Hall effects with phase-space Berry curvatures, *Sci. Adv.* **8**, eabq2765 (2022).
- [18] P. Goswami and S. Tewari, Axionic field theory of $(3 + 1)$ -dimensional Weyl semimetals, *Phys. Rev. B* **88**, 245107 (2013).
- [19] M. Hirschberger, T. Nakajima, S. Gao, L. Peng, A. Kikkawa, T. Kurumaji, M. Kriener, Y. Yamasaki, H. Sagayama, H. Nakao, *et al.*, Skyrmion phase and competing magnetic orders on a breathing kagomé lattice, *Nat. Commun.* **10**, 5831 (2019).
- [20] M. Hirschberger, B. G. Szigeti, M. Hemmida, M. M. Hirschmann, S. Esser, H. Ohsumi, Y. Tanaka, L. Spitz, S. Gao, K. K. Kolincio, *et al.*, Lattice-commensurate skyrmion texture in a centrosymmetric breathing kagome magnet, *npj Quantum Mater.* **9**, 45 (2024).
- [21] Q. Du, M.-G. Han, Y. Liu, W. Ren, Y. Zhu, and C. Petrovic, Room-temperature skyrmion thermopower in Fe_3Sn_2 , *Adv. Quantum Technol.* **3**, 2000058 (2020).
- [22] P. Vir, J. Gayles, A. S. Sukhanov, N. Kumar, F. Damay, Y. Sun, J. Kübler, C. Shekhar, and C. Felser, Anisotropic topological Hall effect with real and momentum space Berry curvature in the antiskyrmion-hosting Heusler compound $\text{Mn}_{1.4}\text{PtSn}$, *Phys. Rev. B* **99**, 140406(R) (2019).
- [23] Z. Addison, L. Keyes, and M. Randeria, Anomalous and topological Hall effects with phase-space Berry curvatures: Electric, thermal, and thermoelectric transport in magnets, *Phys. Rev. B* **112**, 014446 (2025).
- [24] B. Göbel, L. Schimpf, and I. Mertig, Topological orbital Hall effect caused by skyrmions and antiferromagnetic skyrmions, *Commun. Phys.* **8**, 17 (2025).
- [25] A. Ahmad and G. Sharma, Longitudinal magnetoconductance and the planar Hall effect in a lattice model of tilted Weyl fermions, *Phys. Rev. B* **103**, 115146 (2021).
- [26] A. Ahmad, K. V. Raman, S. Tewari, and G. Sharma, Longitudinal magnetoconductance and the planar Hall conductance in inhomogeneous Weyl semimetals, *Phys. Rev. B* **107**, 144206 (2023).
- [27] C.-X. Liu, P. Ye, and X.-L. Qi, Chiral gauge field and axial anomaly in a Weyl semimetal, *Phys. Rev. B* **87**, 235306 (2013).
- [28] A. G. Grushin, Consequences of a condensed matter realization of Lorentz-violating QED in Weyl semimetals, *Phys. Rev. D* **86**, 045001 (2012).
- [29] A. A. Zyuzin and A. A. Burkov, Topological response in Weyl semimetals and the chiral anomaly, *Phys. Rev. B* **86**, 115133 (2012).
- [30] Z. Addison, L. Keyes, and M. Randeria, Theory of topological Nernst and thermoelectric transport in chiral magnets, *Phys. Rev. B* **108**, 014419 (2023).
- [31] A. Knoll, C. Timm, and T. Meng, Negative longitudinal magnetoconductance at weak fields in Weyl semimetals, *Phys. Rev. B* **101**, 201402(R) (2020).

- [32] G. Sharma, P. Goswami, and S. Tewari, Chiral anomaly and longitudinal magnetotransport in type-II Weyl semimetals, *Phys. Rev. B* **96**, 045112 (2017).
- [33] G. Sharma, S. Nandy, and S. Tewari, Sign of longitudinal magnetoconductivity and the planar Hall effect in Weyl semimetals, *Phys. Rev. B* **102**, 205107 (2020).
- [34] A. Ahmad and G. Sharma, Longitudinal magnetoconductance of higher-pseudospin fermions, *Phys. Rev. B* **112**, 045135 (2025).
- [35] A. Ahmad, G. Varma K, and G. Sharma, Geometry, anomaly, topology, and transport in Weyl fermions, *J. Phys.: Condens. Matter* **37**, 043001 (2025).
- [36] A. Ahmad, G. Varma K., and G. Sharma, Chiral anomaly induced nonlinear Hall effect in three-dimensional chiral fermions, *Phys. Rev. B* **111**, 035138 (2025).
- [37] I. Mandal, Chiral anomaly and internode scatterings in multifold semimetals, *Phys. Rev. B* **111**, 165116 (2025).
- [38] I. Mandal, Disentangling contributions to longitudinal magnetoconductivity for Kramers-Weyl nodes, *Sci. Rep.* **15**, 37542 (2025).
- [39] R. Ghosh, F. Haidar, and I. Mandal, Linear response in planar Hall and thermal Hall setups for Rarita-Schwinger-Weyl semimetals, *Phys. Rev. B* **110**, 245113 (2024).
- [40] I. Mandal, Longitudinal magnetoconductivity in multifold semimetals exemplified by pseudospin-1 nodal points, *Eur. Phys. J. Plus* **141**, 75 (2026).
- [41] I. Mandal, Distinguishing features of longitudinal magnetoconductivity for a Rarita-Schwinger-Weyl node, *J. Low Temp Phys.* **222**, 120 (2026).
- [42] I. Mandal, Unconventional magnetoelectric conductivity and electrochemical response from dipole-like sources of Berry curvature, [arXiv:2602.08844](https://arxiv.org/abs/2602.08844).
- [43] B. Yan and C. Felser, Topological materials: Weyl semimetals, *Annu. Rev. Condens. Matter Phys.* **8**, 337 (2017).
- [44] B. Q. Lv, H. M. Weng, B. B. Fu, X. P. Wang, H. Miao, J. Ma, P. Richard, X. C. Huang, L. Zhao, G. F. Chen, *et al.*, Experimental discovery of Weyl semimetal TaAs, *Phys. Rev. X* **5**, 031013 (2015).
- [45] M. Z. Hasan, G. Chang, I. Belopolski, G. Bian, S.-Y. Xu, and J.-X. Yin, Weyl, Dirac and high-fold chiral fermions in topological quantum matter, *Nat. Rev. Mater.* **6**, 784 (2021).
- [46] G. Varma K., M. H. Raza, and A. Ahmad, Chiral anomaly-induced nonlinear Hall effect in spin-orbit coupled noncentrosymmetric metals, *Phys. Rev. B* **113**, 035112 (2026).
- [47] S. Nandy, G. Sharma, A. Taraphder, and S. Tewari, Chiral anomaly as the origin of the planar Hall effect in Weyl semimetals, *Phys. Rev. Lett.* **119**, 176804 (2017).
- [48] S. Nandy, C. Zeng, and S. Tewari, Chiral anomaly induced nonlinear Hall effect in semimetals with multiple Weyl points, *Phys. Rev. B* **104**, 205124 (2021).
- [49] S.-B. Zhang, H.-Z. Lu, and S.-Q. Shen, Linear magnetoconductivity in an intrinsic topological Weyl semimetal, *New J. Phys.* **18**, 053039 (2016).
- [50] S. A. Yang, H. Pan, and F. Zhang, Chirality-dependent Hall effect in Weyl semimetals, *Phys. Rev. Lett.* **115**, 156603 (2015).
- [51] K. Das and A. Agarwal, Berry curvature induced thermopower in type-I and type-II Weyl semimetals, *Phys. Rev. B* **100**, 085406 (2019).
- [52] K. Das, S. K. Singh, and A. Agarwal, Chiral anomalies induced transport in Weyl metals in quantizing magnetic field, *Phys. Rev. Res.* **2**, 033511 (2020).
- [53] S. Das, K. Das, and A. Agarwal, Nonlinear magnetoconductivity in Weyl and multi-Weyl semimetals in quantizing magnetic field, *Phys. Rev. B* **105**, 235408 (2022).
- [54] G. Sharma and S. Tewari, Transverse thermopower in Dirac and Weyl semimetals, *Phys. Rev. B* **100**, 195113 (2019).
- [55] G. Sharma, P. Goswami, and S. Tewari, Nernst and magnetothermal conductivity in a lattice model of Weyl fermions, *Phys. Rev. B* **93**, 035116 (2016).
- [56] C. Zeng, S. Nandy, and S. Tewari, Nonlinear transport in Weyl semimetals induced by Berry curvature dipole, *Phys. Rev. B* **103**, 245119 (2021).
- [57] G. Sharma, C. Moore, S. Saha, and S. Tewari, Nernst effect in Dirac and inversion-asymmetric Weyl semimetals, *Phys. Rev. B* **96**, 195119 (2017).

- [58] S. Nandy, A. Taraphder, and S. Tewari, Planar thermal Hall effect in Weyl semimetals, [Phys. Rev. B **100**, 115139 \(2019\)](#).
- [59] K. Das and A. Agarwal, Thermal and gravitational chiral anomaly induced magneto-transport in Weyl semimetals, [Phys. Rev. Res. **2**, 013088 \(2020\)](#).
- [60] S. Das, K. Das, and A. Agarwal, Chiral anomalies in three-dimensional spin-orbit coupled metals: Electrical, thermal, and gravitational anomalies, [Phys. Rev. B **108**, 045405 \(2023\)](#).
- [61] D. Mandal, K. Das, and A. Agarwal, Chiral anomaly and nonlinear magnetotransport in time reversal symmetric Weyl semimetals, [Phys. Rev. B **106**, 035423 \(2022\)](#).
- [62] I. Mandal and K. Saha, Thermopower in an anisotropic two-dimensional Weyl semimetal, [Phys. Rev. B **101**, 045101 \(2020\)](#).
- [63] I. Mandal and A. Sen, Tunneling of multi-Weyl semimetals through a potential barrier under the influence of magnetic fields, [Phys. Lett. A **399**, 127293 \(2021\)](#).
- [64] T. Schulz, R. Ritz, A. Bauer, M. Halder, M. Wagner, C. Franz, C. Pfleiderer, K. Everschor, M. Garst, and A. Rosch, Emergent electrodynamics of skyrmions in a chiral magnet, [Nat. Phys. **8**, 301 \(2012\)](#).
- [65] Y. Tokura and N. Kanazawa, Magnetic skyrmion materials, [Chem. Rev. **121**, 2857 \(2021\)](#).
- [66] K.-S. Kim, H.-J. Kim, and M. Sasaki, Boltzmann equation approach to anomalous transport in a Weyl metal, [Phys. Rev. B **89**, 195137 \(2014\)](#).
- [67] M. Imran and S. Hershfield, Berry curvature force and Lorentz force comparison in the magnetotransport of Weyl semimetals, [Phys. Rev. B **98**, 205139 \(2018\)](#).
- [68] V. A. Zyuzin, Magnetotransport of Weyl semimetals due to the chiral anomaly, [Phys. Rev. B **95**, 245128 \(2017\)](#).
- [69] K. Das and A. Agarwal, Linear magnetochiral transport in tilted type-I and type-II Weyl semimetals, [Phys. Rev. B **99**, 085405 \(2019\)](#).
- [70] C. Duval, Z. Horváth, P. A. Horvathy, L. Martina, and P. C. Stichel, Comment on “Berry phase correction to electron density of states in solids”, [Phys. Rev. Lett. **96**, 099701 \(2006\)](#).
- [71] H. Bruus and K. Flensberg, *Many-body Quantum Theory in Condensed Matter Physics: An Introduction* (Oxford University Press, Oxford, 2004).
- [72] J. M. Ziman, *Principles of The Theory of Solids* (Cambridge University Press, Cambridge, 1979).
- [73] B. Roy, V. Juričić, and S. Das Sarma, Universal optical conductivity of a disordered Weyl semimetal, [Sci. Rep. **6**, 32446 \(2016\)](#).
- [74] A. Ahmad, P. Bhalla, S. Nandy, and T. Nag, Semiclassical theory of frequency dependent linear magneto-optical transport in Weyl semimetals, [arXiv:2604.11527](#).
- [75] G. Sharma, S. Nandy, K. V. Raman, and S. Tewari, Decoupling intranode and internode scattering in Weyl fermions, [Phys. Rev. B **107**, 115161 \(2023\)](#).
- [76] R.-H. Li, O. G. Heinonen, A. A. Burkov, and S. S.-L. Zhang, Nonlinear Hall effect in Weyl semimetals induced by chiral anomaly, [Phys. Rev. B **103**, 045105 \(2021\)](#).
- [77] C. Zeng, S. Nandy, and S. Tewari, Chiral anomaly induced nonlinear Nernst and thermal Hall effects in Weyl semimetals, [Phys. Rev. B **105**, 125131 \(2022\)](#).
- [78] D. E. Kharzeev, The chiral magnetic effect and anomaly-induced transport, [Prog. Part. Nucl. Phys. **75**, 133 \(2014\)](#).
- [79] P. Kumar, V. Nagpal, Sudesh, and S. Patnaik, Chiral anomaly induced negative magnetoresistance and weak anti-localization in Weyl semimetal $\text{Bi}_{0.97}\text{Sb}_{0.03}$ alloy, [J. Phys.: Condens. Matter **34**, 055601 \(2022\)](#).
- [80] A. Johansson, J. Henk, and I. Mertig, Chiral anomaly in type-I Weyl semimetals: Comprehensive analysis within a semiclassical Fermi surface harmonics approach, [Phys. Rev. B **99**, 075114 \(2019\)](#).
- [81] A. Ahmad, Chiral anomaly and planar Hall conductance in pseudospin-1 fermions, [arXiv:2606.11364](#).
- [82] G. D. Mahan, *Many-Particle Systems* (Princeton University Press, Princeton, 2008).
- [83] E. S. Abers, *Quantum Mechanics* (Pearson Education, 2004).
- [84] T. Morimoto, S. Zhong, J. Orenstein, and J. E. Moore, Semiclassical theory of nonlinear magneto-optical responses with applications to topological Dirac/Weyl semimetals, [Phys. Rev. B **94**, 245121 \(2016\)](#).
- [85] I. Sodemann and L. Fu, Quantum nonlinear Hall effect induced by Berry curvature dipole in time-reversal invariant materials, [Phys. Rev. Lett. **115**, 216806 \(2015\)](#).
- [86] A. K. Dagnino, X. Liu, and T. Neupert, Non-linear transport in multifold semimetals, [arXiv:2512.08891](#).

- [87] D. T. Son and B. Z. Spivak, Chiral anomaly and classical negative magnetoresistance of Weyl metals, [Phys. Rev. B **88**, 104412 \(2013\)](#).
- [88] G. Varma K., A. Ahmad, S. Tewari, and G. Sharma, Magnetotransport in spin-orbit coupled noncentrosymmetric and Weyl metals, [Phys. Rev. B **109**, 165114 \(2024\)](#).
- [89] G. Varma K., A. Ahmad, and G. Sharma, Strain-induced Ettingshausen effect in spin-orbit coupled noncentrosymmetric metals, [Phys. Rev. B **113**, 075118 \(2026\)](#).